Moduli Spaces of Pointed Stable Curves of Genus 0

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1 学期. Minimum Basics on Algebraic Geometry

) Undergraduate Algebraic Geometry course

2 学期. $\overline{M}_{0,n}$

) First Graduate Algebraic Geometry course

3 学期. Construction + cohomology of sm. proj. var.

4 学期. Blowup formula and repr. th.

) Second Graduate Algebraic Geometry course

) Current Research.

§0. Minimum Basics on Algebraic Geometry

k = field of characteristic 0, alg. closed. (e.g. $\mathbb{C}$)

$$\mathbb{A}^n = \mathbb{A}^n_k = k^n = \{(a_1, \cdots, a_n) \mid a_i \in k\}$$ affine n-space

(a = $(a_1, \cdots, a_n)$)

$\mathbb{A}^n_k$ ← $\mathrm{Aut}(\mathbb{A}^n) = GL_n(k) \ltimes k^n$

$$\Gamma \subset k[x_1, \cdots, x_n] \Longrightarrow V(\Gamma) = \{a \in \mathbb{A}^n \mid f(a) = 0 \ \forall f \in \Gamma\} = \bigcap_{f \in \Gamma} V(f) \subset \mathbb{A}^n$$

affine variety

$V(\Gamma)$: Not compact

$V(f)$: hypersurface

$$\mathbb{P}^n = k^{n+1} - \{0\} / k^\times = \{[a_0, \cdots, a_n] \mid a_i \neq 0 \ \exists i\} = \{1\text{-dim'l subspaces of } k^{n+1}\}$$

($[a]$ = $[a_0, \cdots, a_n]$)

$[a_0, a_1, \cdots, a_n] = [t a_0, \cdots, t a_n]$, $\forall t \neq 0$.

projective n-space

$k = \mathbb{C}$: $S^{2n+1} \to \mathbb{P}^n$, $a \mapsto [a]$ $\qquad S^{2n+1}/S^1 = \mathbb{P}^n$

compact

$$\mathbb{P}^n \supset_{\text{open}} \{[a_0, a_1, \cdots, a_n] \mid a_0 \neq 0\} = \{[1, a_1, \cdots, a_n] \mid a_1, \cdots, a_n \in k\} = k^n = \mathbb{A}^n$$

$\in U_0$

$$\mathbb{P}^n - U_0 = \{[0, a_1, \cdots, a_n]\} = \mathbb{P}^{n-1} \Rightarrow \mathbb{P}^n = \mathbb{A}^n \amalg \mathbb{A}^{n-1} \amalg \cdots \amalg \mathbb{A}^1 \amalg p^1$$

$$\mathbb{P}^n = U_0 \cup \cdots \cup U_n \text{ open cover} \qquad U_i = \{[a_0, \cdots, a_n] \mid a_i \neq 0\}$$

$$GL_{n+1}(k) = \{(n+1)\times(n+1) \text{ invertible matrices}\} \subset M^{(n+1)\times(n+1)} = \mathbb{A}^{(n+1)^2}$$

$\det \neq 0$

$$= \mathbb{A}^{(n+1)^2} - V(\det)$$

$$PGL_{n+1}(k) = GL_{n+1}(k) / k^* = \{[A] \mid A \in GL_{n+1}(k)\}$$

$[A] = [tA]$, $t \neq 0$

$PGL_{n+1}(k)$ acts on $\mathbb{P}^n$ by $[A] \cdot [v] = [Av]$.

$$Aut(\mathbb{P}^n) = PGL_{n+1}(k)$$

$$GL_n(k) \ltimes k^n \leq PGL_{n+1}(k). \qquad \begin{pmatrix} B & w \\ 0 & a w B^{-1} \end{pmatrix}$$

$n = 1$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} a x_0 + b x_1 \\ c x_0 + d x_1 \end{pmatrix}$, $z = \frac{x_0}{x_1} \longmapsto \frac{a x_0 + b x_1}{c x_0 + d x_1} = \frac{az + b}{cz + d}$

fractional linear transformations

$\Omega \subset k[x_0, \cdots, x_n]$ homogeneous ideal $\Rightarrow$

$$V(\Omega) = \{ [a] \in \mathbb{P}^n \mid f(a) = 0, \ \forall f \in \Omega \text{ homogeneous} \} \subset \mathbb{P}^n$$

Projective Variety

$V(\Omega) - V(\mathcal{B})$ quasi-proj. var.

$f \in k[x_1, \cdots, x_n] \Rightarrow \tilde{f}(x_0, \cdots, x_n) = x_0^{\deg f} f\left(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}\right)$ homogenization

$h \in k[x_0, \cdots, x_n] \Rightarrow \hat{h}(x_1, \cdots, x_n) = h(1, x_1, \cdots, x_n)$ dehomogenization

$V(\tilde{\Omega}) \subset \mathbb{P}^n$ is a compactification of $V(\Omega) \subset \mathbb{A}^n$.

$\{ V(\Omega) \mid \Omega \subset k[x_0, \cdots, x_n] \text{ homogeneous} \}$ is a topology on $\mathbb{P}^n$

$V(\Omega)$ ← closed set

Zariski topology

$V(1) = \emptyset$. $V(0) = \mathbb{P}^n$

$V(\Omega) \cup V(\mathcal{B}) = V(\Omega \cap \mathcal{B})$

$\bigcap V(\Omega_i) = V(\sum \Omega_i)$

Algebraic Geometry = Study of (quasi-) projective varieties + ...

History.

~ 5^c: Mediterranean studied conics ; geometry $\Rightarrow$ algebra

~ 15^c : Asian mathematicians discovered 0, developed Algebra.

Al-Khowarizmi, "Al-jabr ...!" about solving equations of degree ≤ 2, & special cubic equations
($\gg$ algorithm)

Cardano, Tartalia. Ferari. cubic, quartic equations

1637 Descartes (Fermat) discovered the coordinate plane.

↕ This talented geometer + swordsman invented a systematic approach to geometry.

"Discourse on Method, Optics, Geometry and Meteorology".

