제17회 기하학 겨울학교 강의록



제17회 고등과학원 기하학 겨울학교

	1월 20일 (목)	1월 21일 (금)	1월 24일 (월)	1월 25일 (화)
10:00–11:00	김승혁	김승혁	김준태	이재혁
11:10–12:10	최영준	김준태	김호성	박웅배
토론				
19:00–20:00	최영준	강정수	김호성	김민현
20:10–21:10	김승혁	강정수	이재혁	김성찬

주제강연	입문강연	새싹강연

김승혁 **한양대학교 (1,2,3 강의)**

라플라스 작용소, 소볼레프 부등식과 공형기하학

공형기하학은 미분기하학의 세부학문 중 하나로서 주어진 리만 다양체에서 정의된 등각사상(각도를 보전하는 사상)을 연구하는 학문이다. 이 강연에서는 미분방정식론의 기본 개념인 라플라스 작용소와 소볼레프 부등식이 공형기하학에서 어떠한 의미를 갖는지를 살펴본다.

강연 1

소볼레프 부등식, 연관 성질(대칭성 포함) 및 정리, 유클리드 공간 위에서 정의된 라플라스 작용소의 공형 불변성

강연 2

리만 다양체 위의 공형 라플라스 작용소 정의, 야마베 문제의 개략적 소개, 소볼레프 부등식과의 관련성 탐구

강연 3

야마베 문제의 해법, 양수 질량 정리, (시간이 허락하는 경우) 최근 연구 주제에 대한 간략한 소개

최영준 부산대학교

Yau's answer for Calabi's conjecture

In the study of compact Kähler manifolds, the first Chern class is represented by the Ricci curvature of any Kähler metric. In 1954, E. Calabi raised the following question:

Let (X,ω) be a Kähler manifold. For a given $(1,1)$-form Ω in the first Chern class $c_1(X)$. Is there a Kähler metric $\tilde{\omega}$ which is cohomologous to ω such that $Ric(\tilde{\omega}) = \Omega$?

In 1978, S.-T. Yau solved this conjecture by solving complex Monge-Ampère equations. A direct consequence is the existence of the Kähler-Einstein metrics on compact Kähler manifold with nonpositive first Chern class.

In this talk, we will discuss Calabi's conjecture and Kähler-Einstein metric. We also talk about the sketch of Yau's solution.

김준태 고등과학원

Real Lagrangian tori in symplectic four-manifolds

The study and classification of Lagrangian submanifolds in symplectic manifolds has been a central topic of modern symplectic topology. In this talk, we address its historical results, and explore the topology of real Lagrangian tori in monotone symplectic four-manifolds. In particular, the existence and exoticness of real Lagrangian tori in del Pezzo surfaces are discussed.

강정수 서울대학교

Symplectic geometry and Hamiltonian dynamics

The aim of this series of lectures is to give an (incomplete) overview on the interplay between symplectic geometry and Hamiltonian dynamics. Starting from very basics on symplectic geometry, I will introduce several symplectic measurements for symplectic manifolds and applications thereof. If time permits, I will also mention recent studies on the systolic-type inequality for Hamiltonian periodic solutions.

김호성 IBS-복소기하학 연구단

Cartan-Fubini type extension problem

In the first talk I will define the VMRT associated to a family of minimal rational curves on a Fano manifold of Picard number 1 and describe some properties of it. I will also introduce the Cartan-Fubini type extension problem and J.-M. Hwang and N. Mok's result on the cases with positive dimensional VMRT.

The aim of the second talk is to prove the Cartan-Fubini type extension property for a double cover of projective space appearing in the joint work with J.-M. Hwang. To do this we used a property of the flat projective connection on a projective space and the explicit description of the projective geometry of VMRT.

이재혁 이화여자대학교

Calibrated geometry and Vector cross products

Calibrated geometry is a study of submanifolds characterized by closed forms with volume related conditions. The interesting cases of these forms appear in manifolds with special holonomies such as Calabi-Yau manifolds and G_2-manifolds, and the related calibrated geometry derives more attention thanks to mirror symmetry. In the first talk, we have a brief introduction to calibrated geometry of manifolds with special holonomies and give a uniform approach to these submanifolds along the vector cross products. In the second talk, we discuss the comparison between calibrated geometry of Calabi-Yau 3-folds and G_2-manifolds induced by holonomy reduction and mirror symmetry. This is a joint work with N. C. Leung.

박웅배 University of Pittsburgh

박사학위 : 2021년 Michigan State University, USA, (지도교수: Thomas H. Parker)
연구경력 : 2021년-현재 University of Pittsburgh

Harmonic maps over nodal domain using Deligne-Mumford moduli space

This talk consists of three parts. In the first part I will review basic facts about harmonic maps in Riemann surfaces and consider the situation when a sequence of harmonic maps converges away from finitely many points, called bubble points. Next, I will introduce the brief notion of Deligne-Mumford moduli space. Finally, I will describe how we can capture the bubbles using Deligne-Mumford moduli space.

김민현 Universitat Bielefeld

박사학위 : 2020년 서울대학교 (지도교수: 이기암)
연구경력 : 2020년 중앙대학교
2021년-현재 Universität Bielefeld, Bielefeld, Germany

L_p Minkowski problem for electrostatic $\mathfrak{B}$-capacity

The Minkowski problem asks if a given measure on the unit sphere arises as the surface area measure of a convex body. This problem has been generalized to L_p Minkowski problem for electrostatic $\mathfrak{B}$-capacity. In this talk, we study a recent progress on this problem for various values p and $\mathfrak{B}$.

