MODULI SPACE OF VECTOR BUNDLES

AG - NT SCHOOL

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PART 1.

CH1.

$\mathbb{P}^n$ = PROJ. SPACE / $\mathbb{C}$ $\ni [x_0, x_1, \cdots, x_n]$

$= \mathbb{C}^{n+1} \setminus \{0\} / \sim$

$R = \mathbb{C}[x_0, \cdots, x_n] = \oplus R_d$

DEF

$R \supset \Sigma$: SET OF HOM. POLY.

$\rightsquigarrow V(\Sigma) = \{ x \in \mathbb{P}^n \mid f(x) = 0 \ \ \forall f \in \Sigma \}$

; proj. VAR IF IRRED. ($\rightsquigarrow$ q. proj. var)

Ex.

(1) HYPERSURFACE ; $\mathbb{P}^n \supset V(F)$ ← IRRED.

e.g. $\mathbb{P}^2 \supset V(F) \sim$ C. R w/ $(d-1)(d-2)/2$ HANDLES

(2) COMPLETE INT.

$\mathbb{P}^n \supset V(F_1, \cdots, F_k)$ w/ DIM $= n-k$

(3) GRASSMANNIANS

$Gr(r, n) = Gr(r, V)$. PARAMETRIZING r-DIM'L SUBSP. in V. w/ DIM $V = n$.

CH2.

DEFN

$U \subset X$ open $\rightsquigarrow \mathcal{F}(U)$ AB. GP

w/ $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ FOR $V \subset U$.

① $\mathcal{F}(\phi) = 0$

② $\rho_{UU} = ID_{\mathcal{F}(U)}$

③ $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ FOR $W \subset V \subset U$

④ $s, t \in \mathcal{F}(U)$ w/ $s|_{U_i} = t|_{U_i}$ $\forall i$ $\Rightarrow s = t$

$U = \cup U_i$

⑤ $\{ s_i \in \mathcal{F}(U_i) \}$ w/ $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$

$\Rightarrow \exists s \in \mathcal{F}(U)$ s.t. $s|_{U_i} = s_i$

EX.

$\mathcal{O}_X$ = SHEAF OF REGULAR FNS.

STRUCTURE SHEA OF X.

$\rightsquigarrow \mathcal{O}_X$ – module.

RMK

(1) $\alpha : \mathcal{F} \to \mathcal{G}$ $\rightsquigarrow$ kernel, image, quotient

α = INJ IF $\ker(\alpha) = 0$

SURJ IF $\mathrm{im}(\alpha) = \mathcal{G}$

$\mathcal{O}_X \quad \mathcal{O}_Y$

(2) $f: X \to Y$

CAN DEFINE $f_* \mathcal{F}$ on Y

$$(f_* \mathcal{F})(V) := \mathcal{F}(f^{-1}(V))$$

$f^{-1}\mathcal{G}$ by $(f^{-1}\mathcal{G})(U) := \varinjlim_{f(U) \subset V} \mathcal{G}(V)$

$f^{-1}\mathcal{O}_X$-SHEAF

$$f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X = f^*\mathcal{G}$$

DEF

$$\cdots \to \mathcal{F}_{i-1} \to \mathcal{F}_i \xrightarrow{\alpha_i} \mathcal{F}_{i+1} \to \cdots$$

EXACT IF $\ker \alpha_i = \operatorname{im} \alpha_{i-1}$

$\to$ $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ SHORT EXACT SEQ.

$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X)$

DEF (SHEAF COHOMOLOGY).

$\mathcal{F}$: SHEAF.

$$0 \to \mathcal{F} \to \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots$$ EXACT (RESOLUTION)

w/ $\mathcal{F}_i$ = flasque

$$\mathcal{F}_0(X) \to \mathcal{F}_1(X) \to \mathcal{F}_2(X) \to \cdots$$

$$H^p(X, \mathcal{F}) := \ker(\mathcal{F}_p(X) \to \mathcal{F}_{p+1}(X)) \,/\, \operatorname{Im}(\mathcal{F}_{p-1}(X) \to \mathcal{F}_p(X))$$

$H^0(X, \mathcal{F}) = H^0(\mathcal{F}) = \mathcal{F}(X)$

DEF.