Geometry $\rightleftarrows$ Algebra

Coordinate plane



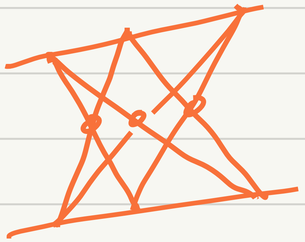
intersection points

$x^2+y^2=1$

$y-x=\frac{1}{2}$

common roots

~1850 Desargue, Pascal, Poncelet, ... developed projective geometry



$\mathbb{R}^2 \subset \mathbb{RP}^2 = \mathbb{R}^2 \cup S^1$ (open; compact; line at ∞)

$\cap$

$\mathbb{C}^2 \subset \mathbb{CP}^2 = \mathbb{C}^2 \cup S^2$

compactification by projective space

complexification

~1860 Riemann [applied complex analysis to study algebraic curves.
initiated intrinsic geometry.

$\dim\{\text{genus } g \text{ curves}\} = 3g-3$

~1930 Castelnuovo, Severi, ... (italian school) applied Riemann's method to surfaces & higher dim'l var. ⇒ classical alg. geo. [not so old]

Dedekind (german school), Kummer, Kronecker, Hilbert, Noether, Krull, ... developed commutative algebra to provide rigorous foundation for AG. ⇒ works for other fields k

Hilbert Nullstellensatz: $\{\text{max. ideals in } k[x_1, \cdots, x_n]\} \xleftrightarrow{1:1} k^n \quad (k = \bar{k})$

Algebra $\Longleftrightarrow$ Geometry

ring / ideal / module theory

1908 Poincaré predicted that methods of AG should apply to number theory.
↑ became systematic scientific

~1970 Weil, Serre, Grothendieck expanded AG from the study of polynomial rings (Noetherian rings) to arbitrary rings by discovering scheme theory, thus arose Arithmetic Geometry ⊂ Algebraic Geometry
∩
Number theory

~present. Atiyah, Donaldson, Witten, ··· found exciting links connecting AG and Physics.

Donaldson theory, Verlinde formula, Gromov-Witten theory, Donaldson-Thomas theory, wall crossing, DT4, ···
↙ Enumerative geometry of vector bundles

~present. Okounkov, Huh, ··· found geometric structures in combinatorics
↖ Hodge structure

A generalization of scheme theory that encompass geometry, ⌋ varieties
number theory, ⌋ schemes
combinatorics? ⌋ ??

NT

AG

$\mathbb{Q} \supset \mathbb{Z} \leftarrow\!\!-\!\!-\!\!-\!\!\rightarrow k[x] \subset k(x)$ (PID)

finite $\cap$ $\cap$ $\cap$ $\cap$ finite

$K \supset R \qquad R = k[C] \subset k(C) = K$

number field — ring of integers — coordinate ring of an affine curve C — function field of C

Compactification of Spec $\mathbb{Z}$??
Spec R

Spec $k[x]$ has compactification $\bar{C} \subset \mathbb{P}^1$ (projective curve)
Spec $R = C$

↑ easier with this!
e.g. Riemann hypothesis !

??

Higher dimensional varieties (infinite extensions)

natural number = product of primes $\longleftrightarrow$ finite numbers of points in line.

Mo,n

§1. $M_{0,n}$

$p_1, \cdots, p_n \in \mathbb{P}^1 \sim p_1', \cdots, p_n' \in \mathbb{P}^1$
$\iff \exists g \in PGl_2(k)$ s.t. $g \cdot p_i = p_i' \ \forall i$.

main object of interest in invariant theory since 19th c.

$$\{ n \text{ distinct points } p_1, \cdots, p_n \text{ in } \mathbb{P}^1 \} / Aut(\mathbb{P}^1) =: M_{0,n}$$

$\| \quad Aut(\mathbb{P}^1) = PGl_2(k) = \{\text{fractional linear transf.}\}$

$\subset (\mathbb{P}^1)^n / PGl_2$

① $(\mathbb{P}^1)^n - \Delta / PGl_2$ $\qquad \Delta = \bigcup_{i \neq j} \Delta_{ij}$, $\Delta_{ij} = (p_i = p_j)$

$\| \leftarrow (\mathbb{P}^1)^n = (k^2 - 0)^n / (k^*)^n \subset (k^2)^n / (k^*)^n = k^2 \otimes k^n / (k^*)^n$

$\| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \quad M^{2\times n} / (k^*)^n$

$k^2 \otimes k^n - \Delta / GL_2(k) \times (k^*)^n$

GL_2 ↷ $k^2 \otimes k^n$, $(k^*)^n$ ↷ k^n

$\|$

② $Gr(2,n) - \Delta / (k^*)^n \subset Gr(2,n) / (k^*)^n$

$GL_2 \backslash M_2^{2\times n} =: Gr(2,n) = \{ 2\text{-dim'l subsp. in } k^n \}$

plücker $\int$ 2×2 minor det.

$\mathbb{P}^{\binom{n}{2}-1}$ image $V(z_{ij}z_{kl} - z_{ik}z_{jl} + z_{il}z_{jk}) =$ Grassmannian variety.

$\bigcup_{i \neq j} (z_{ij} = 0)$ plücker coordinates

③ Cross ratio

$$\lambda(z_1,z_2,z_3,z_4) = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)} \quad \text{cross ratio}$$

$z_1 \to 1,\ z_2 \to \infty,\ z_3 \to 0,\ z_4 \to \lambda$ via $\dfrac{(z_1-z_2)(z-z_3)}{(z_1-z_3)(z-z_2)}$

$$(p_1,\cdots,p_n) \underset{PGL_2}{\sim} (p_1',\cdots,p_n') \iff \lambda(p_1,p_2,p_3,p_i) = \lambda(p_1',p_2',p_3',p_i') \quad \forall\ 4 \le i \le n$$

$$M_{0,n} \xrightarrow{\cong} (\mathbb{P}^1 - \{0,1,\infty\})^{n-3} - \Delta \subset \mathbb{P}^{n-3}$$

$$(p_1,\cdots,p_n) \longmapsto (\lambda(p_1,p_2,p_3,p_4),\cdots,\lambda(p_1,p_2,p_3,p_n))$$

④ Rational normal curves

$\nu_d : \mathbb{P}^1 \to \mathbb{P}^d$, $[x_0,x_1] \mapsto [x_0^d, x_0^{d-1}x_1, \cdots, x_1^d]$ (monomials of degree d)

$\nu_d(\mathbb{P}^1) = \mathbb{V}(z_i^2 - z_{i-1}z_{i+1},\ z_{i-1}z_{i+2} - z_iz_{i+1}) \subset \mathbb{P}^d$ projective curve

$C \sim \nu_d(\mathbb{P}^1)$ PGL_{d+1} rational normal curve of degree d

Let $q_1 = [e_0]$, $q_2 = [e_1]$, $\cdots$, $q_{n-2} = [e_{n-3}]$, $q_{n-1} = [1,1,\cdots,1] \in \mathbb{P}^{n-3}$.