김성찬 고등과학원

박사학위 : 2018년 Universit¨at Augsburg, Germany (지도교수: Urs Frauenfelder)
연구경력 : 2019년-2020년 Université de Neuchâtel, 2021년 Seoul National University
2021년-현재 고등과학원

Transverse foliations in Hamiltonian systems

The modern dynamical systems theory and symplectic geometry have their origin in Poincare's work in Hamiltonian systems. Since then, both research fields have successfully developed independently. Symplectic Dynamics is a new research field trying to bring them closer together again. Namely, it studies Hamiltonian systems utilising highly integrated ideas from both research fields. In this talk, I will give a mild introduction to the field of symplectic dynamics. In particular, I would like to talk about a transverse foliation, which is a powerful tool to study the orbit structure of a given Hamiltonian system, and some consequences of its existence.

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2022년 1월 20일 목요일 오전 9:54

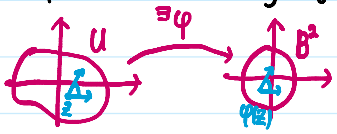
Title: Laplacian, Sobolev inequality, and conformal geometry

Lecture 1. Sobolev ineq. and its properties.

* conformal geometry : the study of conformal (angle-preserving) maps on Riemann manifolds.

e.g.) holomorphic function f in $\mathbb{C}$: conformal
(with $f'(z) \neq 0$ for $\forall z \in \mathbb{C}$)
$\Leftarrow$ local biholomorphism.

→ Riemann mapping theorem
: Any simply connected nonempty open subset u of $\mathbb{C}$ which is not $\mathbb{C}$ itself is conformally equivalent to $\mathbb{B}^2$.



Uniformization theorem
: Any simply connected Riemann surface is conformally equivalent to one of $\mathbb{B}^2$, $\mathbb{C}$, S^2 (= Riemann sphere).

(compact Riemann surface $\simeq \mathbb{B}^2/\sim$, T^2, S^2)
↑ n=2
Gaussian curvature : −1, 0, 1

(cf. Yamabe problem: closed Riemannian manifold with dim ≥ 3 $\simeq$ manifold with constant scalar curvature ?)

Q. conformal geometry in dim ≥ 3?

* Sobolev inequality in $\mathbb{R}^n$. $\|u\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u(x)|^p dx\right)^{1/p}$. $p \geq 1$

$\|u\|_{L^q(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)}$ $p \geq 1$, q, $u \in C_c^\infty(\mathbb{R}^n)$

Find $C > 0$ depending only on $C(n, p, q)$

Fix any $u \neq 0$.

Here, q cannot be arbitrary. Set $u_\lambda(x) = u(\lambda x)$ $\lambda > 0$. (Fix any $u \neq 0$.)

Scaling: conformal map.

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \leq C \|\nabla u_\lambda\|_{L^p(\mathbb{R}^n)}$$

C: independent of λ.

We have: $\|\nabla u_\lambda\|_{L^p(\mathbb{R}^n)} = \lambda^{1-\frac{n}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)}$, $\|u_\lambda\|_{L^q(\mathbb{R}^n)} = \lambda^{-\frac{n}{q}} \|u\|_{L^q(\mathbb{R}^n)}$

$$\Rightarrow \lambda^{-\frac{n}{q}} \|u\|_{L^q(\mathbb{R}^n)} \leq C \lambda^{1-\frac{n}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

$-\frac{n}{q} = 1 - \frac{n}{p} \Rightarrow q = \frac{np}{n-p} = p^*$: Sobolev conjugate

→ Scaling property determines q!

Theorem (Gagliardo – Nirenberg – Sobolev ineq.)

Assume $1 \leq p < n$. $\exists C = C(n,p) > 0$ such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)} \quad \forall u \in C_c^1(\mathbb{R}^n)$$

$\dot{W}^{1,p}(\mathbb{R}^n)$: completion with respect to $\|\nabla u\|_{L^p(\mathbb{R}^n)}$

proof) 'Evans, Partial differential equations' Sec 5.6. ∎

Remark If $p=1$, then the GNS ineq. (Sobolev) is related to the isoperimetric ineq.

(Reference: 'Evans, Measure theory and fine properties of functions' Sec 5.6)

Theorem (Poincaré inequality)

(Ω: a bounded open subset of $\mathbb{R}^n$, $1 \leq p < n$, $1 \leq q \leq p^*$)

$\Rightarrow \exists C = C(p,q,n,\Omega) > 0$ such that

$$\|u\|_{L^q(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}$$

$W_0^{1,p}(\Omega)$: completion of $u \in C_c^1(\Omega)$

$u = 0$ on $\partial\Omega$.

Theorem (Rellich-Kondrachov compact theorem)

(Ω: a bounded open subset of $\mathbb{R}^n$.

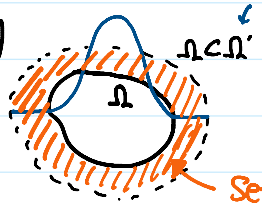
Theorem (Rellich-Kondrachov compact theorem)

(Ω: a bounded open subset of $\mathbb{R}^n$. $1 \le p < n$, $1 \le q < p^*$)

$\{u_m\}$: bounded in $W_0^{1,p}(\Omega)$. $\Leftrightarrow$

If $\{u_m\}_{m\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ satisfies $\sup_m \|\nabla u_m\|_{L^p(\Omega)} \le M$ for some $M>0$, then it has a convergence subsequence in $L^q(\Omega)$.

sketch of the proof)



Let $u_m^\varepsilon = \eta_\varepsilon * u_m \in C_0^\infty(\Omega')$ for small $\varepsilon > 0$.

mollifier or approximation to the identity

$$((\eta_\varepsilon * u_m)(x) = \int_{B(0,\varepsilon)} \eta_\varepsilon(y) u_m(x-y)\,dy = \int_{B(0,1)} \eta(z) u_m(x-\varepsilon z)\,dz)$$

$\eta_\varepsilon(y) = \frac{1}{\varepsilon^n}\eta(\frac{y}{\varepsilon})$, $y = \varepsilon z$

Claim 1 $u_m^\varepsilon \to u_m$ in $L^q(\Omega')$ as $\varepsilon \to 0$ uniformly in $m \in \mathbb{N}$.