E A VBDLE OF RANK r IF

$$\begin{array}{c} E \\ \downarrow \pi \\ X \end{array}$$

① $\exists X = \bigcup U_i$

$$\begin{array}{ccc} E|_{U_i} & \xrightarrow{h_i} & U_i \times \mathbb{C}^r \\ \pi \downarrow & \swarrow t & \\ U_i & & \end{array}$$

LOCAL TRIV.

② $h_i \circ h_j^{-1} : (U_j \cap U_i) \times \mathbb{C}^r \longrightarrow (U_i \cap U_j) \times \mathbb{C}^n$

$$(x, v) \longmapsto (x, g_{ij}(x)(v))$$

w/ $g_{ij} : U_i \cap U_j \longrightarrow GL(r, \mathbb{C})$ alg.

↳ transition FN.

$g_{ij} \cdot g_{jk} = g_{ik} \rightsquigarrow$ ✓ $H^1(X, \underline{GL_r}) = H^1(X, \underline{GL_n})$

$r=1$; $H^1(X, \mathcal{O}_X^*)$

RMK

$s : U \longrightarrow E$ w/ $\pi \circ s = id_U$; (LOCAL) SECTION

$\rightsquigarrow \mathcal{E}(U) = \{$LOCAL SECTIONS OVER $U\}$

$\Rightarrow \mathcal{E}$ = LOCALLY FREE SHEAF OF $\mathcal{O}_X$-MODULES

* CAN DEFINE.

$E \oplus F$, $E \otimes F$, $E^\vee = \mathrm{Hom}(E, \mathcal{O}_X)$, $F \subset E$, E/F

$\Lambda^k E$, $\mathrm{Hom}(E,F) \cong E^\vee \otimes F$, ...

Ex.

T_X = TANGENT BDLE.

$\Omega_X^1 := T_X^\vee$ = COTANGENT BDLE.

$\omega_X = \Lambda^n \Omega_X^1$ = CANONICAL LINE BDLE.

THM (SERRE DUALITY)

$$H^i(X, E) \cong H^{n-i}(X, E^\vee \otimes \omega_X)^\vee \quad \forall i$$

$G = G_r(r, V)$

$$0 \to S \to V \otimes \mathcal{O}_G \to Q \to 0$$

$$T_G \cong \mathrm{Hom}(S, Q) \cong S^\vee \otimes Q$$

$r = 1$. $G = \mathbb{P}^n$ $\quad \hookrightarrow T_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(1) \otimes Q$

$$\begin{array}{ccccccc} 0 & \to & S & \to & V \otimes \mathcal{O}_G & \to & Q \to 0 \\ & & \downarrow & & \| & & \| \\ & & \mathcal{O}_{\mathbb{P}^n}(-1) & & \mathcal{O}_{\mathbb{P}^n}^{n+1} & & T_{\mathbb{P}^n}(-1) \end{array}$$

E = V.BDLE ON X OF RANK r

$$H^0(X,E) \otimes \mathcal{O}_X(U) \longrightarrow E(U)$$
$$s \otimes 1_U \longmapsto s|_U$$

$$ev : H^0(X,E) \otimes \mathcal{O}_X \longrightarrow E \quad \text{surj} \Rightarrow E = \text{GLOBALLY GENERATED.}$$

$$\Rightarrow \varphi_E : X \longrightarrow Gr(r, H^0(X,E)^\vee)$$

($r=1$;) $E = L$ IS VERY AMPLE IF φ_L IS EMBEDDING

$$\varphi_L : X \longrightarrow \mathbb{P}H^0(X,L)^\vee \cong \mathbb{P}^N$$
$$\uparrow$$
$$\mathcal{O}_{\mathbb{P}^N}(1)$$
$$\varphi_L^* \mathcal{O}_{\mathbb{P}^N}(1) = L \cong \mathcal{O}_X(1)$$