($e_0 \to (1,0,\cdots,0)$, $e_1 \to (0,1,0,\cdots)$, $e_{n-3} \to (0,\cdots,0,1)$)

$\forall\ q_n = [\mu_0,\cdots,\mu_{n-3}] \in \mathbb{P}^{n-3}$ with $[\ \mu_i \neq \mu_j\ \ \forall i \neq j,\ \ \mu_i \neq 0\ \ \forall i\ ]$

$(n-1)$ general points in $\mathbb{P}^{n-3}$ $\Leftarrow$ [$\le n-2$ points are linearly indep.]

$\exists!$ rational normal curve $\nu : \mathbb{P}^1 \to \mathbb{P}^{n-3}$ through $q_1,\cdots,q_n$

$[x_0,x_1] \mapsto [f_0(x_0,x_1),\cdots,f_{n-3}(x_0,x_1)]$ lin. indep homogeneous poly. of degree $n-3$

$$f_i(x_0, x_1) = \prod_{j\neq i} (x_0 - \mu_j^{-1} x_1) = \frac{\prod_{j=0}^{n-3} (x_0 - \mu_j^{-1} x_1)}{x_0 - \mu_i^{-1} x_1}$$

$$\rightsquigarrow \left[\begin{array}{lll} f_i(1, \mu_j) = \begin{cases} 0 & \text{if } i \neq j \\ \prod_{j\neq i}(1 - \mu_j^{-1}\mu_i) \neq 0 & \text{if } i = j \end{cases} & & \Rightarrow \nu[1, \mu_j] = [e_j] = q_{j+1} \\ f_i(1, 0) = 1 & \forall i & \Rightarrow \nu[1, 0] = q_{n-1} \\ f_i(0, 1) = -\mu_i \prod_{j=0}^{n-3} (-\mu_j) & \forall i & \Rightarrow \nu[0, 1] = q_n. \end{array} \right.$$

n points in $\mathbb{P}^1$

n points in $\mathbb{P}^{n-3}$

Let $\mu_0 = 1$. Then $\left\{ q_n = [1, \mu_1, \cdots, \mu_{n-3}] \;\middle|\; \begin{array}{l} \mu_i \neq 0, 1 \\ \mu_i \neq \mu_j \;\; \forall i \neq j \end{array} \right\} \xrightarrow{\cong} (k - \{0,1\})^{n-3} - \Delta$

$\quad \parallel \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

$\mathbb{P}^{n-3} - \bigcup_{i \neq j} \left(\text{hyperplane spanned by } \{q_1, \cdots, q_{n-1}\} - \{q_i, q_j\} \right) \qquad M_{0,n}$

$\leftarrow \{\mu_i \neq \mu_j\} = \Delta_{ij}$

(S) $(\mathbb{P}^{n-3})^n_{\text{general}} / PGL_{n-2} = M_{0,n}$

§2. Moduli functor

To compare different realizations of $M_{g,n}$ and construct a natural compactification, it is convenient to set up a moduli problem / functor.

Moduli Problem

1) objects to classify + isomorphism

2) families of objects + equivalence relation

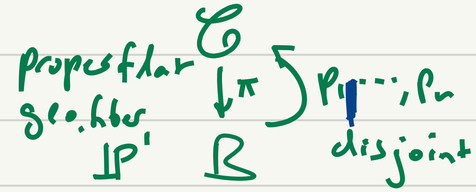
isom. over B = Spec k

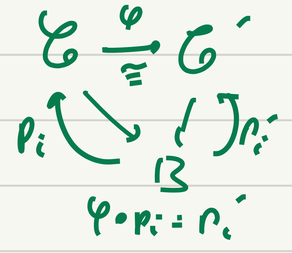
3) pull-back of a family

$M_{g,n}$

$\mathbb{P}^1 \ni p_1, \cdots, p_n$ distinct

projective equivalence PGL_2.







Moduli functor

$$F: (\mathrm{Aff})\mathrm{Sch}/k \longrightarrow \mathrm{Sets}\ (\text{or Groupoids})$$

$$B \longmapsto \{\text{families par. by } B\}/\text{equiv.}$$

$$\mathrm{Spec}\, k \longmapsto \{\text{objects}\}/\text{isom}$$

$$\mathcal{M}_{0,n}(B) = \left\{ \begin{array}{c} \mathcal{C} \\ \pi \downarrow \\ B \end{array} \supset \mathbb{P}^1\text{-bundle } P_1, \dots, P_n \text{ disjoint} \right\} / \text{equiv.}$$

$$\mathcal{M}_{0,n}(\mathrm{Spec}\, k) = M_{0,n}$$

Ex. X = scheme, $\mathcal{C}$ = points of X, $h_X(B) = \mathrm{Mor}(B, X)$. ← h_X determines X up to isom.

Moduli Space

If $F = h_M$ for some variety / scheme / stack M, then M is called a (fine) moduli space

- M is uniquely determined by F.
- $\exists$ universal family $\mathcal{U} \in F(M)$ parameterized by M s.t. $\forall \xi \in F(B)$, $\exists!\ B \xrightarrow{\varphi} M$ s.t. $\xi = F(\varphi)(\mathcal{U})$.

Exercise. ① ~ ⑤ are all fine moduli sp. for $M_{0,n}$. (Hence all isomorphic.)

$$M_{0,3} = pt, \quad M_{0,4} = \mathbb{P}^1 - \{0, 1, \infty\}, \quad M_{0,5} = (\mathbb{P}^1 - \{0,1,\infty\})^2 - \Delta, \cdots$$

Compactification of $M_{0,n}$? Add limit points.

$n = 4$

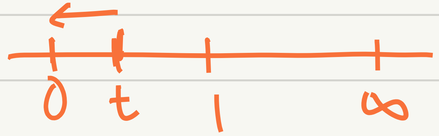
$\overline{M}_{0,4} \simeq \mathbb{P}^1$



Moduli functor for $\overline{M}_{0,4}$?