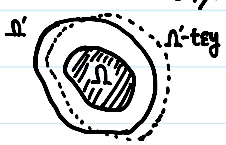
$\to$ If $u_m \in C_0^1(\Omega)$ $\quad u_m^\varepsilon(x) - u_m(x) = \int_{B(0,1)} \eta(y)\,(u_m(x-\varepsilon y) - u_m(x))\,dy$

fundamental theorem of calculus $\nearrow$ $= \int_{B(0,1)} \eta(y) \int_0^1 \frac{d}{dt}(u_m(x-t\varepsilon y))\,dt\,dy$

$= -\varepsilon \int_{B(0,1)} \eta(y) \int_0^1 \nabla u_m(x-t\varepsilon y)\cdot y\,dt\,dy$

$\Rightarrow \int_{\Omega'} |u_m^\varepsilon - u_m|(x)\,dx \le \varepsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_{\Omega'} |\nabla u_m(x-t\varepsilon y)|\,|y|\,dx\,dt\,dy$ ($\eta(y) \le 1$, $x - t\varepsilon y = z$, $dx = dz$)

$\le \varepsilon |B(0,1)| \int_{\Omega'} |\nabla u_m(z)|\,dz$ ($|B(0,1)| \le C$)



$\Rightarrow \|u_m^\varepsilon - u_m\|_{L^1(\Omega')} \le \varepsilon C \|\nabla u_m\|_{L^1(\Omega')} \le \varepsilon C \|\nabla u_m\|_{L^p(\Omega)}$ $\forall u_m \in C_0^1(\Omega)$ (generalized to any $W_0^{1,p}(\Omega')$)

$\Rightarrow$ Interpolation ineq:

$$\|u_m^\varepsilon - u_m\|_{L^q(\Omega')} \le \|u_m^\varepsilon - u_m\|_{L^1(\Omega')}^{\theta}\, \|u_m^\varepsilon - u_m\|_{L^{p^*}(\Omega')}^{1-\theta} \le (\varepsilon C M)^{\theta} (\|\nabla u_m^\varepsilon\|_{L^p(\Omega')} + \|\nabla u_m\|_{L^p(\Omega')})^{1-\theta} \le (CM)^{1-\theta}$$

$\to 0$ as $\varepsilon \to 0$.

$1 \le q < p^* \Rightarrow \frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p^*}$ for $0 < \theta \le 1$

Claim 2] For each $\varepsilon>0$, $\{u_m^\varepsilon\}_m$: unif. bounded & equicontinuous in Ω'

↳ $\{u_m^\varepsilon\}_m$ has a uniformly convergent subseq. in Ω' ($\Rightarrow$ convergent subseq. in $L^2(\Omega')$)

by the Arzela-Ascoli theorem

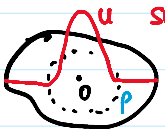
→ 'Claim 1 + Claim 2 + diagonal argument' completes the proof. ∎

Q. Is it the condition $q<p^*$ technical?

A. No! Given some $u\in C_0^1(\Omega)\setminus\{0\}$, consider $\{u_m(x) = u(mx)\cdot m^{\frac{n-p}{p}}\}_{m\in\mathbb{N}}$.

$\operatorname{supp} u_m \subset B(0,\frac{1}{m})$ ↳0 as $m\to\infty$

$n-p \hookrightarrow \infty$ as $m\to\infty$.



$\operatorname{supp} u \subset B(0,\rho) \subset\subset \Omega$ (small)

$$\|\nabla u_m\|_{L^p(\Omega)} = m^{1-\frac{n}{p}}\cdot m^{\frac{n-p}{p}}\|\nabla u\|_{L^p(\Omega)} < \infty$$

$$\|u_m\|_{L^{p^*}(\Omega)} = m^{-\frac{n}{p^*}} m^{\frac{n-p}{p}}\|u\|_{L^{p^*}(\Omega)} < \infty.$$

($-\frac{n}{p^*} = 1-\frac{n}{p}$)

⇒ The compactness theorem does not hold if $q=p^*$.

⇒ loss of compactness becaus of the scaling invariance.

* Kelvin transform (inversion)

$x^* = \frac{x}{|x|^2}$ for some $x\in\mathbb{R}^n$ ⇒ $u^*(x^*) = \frac{1}{|x^*|^{n-2}} u\left(\frac{x^*}{|x^*|^2}\right)$

↳ a conformal map ($I(x) = x^* = \frac{x}{|x|^2}$ ⇒ $I^*(g_{Eucl})_{ij} = g_{Eucl}(I_*(\frac{\partial}{\partial x_i}), I_*(\frac{\partial}{\partial x_j})) = \frac{\delta_{ij}}{|x|^4}$)

$\frac{\partial}{\partial x_i}\left(\frac{x_k}{|x|^2}\right)\frac{\partial}{\partial x_k} = \frac{1}{|x|^2}\left(\delta_{ik} - \frac{2x_i x_k}{|x|^2}\right)\frac{\partial}{\partial x_k}$

(p=2)

Fact 1] $\int_{\mathbb{R}^n}(\nabla u^*\cdot\nabla v^*)\,dx^* = \int_{\mathbb{R}^n}(\nabla u\cdot\nabla v)\,dx$. In particular, $\|\nabla u^*\|_{L^2(\mathbb{R}^n)} = \|\nabla u\|_{L^2(\mathbb{R}^n)}$

proof) $u\in C_c^1(\mathbb{R}^n)$ ⇒ known: $\Delta u^*(x^*) = \frac{1}{|x^*|^{n+2}}(\Delta u)\left(\frac{x^*}{|x^*|^2}\right)$

→ $\int_{\mathbb{R}^n}(\nabla u^*\cdot\nabla v^*)(x^*)\,dx^* = -\int_{\mathbb{R}^n}\Delta u^*(x^*)\,v(x^*)\,dx^*$