L = AMPLE $\rightsquigarrow L^{\otimes m}$ VERY AMPLE

FOR E = V.BDLE. $\rightsquigarrow c(E) = c_0(E) + c_1(E) + \cdots + c_n(E)$
w/ $c_i(E) \in H^{2i}(X,\mathbb{Z})$

$c_1 : Pic(X) \longrightarrow H^2(X,\mathbb{Z})$
$L \longmapsto c_1(L)$

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$
$$f \mapsto e^{2\pi i f}$$
$$0 \to H^1(X,\mathbb{Z}) \to H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X^*) \to H^2(X,\mathbb{Z}) \to \cdots$$

$Pic(X)$

$$0 \to E' \to E \to E'' \to 0$$

$$c(E) = c(E') \cdot c(E'')$$

$$\rightsquigarrow \quad \int_X c_1(E) \wedge c_1(\mathcal{O}_X(1))^{n-1} = :\deg_{\mathcal{O}_X(1)}(E)$$

Rmk

$\dim X = n = 1$; DOES NOT DEPEND ON $\mathcal{O}_X(1) =: H$.

DEF.

$$\text{SLOPE} := \frac{\deg_{\mathcal{O}_X(1)}(E)}{\operatorname{rank}(E)} = \mu_H(E)$$

Hilbert poly $\chi_E(m) = \chi(E(m))$, $E \otimes \mathcal{O}_X(1)^{\otimes m}$

$$h^0(E) - h^1(E) + h^2(E) - \dots = \chi(E)$$

DEF.

E = TORSION FREE COHERENT SHEAF ON X

w/ H = AMPLE.

$$\mu_H(E) = \frac{\deg_H E}{\operatorname{rank} E} \qquad P_E(m) = \frac{\chi(E(m))}{\operatorname{rank}(E)}$$

E = μ-(SEMI)STABLE IF

$$\mu_H(F) \ (\leq) < \mu_H(E) \quad \forall\, 0 \neq F \subsetneq E$$

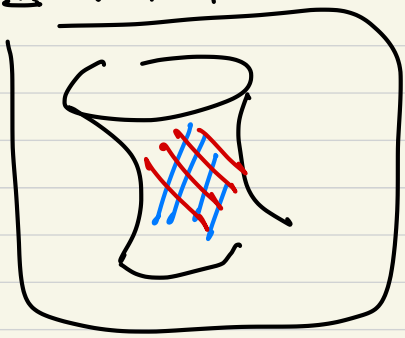
(SEMI)STABLE IF

$$P_F(m) \ (\leq) \ P_E(m) \quad \forall m \gg 0$$

$$<$$

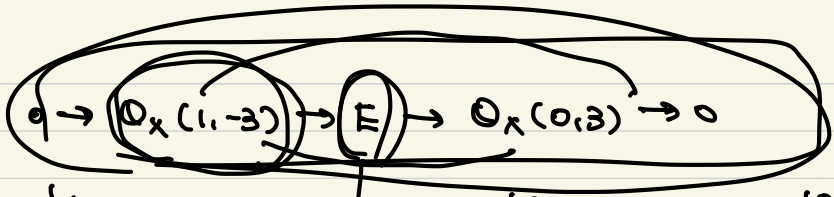
μ-STAB $\Rightarrow$ STAB $\Rightarrow$ SEMIST. $\Rightarrow$ μ-SS.