$p_1 \quad p_2 \quad p_3 \quad p_4$

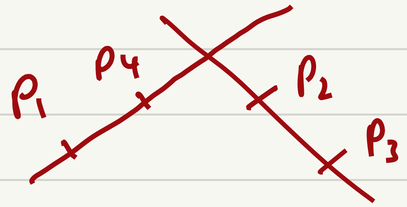
$(0, 1, \infty, t) \sim (0, t^{-1}, \infty, 1)$



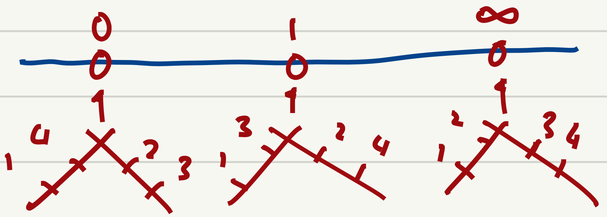


$(p_1 = p_4, p_2, p_3)$ $(p_1, p_2 = p_3, p_4)$

Which one is the correct limit?



Deligne-Mumford



§3. Stable Curves

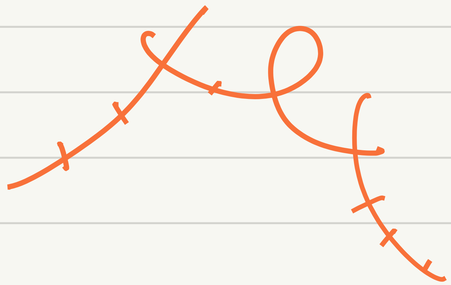
$(C, p_1, \cdots, p_n)$ with $|\mathrm{Aut}| < \infty$

- C: connected curve with $\le$ nodal sing. $h^1(\mathcal{O}_C) = g$
- $p_1, \cdots, p_n$: n distinct smooth points in C

nonsing.: unique tangent line

node: 

2 tangent lines



$|\mathrm{Aut}(C, p_i)| < \infty \iff$ each rational (elliptic) irreducible component has ≥ 3 (≥ 1) special points ← nodes + marked points

$\iff \omega_C^{\log} = \omega_C(p_1 + \cdots + p_n)$ is ample

Exercise. $\boxed{g = 0}$

(1) $|\mathrm{Aut}(C, p_i)| < \infty \iff \mathrm{Aut}(C, p_i) = 1$

(2) C is a tree of $\mathbb{P}^1$'s (no loops) and each $\mathbb{P}^1$ has ≥ 3 special points.



Deligne - Mumford (1969). $\overline{\mathcal{M}}_{g,n}$ is a smooth proper DM stack irreducible of dim $3g-3+n$.

$\rightsquigarrow$ $\overline{M}_{g,n}$ coarse moduli space (orbifold)

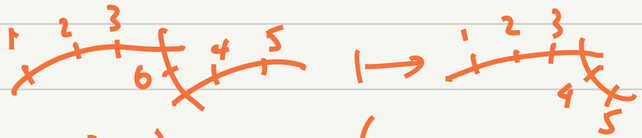
Gieseker ($n=0$, 1983), Ball/Swinarski (2008).

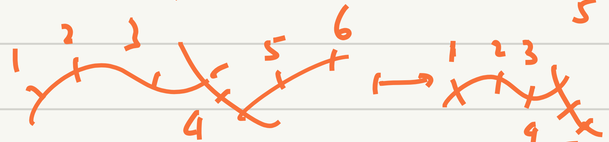
Constructed $\overline{M}_{g,n}$ as projective GIT quotient.

Knudsen ($g=0$, 1983) $\overline{\mathcal{M}}_{0,n} = \overline{M}_{0,n}$ irred. smooth proj. var.

$\cup$

$M_{0,n}$ open dense





$(C, p_1, \dots, p_{n+1}) \mapsto (\bar{C}, p_1, \dots, p_n)$

$\overline{M}_{0,n} - M_{0,n} = \bigcup_{\substack{S \subset [n] \\ 2 \le |S| \le \lfloor \frac{n}{2} \rfloor}} \overline{M}_{0,S\cup *} \times \overline{M}_{0,S^c \cup *}$



smooth normal crossing divisors

$\mathcal{C} = \overline{M}_{0,n+1} \xrightarrow{\text{for}} \overline{M}_{0,n}$ is the universal curve

Bruno - Mella (2013) $\mathrm{Aut}(\overline{M}_{0,n}) = S_n$ for $n \ge 5$

§4. Constructions of $\bar{M}_{0,n}$

Knudsen (1983) — Inductive construction

$\bar{M}_{0,n} = \{(\pi^{-1}(\xi), p_1(\xi), \cdots, p_n(\xi))\}$ (C_ξ)

All members of $\bar{M}_{0,n}$ appear exactly once & functorial property.

$\pi(p_1, \cdots, p_n, p_{n+1}) = (p_1, \cdots, p_n)$

univ. family

$\bar{M}_{0,n+2}$ → blowup along ($\delta = p_i$) → $\tilde{C}$ (singular, $xy = zw$) → $\bar{M}_{0,n+1}$

p_i, δ ; π ; p_i

$\bar{M}_{0,n+1} \xrightarrow{\pi} \bar{M}_{0,n}$

$\tilde{C}$ hard for computation



$Bl_0 \mathbb{A}^r = \{(v, [w]) \mid v \parallel w\} \subset \mathbb{A}^r \times \mathbb{P}^{r-1}$

isom over $\mathbb{A}^r - 0$; $\mathbb{A}^r$

$Z \subset X$ smooth $\Rightarrow$ locally $Z \subset Z \times \mathbb{A}^r$

$Bl_Z X$; $Z \times Bl_0 \mathbb{A}^r$

Keel (1992)

$\bar{M}_{0,n+2} = U_{n-2} \to \cdots \to U_1 \to \bar{M}_{0,n} \times \bar{M}_{0,4}$

blowups

p_i' ; $\gamma = (\pi, f_{1,2,3,n+1})$; π ; $\gamma \circ p_i$

$\bar{M}_{0,n+1} \xrightarrow{\pi} \bar{M}_{0,n}$; p_i

NOT S_n-equivariant

Kapranov (1994)

$\mathbb{P}^{n-3} \ni p_1, \cdots, p_{n-1}$

fixed general points

$[e_0], \cdots, [e_{n-3}], [e_0 + \cdots + e_{n-3}]$

$$M_{0,n} = \mathbb{P}^{n-3} - \text{zero}\left(\prod_{i=0}^{n-3} 2i \prod_{i<j} (z_i - z_j) \right)$$

hyperplane arrangement

Blow up along $p_1, \cdots, p_{n-1}$

proper transforms of $\overline{p_i p_j}$

" " $\overline{p_i p_j p_k}$

$\vdots$

" " $\overline{p_{i_1} p_{i_2} \cdots p_{i_{n-2}}}$

$\Rightarrow \overline{M}_{0,n}$.