$= -\int_{\mathbb{R}^n}\frac{1}{|x|^{2n}}(\Delta u)(x)\,v(x)\,\underbrace{dx^*}_{=|x^*|^{2n}dx} = \int_{\mathbb{R}^n}(\nabla u\cdot\nabla v)\,dx$. ∎

Fact 1 $\int_{\mathbb{R}^n} (\nabla u^* \cdot \nabla v^*)\, dx^* = \int_{\mathbb{R}^n} (\nabla u \cdot \nabla v)\, dx$. In particular, $\|\nabla u^*\|_{L^2(\mathbb{R}^n)} = \|\nabla u\|_{L^2(\mathbb{R}^n)}$

proof) $u \in C_c^1(\mathbb{R}^n) \Rightarrow$ known: $\Delta u^*(x^*) = \frac{1}{|x^*|^{n-2}} (\Delta u)\left(\frac{x^*}{|x^*|^2}\right)$

$\rightarrow \int_{\mathbb{R}^n} (\nabla u^* \cdot \nabla v^*)(x^*)\, dx^* = -\int_{\mathbb{R}^n} \Delta u^*(x^*)\, v(x^*)\, dx^*$

$= -\int_{\mathbb{R}^n} \frac{1}{|x^*|^{2n}} (\Delta u)(x)\, v(x)\, \underbrace{dx^*}_{\frac{1}{|x^*|^{2n}} dx} = \int_{\mathbb{R}^n} (\nabla u \cdot \nabla v)\, dx.$ ∎

Fact 2 $\|u^*\|_{L^{2^*}(\mathbb{R}^n)} = \|u\|_{L^{2^*}(\mathbb{R}^n)}$ where $2^* = \frac{2n}{n-2}$ $(n \geq 3)$.

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2022년 1월 20일 목요일 오후 1:01

제 17회 기하학 겨울학교

Title: Laplacian, Sobolev inequality, and conformal geometry

Lecture 2. Conformal Laplacians and the Yamabe problem

Given a Riemannian manifold (M^n, g), the conformal Laplacian L_g on (M, g) is

$$L_g u = -\Delta_g u + \frac{n-2}{4(n-1)} R_g u$$

(R_g: scalar curvature.)

Laplace-Beltrami operator

(locally, $\Delta_g u = \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|}\, g^{ij} \partial_j u)$)

In $\mathbb{R}^n$, $L_g u = -\Delta u = -\sum_{i=1}^{n} \partial_i u$

good transformation laws under conformal changes

(translations, scalings, inversion.)

* conformal covariance property of L_g $(n \geq 3)$

Theorem If $g' = \varphi^{\frac{4}{n-2}} g$ (conformal change.), then $L_{g'}(u) = \varphi^{-\frac{n+2}{n-2}} L_g(\varphi u)$ on M.

($\varphi > 0$ on M, smooth)

$\mapsto g_{jm,i} = (e^f g_{jm})_i = (\partial_i f\, g_{jm} + g_{jm,i}) e^f$

proof) Assume that $g' = e^f g$.

Γ'^{k}_{ij} Γ^{k}_{ij} : Christoffel symbols.

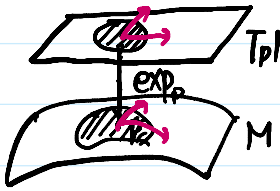
Note that $\Gamma^{k}_{ij} = \frac{1}{2} g^{km} (g_{jm,i} + g_{mi,j} - g_{ij,m})$

$$\Rightarrow \Gamma'^{k}_{ij} = \frac{1}{2}(g^{km} e^{-f}) e^f (\partial_i f\, g_{jm} + \partial_j f\, g_{mi} - \partial_m f\, g_{ij}) + \Gamma^{k}_{ij}$$

$$= \frac{1}{2}(\delta^k_j \partial_i f + \delta^k_i \partial_j f - g^{km} \partial_m f\, g_{ij}) + \Gamma^{k}_{ij}$$

$$= \frac{1}{2}(\delta^k_j \partial_i f + \delta^k_i \partial_j f - \underbrace{g^{km}\partial_m f}_{\nabla^k f} g_{ij}) + \Gamma^k_{ij}$$

To simplify the computations, we take normal coordinates at $p \in M$ with respect to g.



$\Rightarrow g_{ij}(p) = \delta_{ij}, \quad g_{ij,k}(p) = 0.$

$\Gamma^k_{ij}(p) = 0.$

At p, $R'_{ij} = R'^{k}{}_{ikj} = \partial_k \Gamma'^{k}_{ij} - \partial_j \Gamma'^{k}_{ik} + \Gamma'^{k}_{km}\Gamma'^{m}_{ij} - \Gamma'^{k}_{jm}\Gamma'^{m}_{ik}$ (g')

$\frac{1}{2}(\delta^k_i \partial_j f + \delta^k_j \partial_i f - \nabla^k f\, g_{ij}) + \Gamma^k_{ij}$

$\partial_i f(p) = \delta^{kl}\partial_l f(p)\delta_{ik} = (g^{kl}\partial_l f\, g_{ik})(p)$

$$= R_{ij} + \underbrace{\left[\frac{1}{2}\partial_k(\delta^k_j \partial_i f + \delta^k_i \partial_j f - \nabla^k f\, g_{ij}) - \frac{1}{2}\partial_j(\delta^k_i \partial_k f + \delta^k_i \partial_k f - \nabla^k f\, g_{ik})\right]}_{①}$$

$$\underbrace{\left[\begin{array}{l} + \frac{1}{4}(\delta^k_m \partial_k f + \delta^k_k \partial_m f - \nabla^k f\, g_{km})(\delta^m_j \partial_i f + \delta^m_i \partial_j f - \nabla^m f\, g_{ij}) \\ - \frac{1}{4}(\delta^k_m \partial_j f + \delta^k_j \partial_m f - \nabla^k f\, g_{jm})(\delta^m_k \partial_i f + \delta^m_i \partial_k f - \nabla^m f\, g_{ik}) \end{array}\right]}_{②}$$