EX $X = \mathbb{P}^1 \times \mathbb{P}^1$



$\mathbb{P}^n \rightsquigarrow \mathcal{O}_{\mathbb{P}^n}(k)$

$\mathcal{O}_X(a,b) \cong \mathcal{O}_{\mathbb{P}^1}(a) \boxtimes \mathcal{O}_{\mathbb{P}^1}(b)$



$\mathrm{Ext}^1(\mathcal{O}_X(0,3), \mathcal{O}_X(1,-3)) \cong H^1(\mathcal{O}_X(1,-6)) \cong \mathbb{C}^{10}$

$c_1(E) = (1,0), \quad c_2(E) = 3$

$\Rightarrow$ E = μ-STABLE w.r.t. $\mathcal{O}_X(1,5)$

NOT μ-SS w.r.t. $\mathcal{O}_X(1,7)$

$\mu(E) = 7/2.$

$\mu(\mathcal{O}_X(1,-3)) = 7-3 = 4$

RMK

FOR DIM = 1,

μ-(S)S $\Leftrightarrow$ (S)S.

DEF

FOR FIXED $P(t) \in \mathbb{Q}[t]$, DEFINE

$$\mathcal{M}_X^{S.H.P} : (Sch/\mathbb{C}) \longrightarrow (SETS)$$

$$S \longmapsto \mathcal{M}_X^{S.H.P}(S)$$

$$\| $$

$$\{ \text{S-flat families } \mathcal{F} \text{ of v.BDLES} \downarrow X \times S$$

WHOSE FIBRES ARE μ-STABLE

w.r.t. H & Hilbert poly P $\}$ $/\sim$

$\bigsqcup M^s_{X,H}(r; c_1, \dots, c_{min(r,n)})$

THM

$\mathcal{M}_X^{S.H.P}$ HAS THE COARSE MODULI SPACE $M_X^{S.H.P}$

i.e. $\exists\, \Phi : \mathcal{M}_X^{S.H.P} \longrightarrow Hom(-, M_X^{S.H.P})$

① bijective for any pt x_0

② $\forall\, \Psi : \mathcal{M}_{X(-)}^{S.H.P} \to Hom(-, N)$

$\exists!\, \psi : M_X^{S.H.P} \to N$

$\mathrm{Hom}(E,-)$

THM

$T_{[E]} M^S_{X/M}(r; c_1, c_2) \cong \underline{\mathrm{Ext}^1(E,E)}$

$\mathrm{Ext}^2(E,E) = 0 \Rightarrow$ SMOOTH AT $[E]$

Ex. $(X = \mathbb{P}^1 \times \mathbb{P}^1)$

$M_{X, \mathcal{O}(1,b)}(2; \mathcal{O}(1,0), 3) = \mathbb{P}^9$

$\downarrow$

$E \rightsquigarrow \mu = b/2$

$\rightarrow \chi(E(-1,3)) = 1$

$\|$

$h^0 - h^1 + h^2 > 0$

$s + |z| = 3$

$0 \to \mathcal{O}(1,-3) \otimes \mathcal{O}_X(D) \to E \to I_Z \otimes \mathcal{O}(0,3) \otimes \mathcal{O}_X(-D)$

$\|$

$\mathcal{O}_X(a,b)$

$\|$

$\mathcal{O}_X$

$0 \to \mathcal{O}_X(1,-3) \to E \to \mathcal{O}_{\ell}(0,3) \to 0$

$E \in \mathbb{P}\mathrm{Ext}^1(\mathcal{O}(0,3), \mathcal{O}(1,-3)) \cong \mathbb{P}^9$

Ex

C = elliptic curve.

$$M_C^S(2,h) \quad M_C^{S'}(2,1) \quad \ni E$$

$$L \in \mathrm{Pic}^1(C) \ni \det(E) = \Lambda^2 E$$

$\deg L = 1$. $\rightsquigarrow M_C^S(2,h) = \{ 0 \to \mathcal{O}_C \to E \to L \}$