$\overline{M}_{0,n}$ = Chow or limit Mumford quotient of $(\mathbb{P}^1)^n$ by PGL_2

= Hilbert, Chow or limit Mumford quotient of $Gr(2,n)$ by $\mathbb{G}_m^{n-1}$.

S_{n-1} - equivariant.

Can compute $H^*(\overline{M}_{0,n})$ as S_{n-1}-representation.

NOT S_n - equivariant.

Hassett (2003)

$A = (a_1, \cdots, a_n) \in (\mathbb{Q} \cap [0,1])^n$. $\sum_{i=1}^n a_i > 2$

$(C, p_1, \cdots, p_n)$ is A-stable if i) $\omega_C(\sum a_i p_i)$ is ample $\left(\forall C_i \subset C \text{ irred. comp.},\ \#\text{nodes in } C_i + \sum_{p_j \in C_i} a_j > 2\right)$

C: $\leq$ nodal sing. $p_1, \cdots, p_n$: smooth points (not nec. distinct)

ii) $p_{i_1} = \cdots = p_{i_k} \Rightarrow a_{i_1} + \cdots + a_{i_k} \leq 1$.

$\exists$ irreducible projective coarse moduli space $\overline{M}_{g,A}$ of A-stable curves. Smooth if $g=0$.

If $A \geq B$, $\exists$ morphism $\overline{M}_{g,A} \xrightarrow{\rho_B^A} \overline{M}_{g,B}$, $\rho_C^B \circ \rho_B^A = \rho_C^A$.

Ex.

$A = (1, 1, \cdots, 1) \Rightarrow \overline{M}_{0,A} = \overline{M}_{0,n}$

$A_0 = (1, \frac{1}{n-2}, \cdots, \frac{1}{n-2}) \Rightarrow \overline{M}_{0,A_0} = \mathbb{P}^{n-3}$

$A_1 = (1, \frac{1}{n-3}, \cdots, \frac{1}{n-3})$ $\quad \overline{M}_{0,A_1} = bl_{\{p_i\}} \overline{M}_{0,A_0}$

$A_2 = (1, \frac{1}{n-4}, \cdots, \frac{1}{n-4})$ $\quad \overline{M}_{0,A_2} = bl_{l_{ij}} \overline{M}_{0,A_1}$

$\vdots$ $\qquad \vdots$

$A_{n-4} = (1, \frac{1}{2}, \cdots, \frac{1}{2})$ $\quad \overline{M}_{0,A_{n-4}} = bl\ \overline{M}_{0,A_{n-5}}$

$[M_0, M_1, \cdots, M_{n-3}] \longleftrightarrow \infty, 0, M_0, \cdots, M_{n-3} \in \mathbb{P}^1$, $\mathbb{P}^{n-3} \hookrightarrow \mathbb{P}^{n-2}$

Kapranov's blowups! S_{n-1}-equivariant.

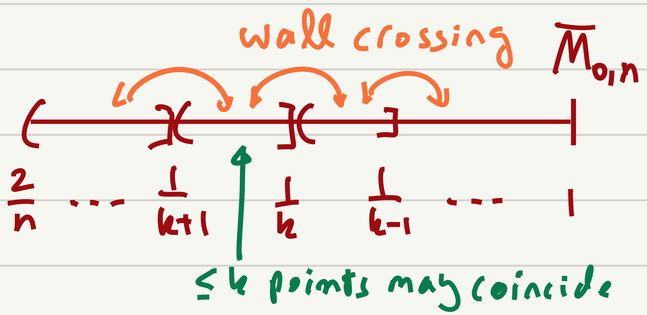
l_{ij} = proper transf. of $\overline{p_i p_j}$.

$\vdots$

$l_{i_1 \cdots i_{n-4}}$ = prop. transf. of $\overline{p_{i_1} \cdots p_{i_{n-4}}}$.

Ex. $\overline{M}_{0,n\cdot\varepsilon} = \overline{M}_{0,(\varepsilon,\cdots,\varepsilon)}$ $\quad \frac{2}{n} < \varepsilon \leq 1$ Symmetric type

$S_n \curvearrowright$



$\overline{M}_{0,n} = \overline{M}_{0,n\cdot\frac{1}{2}} \to \overline{M}_{0,n\cdot\frac{1}{3}} \to \overline{M}_{0,n\cdot\frac{1}{4}} \to \cdots \to \overline{M}_{0,n\cdot\frac{1}{\lfloor\frac{n-1}{2}\rfloor}}$

S_n-equivariant blowups

Last term

When $n = 2m+1$, $\overline{M}_{0,n\cdot\varepsilon} = (\mathbb{P}^1)^n /\!/ PGL_2$ for $\frac{2}{n} < \varepsilon \leq \frac{1}{m}$

H^* computable by Atiyah-Bott-Kirwan

[K.-Moon 2011]

When $n = 2m$, $\overline{M}_{0,n\cdot\varepsilon} = \widetilde{(\mathbb{P}^1)^n /\!/ PGL_2}$ for $\frac{2}{n} < \varepsilon \leq \frac{1}{m-1}$

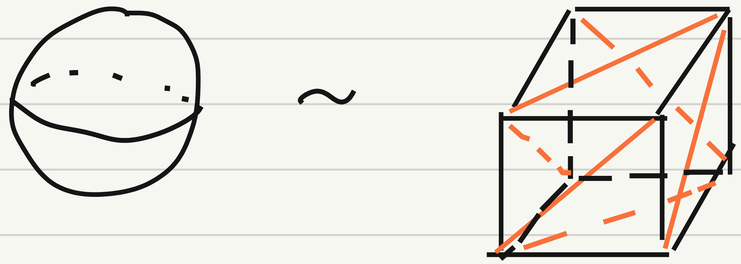
Partial desingularization of $(\mathbb{P}^1)^n /\!/ PGL_2$

Blowup centers are not symmetric like $\overline{M}_{0,(1,\varepsilon,\cdots,\varepsilon)} \times \overline{M}_{0,(1,\varepsilon,\cdots,\varepsilon)}$.
Can compute S_n-representation on $H^*(\overline{M}_{0,n})$ but it is very complicated!!
[Bergstrom-Minabe]

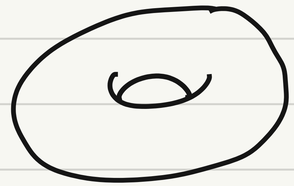
Moon (2013) $\overline{M}_{0,n}$ are log minimal models of $\overline{M}_{0,n}$.