($\partial_m f$, $\delta_{ij}\partial_m f$)

$$① = \frac{1}{2}[\partial_{ji} f + \partial_{ij} f - \partial_k(\partial_k f\, \delta_{ij}) - \delta^k_k \partial_{ij} f - \partial_{ji} f + \partial_j(\partial_k f\, \delta_{ik})] = \frac{2-n}{2}\partial_{ij} f - \partial_{kk} f\, \delta_{ij}\cdot\frac{1}{2}$$

($\partial_{ij} f$)

$$② = \frac{1}{4}[(\partial_m f + n\partial_m f - \partial_m f)(\delta^m_j \partial_i f + \delta^m_i \partial_j f - \delta_{ij}\partial_m f) - (\delta^k_m \partial_j f + \delta^k_j \partial_m f - \delta_{jm}\partial_k f)(\delta^m_k \partial_i f + \delta^m_i \partial_k f - \delta^k_i \partial_m f)]$$

$2n\,\partial_i f\,\partial_j f - n(\partial_m f)^2 \delta_{ij}$ $\qquad$ $\partial_i f\,\partial_j f(n+2) - 2\delta_{ij}(\partial_m f)^2$

$$= \frac{1}{4}\underbrace{(2n-(n+2))}_{n-2}\partial_i f\,\partial_j f - \frac{n-2}{4}\delta_{ij}(\partial_m f)^2.$$

$$\Rightarrow R'_{ij} = R_{ij} - \frac{n-2}{2}\partial_{ij} f + \frac{n-2}{4}\partial_i f\,\partial_j f - \frac{1}{2}(\partial_{kk} f + \frac{n-2}{2}(\partial_m f)^2)\delta_{ij}$$

$$\Rightarrow R' = \underbrace{g'^{ij}}_{e^{-f}g^{ij}(p) = e^{-f}\delta^{ij} \text{ at } p} R'_{ij} = e^{-f} R'_{ii} = e^{-f}\Big[R - \underbrace{(n+1)\partial_{ii} f}_{\Delta_g f} + \underbrace{\frac{(n+1)(n-2)}{4}(\partial_i f)^2}_{|\nabla f|^2}\Big] = e^{-f}[R - (n+1)\Delta_g f - \frac{(n+1)(n-2)}{4}|\nabla f|^2]$$

If $n \geq 3$, then $\frac{4(n+1)}{n-2} e^{-\frac{n-2}{4}f}\Delta_g(e^{\frac{n-2}{4}f}) = (n+1)\partial_{ii} f + \frac{(n+1)(n-2)}{4}(\partial_i f)^2$.

(normal coord. at p)

$$\Rightarrow R' = e^{-f}\left[R - \frac{4(n+1)}{n-2} e^{-\frac{n-2}{4}f}\Delta_g(e^{\frac{n-2}{4}f})\right]$$

If $e^{-f} = m^{\frac{4}{n-2}}$ $\Rightarrow R' = m^{-\frac{4}{n-2}}\left[R - \frac{4(n+1)}{n-2}(m^{-1}\Delta m)\right]$

$$\Rightarrow R' = e^{-f}\left[R - \frac{4(n-1)}{n-2} e^{-\frac{n+2}{4}f}\Delta_g\left(e^{\frac{n-2}{4}f}\right)\right]$$

If $e^f = \varphi^{\frac{4}{n-2}}$ $\Rightarrow R' = \varphi^{-\frac{4}{n-2}}\left[R - \frac{4(n-1)}{n-2}\varphi^{-1}\Delta_g\varphi\right]$

$$\Rightarrow L_{g'}(u) = -\Delta_{g'}(\varphi^{-1}v) + \frac{n-2}{4(n-1)}R'\varphi^{-1}v = \varphi^{-\frac{2n}{n-2}}\left[(\Delta_g\varphi)v - \varphi\Delta_g v\right] + \frac{n-2}{4(n-1)}\varphi^{-\frac{4}{n-2}}\left[R - \frac{4(n-1)}{n-2}\varphi^{-1}\Delta_g\varphi\right]v$$

($\varphi^{-1}v = u$)

$$= \varphi^{-\frac{2n}{n-2}}\left(-\Delta_g v + \frac{n-2}{4(n-1)}Rg\,v\right) = \varphi^{-\frac{2n}{n-2}}L_g(v) = \varphi^{-\frac{2n}{n-2}}(\varphi u)$$

($\Delta_{g'}(\varphi^{-1}v) = u$)

$$= \frac{1}{\sqrt{|g|}}\partial_i\left(\underbrace{\sqrt{|g|}(g')^{ij}}_{\varphi^2\sqrt{|g|}g^{ij}}\partial_j\underbrace{u}_{\varphi^{-1}v}\right) = \varphi^{-\frac{2n}{n-2}}\frac{1}{\sqrt{|g|}}\partial_i\left[\sqrt{|g|}g^{ij}\underbrace{(-\partial_j\varphi v + \varphi\partial_j v)}_{(-\varphi^{-2}\partial_j\varphi v + \varphi^{-1}\partial_j v)}\right]$$

$$= \varphi^{-\frac{2n}{n-2}}\left[(-\Delta_g\varphi)v - g^{ij}\partial_i\varphi\partial_j v + g^{ij}\partial_i\varphi\partial_j v + \varphi\Delta_g v\right]$$

$g'_{ij} = \varphi^{\frac{4}{n-2}}g_{ij}$ $\quad \sqrt{|g'|} = \varphi^{\frac{4n}{n-2}\cdot\frac{1}{2}}\sqrt{|g|}$

$\Rightarrow \sqrt{|g'|} = \varphi^{\frac{2n}{n-2}}\sqrt{|g|}$. ($\frac{2n}{n-2} = p^*$)

$p^* = \frac{np}{n-p}$

* Yamabe problem (on closed Riemannian manifolds)

(M^n, g) : a closed R. manifold $(n \geq 3)$

$\Rightarrow$ Is there a metric $\underbrace{g' = \varphi^{\frac{4}{n-2}}g}_{g'}$ on M such that $R' = \text{const.}$ on M?

g' : conformal to g. ($\rightarrow$ suffices to find φ)

Note! If $n=2$, the answer is yes by the uniformization theorem.

