EXERCISES - ALGEBRAIC GEOMETRY

JANUARY 19, 2022

(1) The Grassmannian $\mathrm{Gr}(r,n) = \mathrm{Gr}(r,V)$ parametrizes the r-dimensional linear subspaces of a vector space V over $\mathbb{C}$ with $\dim V = n$.

   (a) Prove that $\mathrm{Gr}(r,V)$ is a projective variety.

   (b) It is equipped with tautological vector bundles $\mathcal{S}$ and $\mathcal{Q}$, fitting into an exact sequence
   $$0 \longrightarrow \mathcal{S} \longrightarrow V \otimes \mathcal{O}_{\mathrm{Gr}(r,V)} \longrightarrow \mathcal{Q} \longrightarrow 0.$$
   Express the tangent vector bundle of $\mathrm{Gr}(r,V)$ in terms of $\mathcal{S}$ and $\mathcal{Q}$.

(2) Compute the Chern classes $c_i(\mathcal{T}_{\mathbb{P}^n}(-1))$ for $i = 0, 1, \ldots, n$.

(3) Prove that $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}\langle \mathcal{O}_{\mathbb{P}^n}(1) \rangle$.

(4) For an $\mathcal{O}_X$-module $\mathcal{F}$ on a projective variety X, the functors $\mathrm{Ext}^i_X(\mathcal{F}, -)$ are the right derived functors of $\mathrm{Hom}_X(\mathcal{F}, -)$. Prove that there exists a one-to-one correspondence between isomorphism classes of extension of $\mathcal{F}'$ by $\mathcal{F}$:
   $$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0,$$
   and elements of $\mathrm{Ext}^1_X(\mathcal{F}'', \mathcal{F}')$.

(5) Prove that any vector bundle $\mathcal{E}$ of rank r over $\mathbb{P}^1$ is a direct sum of line bundles, i.e.
   $$\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$$
   with integers $a_1 \geq a_2 \geq \cdots \geq a_r$.

(6) Let $\mathcal{E}$ be a vector bundle, fitting into the following non-trivial extension
   $$0 \longrightarrow \mathcal{O}_X(1,-3) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(0,3) \longrightarrow 0,$$
   on a smooth quadric surface X. Prove that $\mathcal{E}$ is μ-stable with respect to $\mathcal{O}_X(1,5)$.

(7) If $\mathcal{E}$ is an indecomposable vector bundle of rank two with even degree on an elliptic curve X, then it admits an exact sequence
   $$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$$
   for some line bundle $\mathcal{L}$ on X.

(8) Let $\mathcal{E}$ be a vector bundle of rank two on $\mathbb{P}^n$. Then $\mathcal{E}$ is μ-stable (resp. μ-semistable) if and only if $\mathrm{H}^0(\mathcal{E}_{\mathrm{norm}}) = 0$ (resp. $\mathrm{H}^0(\mathcal{E}_{\mathrm{norm}}(-1)) = 0$).
In general, let $\mathcal{E}$ be a vector bundle of rank r on a smooth projective variety X with $\mathrm{Pic}(X) \cong \mathbb{Z}\langle\mathcal{O}_X(1)\rangle$. Then we have

(a) If $\mathrm{H}^0(X, (\wedge^q\mathcal{E})_{\mathrm{norm}}) = 0$ for $1 \leq q \leq r-1$, then $\mathcal{E}$ is μ-stable.

(b) If $\mathrm{H}^0(X, (\wedge^q\mathcal{E})_{\mathrm{norm}}(-1)) = 0$ for $1 \leq q \leq r-1$, then $\mathcal{E}$ is μ-semistable.

(9) Give an example to show that the condition in (8) is not necessary.

(10) Prove that $\Omega^1_{\mathbb{P}^n}$ and $\mathcal{T}_{\mathbb{P}^n}$ are μ-stable.

(11) Using Hirzebruch-Riemann-Roch's theorem, prove that

$$\chi(\mathcal{E}(-1,3)) = 1$$

for a μ-stable vector bundle $\mathcal{E}$ of rank two on $\mathbb{P}^1 \times \mathbb{P}^1$ with $c_1 = (1,0)$ and $c_2 = 3$.

(12) Let X be a smooth projective surface and $Z = \{p_1, \ldots, p_s\} \subset X$ with s-distinct points. Fix two line bundles $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}(X)$. Then there exists a vector bundle $\mathcal{E}$ of rank two on X, fitting into an exact sequence

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{Z,X} \otimes \mathcal{L}_2 \longrightarrow 0,$$

if and only if every section of $\mathcal{L}_1^\vee \otimes \mathcal{L}_2 \otimes \omega_X$ vanishing at all but one of the p_i's vanishes at the remaining point as well. Here, $\mathcal{I}_{Z,X}$ is the ideal sheaf of $Z \subset X$.

(13) In (12), prove that $c_1(\mathcal{E}) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2)$ and $c_2(\mathcal{E}) = c_1(\mathcal{L}_1)\dot{c}_1(\mathcal{L}_2) + s$.

(14) Prove that $\mathbf{M}_{\mathbb{P}^2}(2; -1, 1)$ is a single point space with $\Omega^1_{\mathbb{P}^2}(1)$ as its unique point.