§5. Cohomology of $\bar{M}_{0,n}$ as a Vector space

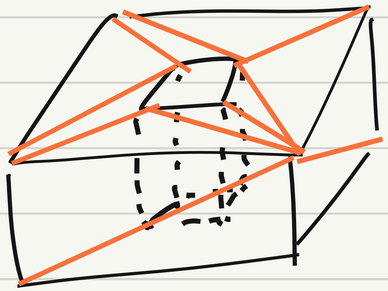
X = top. space + triangulation + orientation



$\sim$



$\sim$



$A_i(X)$ = $\mathbb{C}$- vector space spanned by i-dim'l simplices

$\downarrow \partial_i$

$A_{i-1}(X)$



$\partial_{i-1} \circ \partial_i = 0$

$H_i(X) := \ker \partial_i \,/\, \operatorname{im} \partial_{i+1}$ homology of X

$A^i(X) = A_i(X)^\vee$

$\uparrow \partial_i^\vee = \partial^{i-1}$

$A^{i-1}(X)$

- same as de Rham coh.
- ring str. by $\wedge$ product of diff. forms

$\rightarrow H^i(X) = \ker \partial^i \,/\, \operatorname{im} \partial^{i-1}$ cohomology of X

$\| \; H^i(X, \mathbb{C})$

Indep. of triangulation

X = smooth projective variety / $\mathbb{C}$, $n = \dim_{\mathbb{C}} X$ ($\dim T_p X = \dim_p X \ \forall p \in X$)

$H^*(X, \mathbb{C})$ satisfies

1) Poincaré duality $H^i(X, \mathbb{C}) \cong H^{2n-i}(X, \mathbb{C})^\vee$



dual triangulation

2) Hodge decomposition

$H^i(X, \mathbb{C}) = \bigoplus_{\substack{p+q=i \\ p,q \geq 0}} H^{p,q}(X)$, $H^{p,q}(X) = \overline{H^{q,p}(X)}$

$H^{p,q}(X)$: harmonic (p,q)-forms

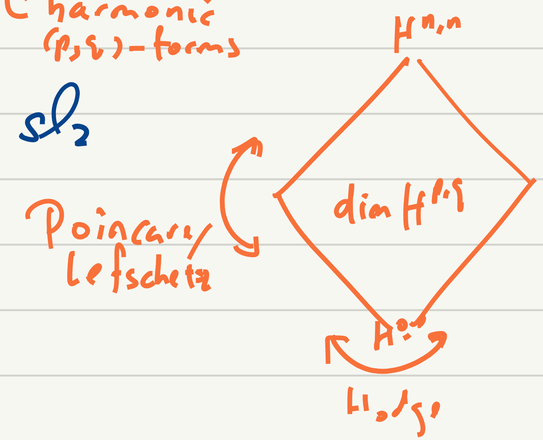
$(\operatorname{im} d)^\perp = \ker d^*$: harmonic forms

3) Hard Lefschetz decomposition

$\exists\, L : H^{p,q}(X) \to H^{p+1,q+1}(X)$

$L^{n-p-q} : H^{p,q}(X) \xrightarrow{\cong} H^{n-q,n-p}(X)$

$H^*(X, \mathbb{C}) \circlearrowleft sl_2$



4) Lefschetz hyperplane theorem

5) Hodge index theorem

Virtual Poincaré Polynomial

Mixed Hodge Structure

Y = q-proj. var./$\mathbb{C}$ $\overset{\text{Deligne}}{\rightsquigarrow}$ $H_c^*(Y,\mathbb{C})$ admits a mixed Hodge str.

weight & Hodge filtration

$h^{p,q;i} = \dim H^{p,q}(H_c^i(Y,\mathbb{C}))$

$\Rightarrow$ $E_{u,v}(Y) = \sum_{p,q,i} (-1)^i h^{p,q;i}(Y) u^p v^q$ ← E-polynomial of Y

$P_t(Y) = E_{-t,-t}(Y) = \sum (-1)^{i+p+q} h^{p,q;i}(Y) t^{p+q}$ ← virtual Poincaré poly of Y

- If Y is smooth & projective, MHS is pure & $P_t(Y) = \sum_i t^i \dim H^i(Y)$
  ↑ [$h^{p,q;i} = 0$ unless $p+q=i$, $p,q \geq 0$.]
- $Y \supset Z$ closed $\Rightarrow$ $P_t(Y) = P_t(Z) + P_t(Y-Z)$
  $P_t(Y \times Y') = P_t(Y) \cdot P_t(Y')$
  ↑ [$\cdots \leftarrow H_c^i(Z) \leftarrow H_c^i(Y) \leftarrow H_c^i(Y-Z) \leftarrow \cdots$ respects MHS.]

Manin (1995) $\overline{M}_{0,n} \xrightarrow{\varphi} M_{0,n} \cup \left(\bigcup_{\substack{S \subset [n] \\ 2 \le |S| \le [\frac{n}{2}]}} \overline{M}_{0,S\cup *} \times \overline{M}_{0,S^c \cup *} \right) = M_{0,n} \cup \left(\bigcup_S (\quad) \times (\quad) \right)$

$= \cdots$

Knudsen

D_S divisor

$P_t(M_{0,n}) + \text{combinatorics} \Rightarrow P_t(\overline{M}_{0,n})$

$M_{0,n+1} = M_{0,n} \times \mathbb{P}^1 - \bigcup_{i=1}^{n} p_i(M_{0,n}) \Rightarrow P_t(M_{0,n+1}) = (1+t^2) P_t(M_{0,n}) - n P_t(M_{0,n})$

$M_{0,n}$ ← $p_1, \dots, p_n$

$$P_t(\overline{M}_{0,n+1}) = (1+t^2) P_t(\overline{M}_{0,n}) + \frac{t^2}{2} \sum_{\substack{i+j=n \\ i \ge 2}} \binom{n}{i} P_t(\overline{M}_{0,i+1}) P_t(\overline{M}_{0,j+1})$$

Getzler (1995) Can refine this to calculate the characters of $H^*(\overline{M}_{0,n})$ as S_n-representation.