$L_{g'}(u) = \varphi^{-\frac{n+2}{n-2}}L_g(\varphi u)$ on M. $\leftarrow$ Take $u=1$.

$$\| $$

$$-\Delta_g u + \frac{n-2}{4(n-1)}Rg'u^{1}$$

($\Delta_g u = 0$, $\frac{n-2}{4(n-1)}Rg' = \text{const.} = c$)

⇒ It is enough to find a smooth solution to

$$\begin{cases} -\Delta_g \varphi + \frac{n-2}{4(n-1)} R_g \varphi = C \cdot \varphi^{\frac{n+2}{n-2}} & \text{on } M. \\ \varphi > 0 & \text{on } M. \end{cases}$$

second-order semi-linear elliptic operator

nonlinear

Q. How to solve? ← variational method.

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제 17회 기하학 겨울학교

Title: Laplacian, Sobolev inequality, and conformal geometry

Lecture 3. Solvability of the Yamabe problem.

* Yamabe problem

(M^n, g) : a closed R. mfd. $n \geq 3$.

Q. $g' = \varphi^{\frac{4}{n-2}} g$ such that $R_{g'}$: const?

$$\Rightarrow \begin{cases} -\Delta_g \varphi + \frac{n-2}{4(n-1)} R_g \varphi = c\varphi^{\frac{n+2}{n-2}} & \text{on } M \\ \varphi > 0 & \text{on } M \end{cases}$$

↑

- variational method.
- Sobolev inequality. (p=2)

$$\|u\|_{L^{\frac{2n}{n-2}}(M)} \leq C \|\nabla u\|_{L^2(M)}$$

* variational methods

Consider a constrained minimization problem

$$Y(M,g) = \inf \left\{ \underbrace{\int_M \left(|\nabla_g u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) dv_g}_{I_g(u)} : \underbrace{\|u\|_{L^{\frac{2n}{n-2}}(M)} = 1}_{J_g(u) = \int_M |u|^{\frac{2n}{n-2}} dv_g - 1 = 0}, \ u \in W^{1,2}(M) \right\} < \infty$$

$I_g(u) : W^{1,2}(M) \to \mathbb{R}$

$W^{1,2}(M)$ Hilbert space

completion of $C^\infty(M)$ with $\|u\|_{W^{1,2}(M)}$

$\|u\|_{W^{1,2}(M)} = \|\nabla u\|_{L^2(M)} + \|u\|_{L^2(M)}$.

Lagrange multipier $\Rightarrow$ φ: minimizer

$\to I_g'(\varphi) = \lambda J_g'(\varphi)$ for some $\lambda \in \mathbb{R}$.

$\to I_g'(\varphi)\psi = \lambda J_g'(\varphi)\psi$ for some $\lambda \in \mathbb{R}$. $\psi \in W^{1,2}(M)$.

↑ directional derivative

$$I_g'(\varphi)\psi = \lim_{t\to 0}\frac{I_g(\varphi+t\psi)-I_g(\varphi)}{t} = 2\int_M \left(\langle \nabla_g\varphi, \nabla_g\psi\rangle_g + \frac{n-2}{4(n-1)}R_g\varphi\psi\right) dv_g$$

$$J_g'(\varphi)\psi = \frac{2n}{n-2}\int_M |\varphi|^{\frac{4}{n-2}}\varphi\psi\, dv_g.$$

$$\Rightarrow \quad 2\int_M \left(\langle \nabla_g\varphi, \nabla_g\psi\rangle_g + \frac{n-2}{4(n-1)}R_g\varphi\psi\right)dv_g = \frac{2n}{n-2}\lambda\int_M |\varphi|^{\frac{4}{n-2}}\varphi\psi\, dv_g$$

$\lambda \in \mathbb{R}$; $\Leftarrow$ if φ: smooth ; $\Leftarrow$ if $\varphi > 0$ on M.

$$\int_M \left(-\Delta_g\varphi + \frac{n-2}{4(n-1)}R_g\varphi\right)\psi\, dv_g \qquad c\int_M \varphi^{\frac{n+2}{n-2}}\psi\, dv_g$$

$$\Rightarrow \quad -\Delta_g\varphi + \frac{n-2}{4(n-1)}R_g\varphi = c\cdot\varphi^{\frac{n+2}{n-2}} \quad \text{on } M.$$

$(M,g) = (\mathbb{R}^n, g_{Eucl.}) \Rightarrow \inf\left\{ \int_{\mathbb{R}^n} |\nabla u|^2 : \|u\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} = 1 \right\}$

$(M,g) = (\mathbb{R}^n, g_{Eucl.})$: non-compact, but conformally equivalent to $S^n \setminus \{\text{north pole}\}$ (stereographic projection)

$$\inf_{u\neq 0}\frac{\int_{\mathbb{R}^n}|\nabla u|^2}{\|u\|^2_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}} =: S^2$$

$$\Rightarrow \quad \frac{\|\nabla u\|_{L^2(\mathbb{R}^n)}}{\|u\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}} \geq S \quad \text{for } \forall u \in \dot{W}^{1,2}_0(\mathbb{R}^n)\setminus\{0\}.$$

S: optimal (largest) constant.

$\Uparrow$ Sobolev ineq. ($p=2$).

$\Rightarrow$ If $(M,g) = (\mathbb{R}^n, g_{Eucl.})$, then (*) is reduced to finding the optimal constant of the Sobolev ineq. ($p=2$) and functions achieving the equality.