Ex $(X = \mathbb{P}^2)$

$$M^S_{\mathbb{P}^2}(2,-1,1) \simeq \{ \Omega^1_{\mathbb{P}^2}(1) \}$$

$$M^S_{\mathbb{P}^2}(2,-1,2) = (\mathbb{P}^{2\vee} \times \mathbb{P}^{2\vee} \setminus \Delta)/S_2$$

$$\mathbb{P}^{2*} \xrightarrow{v_2} \mathbb{P}^{5*}$$

$\mathrm{Im}(v_2) = S \subset \mathbb{P}^{5*}$

$V = \mathrm{Sec}(S) \subset \mathbb{P}^{5*}$

$V(F)$ $\quad V \setminus S$

THM FOR

$$\mathcal{M}_X^{H\text{-}P} = (Sch/\mathbb{C}) \longrightarrow (SETS)$$

$$S \longmapsto \mathcal{M}_X^{H\text{-}P}(S)$$

$$\| $$

$$\left\{ S\text{-flat families } \mathcal{F} \text{ of } \mathcal{F} \downarrow X \times S \right.$$

$$\left. \text{SEMISTABLE SHEAVES w.r.t. } H \right\} / \sim$$

$\exists$ COARSE MODULI SCHEME $M_X^{H\text{-}P}$ ∎

OP1.

$G = Gr(2,4) \subset \mathbb{P}^5$ $\quad \mathcal{O}_{\mathbb{P}^5}(1)$

$\forall \quad 0 \to \mathcal{N} \to \mathcal{O}_G^{\oplus 4} \to \mathcal{Q} \to 0$

S = SMOOTH NONDEG SURF

$\mathcal{N}|_S, \quad \mathcal{Q}|_S \qquad \mathcal{O}_S(1)$

SEMISTABLE ?

DEF. (JORDAN-HÖLDER)

E = SEMISTABLE SHEAF

$\Rightarrow \exists \; 0 = E_0 \subset E_1 \subset \cdots \subset E_l = E$

s.t. $gr_i(E) = E_i / E_{i-1}$ STABLE w. $P_E(m)$

$\rightsquigarrow gr(E) := \bigoplus gr_i(E)$

E_1, E_2 S-EQUIV $\Leftrightarrow gr(E_1) \cong gr(E_2)$

THM

THM^A.

$\mathcal{M}_X^{H,P} : (Sch/\mathbb{C}) \longrightarrow (SETS)$

$S \longmapsto \{$ S-flat families $\mathcal{F}$ of $\mathcal{F}_{X\times S}$

SS. SHEAVES w.r.t. $H \sim \}/\sim$

, $\exists$ COARSE MODULI SP.

$\left(\begin{array}{ll} 0 \to E' \to E \to E'' \to 0 & \\ E' \oplus E'' \ncong E & \mathrm{Ext}^1(E'', E) \end{array} \right)$

WHOSE CLOSED PTS ARE IN $1\!-\!1$ CORRESP.

TO S-EQUIV. CLASSES OF SS SHEAVES

Ex ($\dim X = 1$).

$$U_X^S(r,d) \subset U_X(r,d) \supset SU_X(r,L) \supset SU_X^S(r,L)$$

$U_X^S(r,d)$ ↓

$U_X(r,d) \to \mathrm{Pic}^d(X) \ni L$

$SU_X(r,L) \to \{L\}$

$\mathrm{Ext}^2(E,E) = 0 \Rightarrow SU_X^S(r,L) = \text{SMOOTH}$

$tr : \mathcal{E}nd(E) \to \mathcal{O}_X$

$tr^1 : \mathrm{Ext}^1(E,E) \to H^1(\mathcal{O}_X)$

$$\dim\left(\ker = \mathrm{Ext}^1(E,E)_0\right) = -\chi(E,E) + \chi(\mathcal{O}_X)$$

$$= (r^2-1)(g-1) \longrightarrow$$

① $(r,d) = 1 \Rightarrow SU_X(r,L) = SU_X^S(r,L)$

SMOOTH. FINE MODULI SP. PROJECTIVE

e.g. $\mathrm{rk}\,2$, $d = 2g-1$ $\rightsquigarrow$ $0 \to \mathcal{O} \to E \to L \to 0$

② $(r,d) > 1 \Rightarrow SU_X(r,L)$ IS SINGULAR

except $(g,r,d) = (2,2,\text{odd})$ $\left(\Rightarrow SU_X(2,L) \cong \mathbb{P}^3\right)$

E.g. $SU_X(2,\omega_X) \supset \mathrm{Pic}^{g-1}(X)/\iota_2$: KUMMER VAR

$$[A \oplus \omega_X A^{-1}]$$

$A \in \mathrm{Pic}^{g-1}(X)$

G-DIM'L

OP2.