[NOT really practical since MHS is hard to see.]

§6. More on the cohomology of $\overline{M}_{0,n}$

Blowup formula

$$Z \underset{\text{smooth}}{\subset} X, \qquad \tilde{X} = bl_Z X \quad \text{blowup of } X \text{ along } Z$$

$$\Rightarrow \quad H^*(\tilde{X}) = H^*(X) \oplus H^*(E) \Big/ H^*(Z)$$

$$\begin{array}{ccc} \mathbb{P}N_{Z/X} = E & \hookrightarrow & \tilde{X} \\ \downarrow & & \downarrow e \\ Z & \hookrightarrow & X \end{array}$$

$$H^*(E) = H^*(Z)[\xi] \Big/ \xi^r + c_1(N)\xi^{r-1} + \cdots + c_r(N)$$

[Can compute the cohomology ring of $\tilde{X}$ from $H^*(X), H^*(Z), c_i(N_{Z/X})$.
Same holds for other coh. theories like chow, K, ⋯.

Keel (1992) Using the inductive construction of $\bar{M}_{0,n}$ by blowing up $\bar{M}_{0,n-1} \times \mathbb{P}^1$, Keel proved that

$$A^*(\bar{M}_{0,n}) = H^*(\bar{M}_{0,n}) = \text{free } \mathbb{Z}\text{-algebra gen. by } \{[D_S] \mid 2 \le |S| \le n-2,\ S \subset [n]\} \Big/ I$$

Coh. theory spanned by subvarieties

boundary divisors



I is generated by
- $[D_S] = [D_{S^c}]$
- $[D_S] \cdot [D_{S'}] = 0$ if $S \cap S' = \emptyset$
- WDVV

$\forall |S| = 4$ $\quad \bar{M}_{0,n} \xrightarrow{f} \bar{M}_{0,S} \cong \mathbb{P}^1$, $\quad f^*[0] = f^*[\infty]$

Witten (1992)
Kontsevich

$$\int_{\bar{M}_{0,n}} \prod_{i=1}^{n} \psi_i^{k_i} = \binom{n-3}{k_1, \cdots, k_n}$$

$\psi_i = c_1(p_i^* \Omega_{\mathcal{C}/\bar{M}_{0,n}})$.

Problem. $\mathrm{Aut}(\bar{M}_{0,n}) = S_n$ acts on $H^*(\bar{M}_{0,n})$, $K_0(\bar{M}_{0,n})$, ...

Compute the character of the S_n-representation.

Getzler, Bergstrom-Minabe ⇒ algorithms. [Extremely hard for practical computation]
MHS Hassett

Need to compute $H^*(\bar{M}_{0,A})$ as S_A-module

No closed formula was known before.

Question (Manin, Orlov). Is the representation $H^{2k}(\bar{M}_{0,n})$ of S_n a permutation representation?

Ramadas-Silversmith: Yes for $n \le 10$ or $k \le 2$ ($k \ge n-5$).

Castravet-Tevelev: $D^b(\bar{M}_{0,n})$ has a full exceptional collection

$\rightsquigarrow$ $K_0(\bar{M}_{0,n})$ is a permutation representation

$\rightsquigarrow$ $H^*(\bar{M}_{0,n})$ is a perm. repr. ($\not\Leftarrow$ $H^{2k}(\bar{M}_{0,n})$ perm. repr. for each k)

[$S_n \to GL_N$ factors through $\exists\ S_N \hookrightarrow GL_N$, $N = \dim H^{2k}(\bar{M}_{0,n})$]

§7. A New Construction by Stable Pairs

point $p \in C$ = $(\mathcal{O}_C(p), s)$

C = projective curve, $\mathcal{O}_C(1)$ = ample line bundle, $P(t) = rt + \chi \in \mathbb{Z}[t]$

$\alpha : \mathcal{O}_C^{\oplus m} \to E$, E = coherent sheaf on C with Hilb. poly $P(t)$, $\delta \in \mathbb{Q}_{\geq 0}$.

Def (E, α) is δ-(semi)stable w.r.t. $\mathcal{O}_C(1)$ if

$\forall\, 0 \neq E' \subsetneq E$, the Hilbert poly $\chi(E'(t)) = r't + \chi'$ satisfies

i) $r' \neq 0$, ii) $\dfrac{\chi' + \theta(E', \alpha)\delta}{r'} \lesssim \dfrac{\chi + \delta}{r}$

where $\theta(E', \alpha) = \begin{cases} 1 & \text{if } \alpha \text{ factors through } E' \\ 0 & \text{if not.} \end{cases}$

Thm. (Le Potier, Simpson, Huybrechts-Lehn)

$\exists$ projective coarse moduli space of (S-equivalence classes of) δ-semistable pairs

$\left(\because \text{Incidence} /\!/ PGL(P(t)) \subset Quot\left(\mathbb{C}^{P(t)} \otimes \mathcal{O}_C(-t), P\right) \times \mathbb{P}\mathrm{Hom}\left(H^0(\mathcal{O}_C(t))^{\oplus m}, \mathbb{C}^{P(t)}\right) /\!/ PGL(P(t)) \right)$

$\mathcal{C}$ univ. family

$p_1, \dots, p_n$ ($\downarrow \pi$

$\bar{M}_{g,n} = M$

$\omega_{\mathcal{C}/M}^{\log}$ relatively ample $\Longrightarrow$

$Q^{\delta}_{g,n,d,m}$

$\downarrow$

$\bar{M}_{g,n}$

projective moduli sp. of (S-equiv. classes of) δ-semistable pairs (E, α) on stable curves $C \in \bar{M}_{g,n}$ w.r.t. $\omega_C^{\log}$.