"extremizers"

* bubbles

Conformal invariance : translation, scaling, inversion

Talenti (1976) proved that extremizers of $\|\nabla u\|_{L^2(\mathbb{R}^n)} \geq S\|u\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}$

must have the form $W_{\lambda,\xi}(x) = c_n\left(\frac{\lambda}{\lambda^2+|x-\xi|^2}\right)^{\frac{n-2}{2}}$ $x\in\mathbb{R}^n$.

$(n-2)$ ← inversion.

$= \lambda^{-\frac{n-2}{2}} W_{1,0}(\lambda^{-1}(x-\xi))$

$\cdots \to$ (n-2) ← inversion.

$= \lambda^{-\frac{n-2}{2}} W_{\lambda,0}(\lambda^{-1}(x-\beta))$

scaling, translation

cf. $u^*(x^*) = \frac{1}{|x^*|^{n-2}} u\left(\frac{x^*}{|x^*|^2}\right)$

They are solutions of $\begin{cases} -\Delta\varphi = \varphi^{\frac{n+2}{n-2}} \text{ in } \mathbb{R}^n \\ \varphi > 0 \end{cases}$ (Yamabe eq. in $\mathbb{R}^n$).

* History of the Yamabe problem.

Yamabe (1960), Trudinger (1968), Aubin (1976),

$c<0$ $\quad$ $c=0$ $\quad$ $c>0$ $n \geq 6$ & M: non locally conformally flat

$\Leftrightarrow$ Weyl(p) $\neq 0$ at some point $p \in M$

Schoen (1984)

$c>0$ $\begin{cases} n = 3,4,5 \\ n \geq 6. \ M: \text{l.c.f.} \end{cases}$ ← positive mass theorem.

* Aubin's approach.

Theorem (M,g): a closed n-dim'l R. manifold

('energy' of the bubbles)

i) $Y(M,g) \leq Y(\mathbb{R}^n, g_{Eucl.}) = Y(S^n, g_c) = S^2$

ii) $Y(M,g) < Y(\mathbb{R}^n, g_{Eucl.}) \Rightarrow$ (*) has a minimizer.

(namely, a minimizing sequence is convergent up to a subseq.)

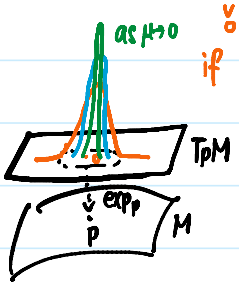
↑

recovery of the Rellich-Kondrachov compactness theorem for the 'critical' case

iii) $n \geq 7 \rightarrow Y(M,g) \leq \dfrac{I_g(\tilde{w}_{\mu,p})}{\|\tilde{w}_{\mu,p}\|_{L^{\frac{2n}{n-2}}(M)}} = Y(\mathbb{R}^n, g_{Eucl}) - C_n \mu^4 |Weyl|^2(p) + o(\mu^4) < Y(\mathbb{R}^n, g_{Eucl})$ for $\mu > 0$ small

$C_n > 0$ if $n \geq 7$

$\tilde{w}_{\mu,p} = \chi \cdot w_{\mu,0}$ ← a suitable cut-off function



$|Weyl|(p) \neq 0$. ← M: non l.c.f.

· Reference : Lee, Parker (1987) : The Yamabe problem
Schoen (1988) : Scalar curvature notes (note of Pollack)

* Variants of Yamabe problems

- Yamabe flow – a parabolic version of the Yamabe problem
- boundary Yamabe problem – (M,g): a compact manifold with boundary
- Q-curvature problem – fourth-order problem
- fractional Yamabe problem. – γ-order problem (mainly, $\gamma \in (0,2)$)

Yau's Answer for Calabi's conjecture

기하학 세미나

2022. 1. 20.

부산대 최영준

References

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- Griffith-Harris : Principles in algebraic geometry
- Blocki : The Calabi-Yau theorem
  [Ch5 in Complex Monge-Ampère Equations and Geodesics in the space of Kähler metrics (Springer LN 2038)]
- Székelyhidi : An introduction to extremal Kähler metrics.

- Song-Weinkove: An Introduction to the Kähler-Ricci flow

  [Ch3 in An Introduction to the Kähler-Ricci flow Springer LN2086]

- M a complex manifold. $z^1 \cdots z^n$, $z^1 = x^1 + iy^1$. $J\left(\frac{\partial}{\partial x^1}\right) = \frac{\partial}{\partial y^1}$.
- J the almost complex str. (integrable) $J\left(\frac{\partial}{\partial y^1}\right) = -\frac{\partial}{\partial x^1}$.
- g a Riemannian metric on M s.t.

$$g(X,Y) = g(JX, JY) \quad \text{for } \forall X, Y \in TM.$$

"hermitian metric"

- $\omega(X,Y) := g(X, JY)$.

$\Rightarrow \omega$ is skew-symmetric.

"associated form of g".

(ex) $\mathbb{C}^n \Rightarrow (x^1, y^1, \cdots, x^n, y^n) \cong (z^1, \cdots, z^n)$

- J is the standard complex str. defined by

$$J\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y} \qquad J\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial x}$$

- $g = \sum_{j=1}^{n} \left(dx^j \otimes dx^j + dy^j \otimes dy^j \right)$

$\Rightarrow J^* g = g$ hermitian metric.

- $\omega = \sum_{j=1}^{n} dx^j \wedge dy^j$ the associated hermitian form.

Adapting $dz^j = dx^j + i\,dy^j$

$dz^{\bar{j}} = dx^j - i\,dy^j$,

complexification : $TM \otimes \mathbb{C}$.

we have

(1,1)-form

- $g = \sum_{j=1}^{n} dz^j \otimes dz^{\bar{j}}$ & $\omega = i \sum_{j=1}^{n} dz^j \wedge dz^{\bar{j}}$ ⌟

On a complex mfd, there is an 1-1 correspondence between

{ hermitian metrics } & { associated hermitian forms }

g $\longleftrightarrow$ ω

More precisely, hermitian

$$g = \sum g_{j\bar{k}}\, dz^j \otimes d\bar{z}^k \quad \Rightarrow \quad \begin{pmatrix} g_{1\bar{1}} & \cdots & g_{1\bar{n}} \\ \vdots & & \\ g_{n\bar{1}} & \cdots & g_{n\bar{n}} \end{pmatrix}$$

$$\omega = i\sum g_{j\bar{k}}\, dz^j \wedge d\bar{z}^k$$

Def. ω is Kähler if $d\omega = 0$

Remark. On a hermitian mfd, there are two well-known connections. ∇^{LC}, ∇^{H}.