$g(X)=3 \rightsquigarrow SU_X(2, K_X)$ RATIONAL ?

$\curvearrowright \mathbb{P}^7$; $\hookrightarrow$ COBLE QUARTIC.

Ex $(X=\mathbb{P}^2)$

$M_{\mathbb{P}^2}(r; c_1, c_2)$ = MODULI SP. OF SS SHEAVES $(c_1, c_2) \in \mathbb{Z}^2$

$\cup$

$M^0_{\mathbb{P}^2}(r; c_1, c_2)$ = " μ-STABLE.

OP3 : IS $M_{\mathbb{P}^2}(r; c_1, c_2)$ RATIONAL ?

Yes for $r=2$; $c_1=0$; BARTH

$c_1=-1$; HULEK.

THM

E = μ-SS OF RANK r on X

$\Rightarrow \Delta(E) = (2r\, c_2(E) - (r-1)\, c_1^2(E)) \cdot H^{n-2} \geq 0$

<PF>

E = V.BDLE. $c_1(E)=0$ X=Sm f. $E \in E^\vee$

$\rightsquigarrow S^m E \rightsquigarrow 0 \to S^m E(-C) \to S^m E \to S^m E|_C \to 0$

$\rightsquigarrow \chi(S^m E) = -\frac{\Delta(E)}{2r} \cdot \frac{m^{r+1}}{(r+1)!} + \cdots$; deg r

Ex $(X = \mathbb{P}^2,\ r = 2\mathfrak{z},\ c_1 = -1)$

$M_{\mathbb{P}^2}(2; c_1, c_2) \neq \emptyset \iff c_1^2 - 4c_2 \geq 0$

① $M_{\mathbb{P}^2}(2; -1, 1) \simeq \{\Omega_{\mathbb{P}^2}^1(1)\}$

② $M_{\mathbb{P}^2}(2; -1, 2) \simeq M^S_{\mathbb{P}^2}(2; -1, 2) \sqcup \partial M$

$M^S_{\mathbb{P}^2}(2; -1, 2) = \mathbb{P}(V(F)^\vee \otimes S) \subset \mathbb{P}^5$

$\partial M \simeq \mathbb{P}^1$

$= B\ell_S V(F)$

SET OF JUMPING LINES OF 2nd kind.

Beilinso SPECTRAL THM

MONADOLOGY

DERIVED CATEGORY

$$0 \to F \to F^{\vee\vee} \to Q \to 0$$

F: $\partial M(-1,2)$

$F^{\vee\vee}$: $M(-1,1)$

Q: $\mathcal{O}_p$

$\Omega(1)$

$\partial M(-1,2) \to \mathbb{P}(\Omega(1))$

Op 4.

On $\mathbb{P}^n$. $F = L_1 \cdots L_m \rightsquigarrow V(F) = V(L_1) \cup V(L_2) \cdots \cup V(L_m)$

L ↓ SNL

$$0 \to T(-\log V(F)) \to \mathcal{O}(1)^{n+1} \to \mathcal{O}(m) \to 0$$

$$(l_1, \dots, l_{n+1}) \mapsto l_1 F_{x_0} + \cdots + l_{n+1} F_{x_{n+1}}$$

↓ rank n v. BDLT

$$T(-\log V(F))|_L \cong \mathcal{O}_L(a_1) \oplus \cdots \oplus \mathcal{O}_L(a_n)$$

$$a_1 \geq \cdots \geq a_n$$

$$a_1 - a_n \leq 1 \ ?$$

FOR GENERAL L

□

$(0, -1)$

$(0, -1, -2)$