Hilb. poly $(2g+n-2)t + d+1-g$

$$\boxed{g = 0,\ d = 1,\ m = 1}$$

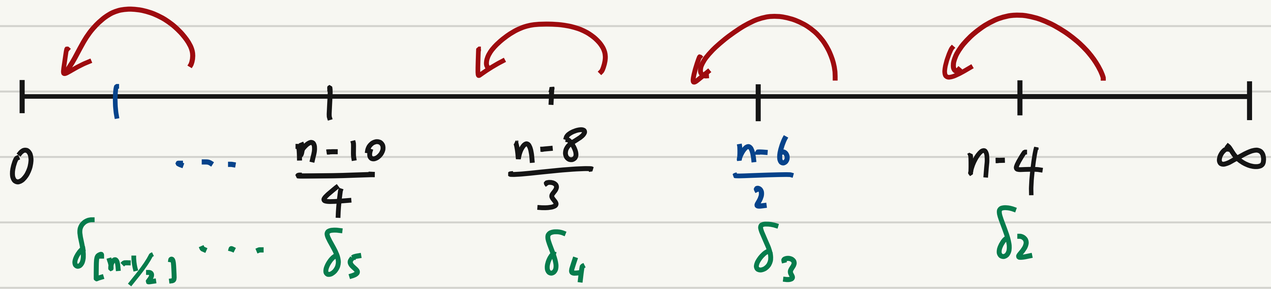
$$Q^{\delta}_{0,n,1,1} = Q^{\delta} = \left\{ (C, L, s) \;\middle|\; \begin{array}{l} C \in \bar{M}_{0,n} \\ L = \text{pure sheaf on } C \text{ of rank 1, deg 1} \\ s \in H^0(C, L) \\ (L, s) \text{ } \delta\text{-semistable w.r.t. } \omega_C^{\log} \end{array} \right\}$$

Q^{δ} changes at $\delta_\ell = \frac{n-2}{\ell-1} - 2 > 0$

$\exists\ 2 \le \ell < \frac{n}{2}$.

The walls are $n-4,\ \frac{n-6}{2},\ \frac{n-8}{3},\ \frac{n-10}{4},\ \dots,\ \frac{1}{k-1}$ if $n = 2k+1$.

$\frac{2}{k-2}$ if $n = 2k$.



S_n - equivariant !

(1) The wall crossing at δ_ℓ is the blowup along $\coprod_{|S|=\ell} D_S$ $\left(D_S = \bar{M}_{0,S\cup *} \times \bar{M}_{0,S^c\cup *}\right)$

(2) $\delta = \infty$ - stable $\iff$ $\dim \operatorname{coker}(s: \mathcal{O}_C \to L) = 0$

$\mathcal{O}_p$ $\exists p \in C$

$$Q^{\delta=\infty}_{0,n,1,1} = \overline{M}_{0,n+1}.$$

(3) $\delta = 0$ - stable $\iff$ L is stable on C w.r.t. $\omega_C^{\log}$

$$Q^{\delta=0}_{0,n,1,1} = \mathbb{P}\pi_* \mathcal{L} \quad \text{if } n \text{ is odd.}$$

$\downarrow$ $\mathbb{P}^1$ - bundle

$\bar{M}_{0,n}$

$$Q^{\delta=0^+}_{0,n,1,1} = \text{blowup of } \mathbb{P}^1\text{- bundle on } \bar{M}_{0,n} \text{ along } \coprod_{|S|=n/2} D_S \quad \text{if } n \text{ is even.}$$

Thm (최진현 - K. - 이동건)

(1) A closed formula for the character of $H^k(\bar{M}_{0,n})$ as S_n-repr.

(2) $H^{2k}(\bar{M}_{0,n}) \oplus H^{2k+2}(\bar{M}_{0,n})$ is a perm. repr. $\forall k, n$

(3) $H^{2k}(\bar{M}_{0,n})$ is a perm. repr. of S_n for $k \le 3$ or $k \ge n-6$.

$\bar{M}_{0,n+1} = Q^{\delta=\infty} \xrightarrow{\text{blowup}} \cdots \xrightarrow{\text{blowup}} Q^{\delta=0} \xrightarrow{\mathbb{P}^1} \bar{M}_{0,n} \Rightarrow S_n\text{-repr. } H^*(\bar{M}_{0,n+1}) \sim S_n\text{-repr. } H^*(\bar{M}_{0,n})$

Kapranov : $\bar{M}_{0,n+1} \to \cdots \to \mathbb{P}^{n-3} \Rightarrow S_n$ - repr. of $H^*(\bar{M}_{0,n+1}) \Rightarrow S_n$ - repr. $H^*(\bar{M}_{0,n})$

- Same method works for K-groups, ...
- Generalization to moduli of spin curves.
- $m=2$: Rep. th. of $H^*(\mathbb{P}^1[n])$ $\xleftrightarrow{\text{theta characteristics}}$ Rep. th. of $H^*(\bar{M}_{0,n})$.

Theorem 2.5. *Let p be a nonnegative integer. Then $A^p(\overline{M}_{0,n})$ is written as virtual permutation representation in the following forms.*

(1) *If n is odd, then*

$$\chi_{A^p(\overline{M}_{0,n})} = \sum_{\substack{T\in\mathcal{T}_{n,p-k}/S_n \\ k\geq 0}} (-1)^k \chi_{U_T} + \frac{1}{2} \sum_{\substack{2\leq h\leq n-2 \\ i+j\leq p-1}} \sum_{\substack{T_1\in\mathcal{T}_{h,i}/S_h \\ T_2\in\mathcal{T}_{n-h,j}/S_{n-h}}} (-1)^{p-i-j} \chi_{U_{T_1,T_2}}.$$

(2) *If n is even, then*

$$\chi_{A^p(\overline{M}_{0,n})} = \sum_{\substack{T\in\mathcal{T}_{n,p-k}/S_n \\ k\geq 0}} (-1)^k \chi_{U_T} + \frac{1}{2} \sum_{\substack{2\leq h\leq n-2 \\ i+j\leq p-1}} \sum_{\substack{T_1\in\mathcal{T}_{h,i}/S_h \\ T_2\in\mathcal{T}_{n-h,j}/S_{n-h} \\ T_1\neq T_2}} (-1)^{p-i-j} \chi_{U_{T_1,T_2}}$$

$$+ (-1)^p \sum_{\substack{T\in\mathcal{T}_{n/2,i}/S_{n/2} \\ 2i\leq p-1}} \left(\chi_{U_{T,T}} - \chi_{U_{T,T}^{S_2}}\right),$$

where $T_1 \neq T_2$ means T_1, T_2 are not isomorphic up to permutations of inputs.