On a Kähler mfd, $\nabla^{LC} = \nabla^{H}$

• Ricci curvature.

(1.1) $\omega = i g_{j\bar{k}} dz^j \wedge d\bar{z}^k$.

$2\, dx \wedge dy$ — Euclidean volume form.

(n.n) $\omega^n = \omega \wedge \cdots \wedge \omega = \det(g_{j\bar{k}})\, i dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge i dz^n \wedge d\bar{z}^n$.

Def. The Ricci form (tensor) $\mathrm{Ric}(\omega)$ of ω is defined by

$\det(g_{j\bar{k}}) = G_T$

(1.1) $\mathrm{Ric}(\omega) = -i\partial\bar{\partial} \log \det(g_{j\bar{k}}) = -i \frac{\partial^2}{\partial z^j \partial \bar{z}^k} \log G_T \, dz^j \wedge d\bar{z}^k$

$=: -i\partial\bar{\partial} \log \omega^n$

Remark. $[\mathrm{Ric}(\omega)] = c_1(M) \in H^2(M,\mathbb{R}) \cap H^{1,1}(M,\mathbb{C}) =: H^{1,1}(M,\mathbb{R})$

de Rham cohomology

Dolbeault cohomology

$\omega = \bar{\omega}$

$\Downarrow$

Since ω is real & d-closed

$$[\omega] \in H^{1,1}(M,\mathbb{R}).$$

Suppose that $\omega_1, \omega_2 \in [\omega]$.

$\Rightarrow \omega_1 = \omega_2 + d\theta$ for some $\theta \in \Omega^1(M)$

Lemma ($\partial\bar\partial$-lemma) $d\theta$

$$d = \partial + \bar\partial \qquad d^c = \frac{1}{i}(\bar\partial - \partial)$$

Let α be a real, d-exact $(1,1)$-form on a cpt Kähler manifold. Then α is $dd^c \left(= i\partial\bar\partial\right)$-exact, i.e.,

$$\alpha = dd^c\eta \quad \text{for some } \eta \in C^\infty(M)$$

$$\Rightarrow \omega_1 = \omega_2 + d\theta$$

$$= \omega_2 + dd^c\eta$$

Therefore, $[\omega] \in H^{1,1}(M,\mathbb{R})$ can be written as

$$[\omega] = \left[\, \omega + dd^c\eta \;\middle|\; \eta \in C^\infty(M) \right].$$

$$\omega_1 \sim \omega_2 \iff \omega_1 = \omega_2 + dd^c\eta.$$

Prop. $[\mathrm{Ric}(\omega)] = c_1(M) \in H^{1,1}(M,\mathbb{R})$

It does NOT depend on ω.

top'l Inv

Proof ① See [GH] Ch 1

② ω_1 ω_2 two Kähler metrics

$$\mathrm{Ric}(\omega_1) - \mathrm{Ric}(\omega_2) = -dd^c \log \omega_1^n + dd^c \log \omega_2^n$$

$$= -dd^c \log \frac{\omega_1^n}{\omega_2^n}.$$

$$\Rightarrow [\mathrm{Ric}(\omega_1)] = [\mathrm{Ric}(\omega_2)].$$

Calabi conjecture. 1954.

(M, ω) a compact Kähler manifold.

$$\hat{R} \in c_1(M) = [Ric(\omega)] \in H^{1,1}(M, \mathbb{R})$$

$$\Rightarrow \exists ?\ \hat{\omega} \in [\omega] \quad s.t.$$

$$Ric(\hat{\omega}) = \hat{R}$$

How to reformulate this in terms of PDE?

$$\tilde{R} \in [R_\omega] \rightleftarrows \tilde{R} = R_\omega + dd^c \eta$$

$$\exists \quad \tilde{\omega} \in [\omega] \quad s.t \quad \underbrace{Ric(\tilde{\omega})}_{R_{\tilde{\omega}}} = \tilde{R} \quad \rightleftarrows \quad \exists\, \tilde{\omega} = \omega + dd^c \varphi \quad s.t \quad R_{\tilde{\omega}} = R_\omega + dd^c \eta$$

$$R_{\tilde{\omega}} - R_\omega = dd^c \eta$$

$$\rightleftarrows \; -dd^c \left(\log \frac{\tilde{\omega}^n}{\omega^n} - \eta \right) = 0$$

$$\rightleftarrows \quad \log \frac{\tilde{\omega}^n}{\omega^n} - \eta = c \qquad \text{for} \quad c \in \mathbb{R}$$

$$\rightleftarrows \quad \tilde{\omega}^n = e^{\eta + c} \omega^n$$

$$\rightleftarrows \quad (\omega + dd^c \varphi)^n = e^{\eta + c} \omega^n$$

Thm (Yau 1978)

$f \in C^\infty(M)$ with $\int_M e^f \omega^n = \int_M \omega^n$

$\Rightarrow \exists ! \varphi \in C^\infty(M)$ s.t.

$$(\omega + dd^c \varphi)^n = e^f \omega^n$$

- Kähler – Einstein metric

<u>Def</u> ω is KE if $Ric(\omega) = \lambda\omega$ for some $\lambda \in \mathbb{R}$.

Suppose that M admits a KE metric.

$\Rightarrow c_1(M) = [Ric(\omega)] = [\lambda\omega]$

$\Rightarrow c_1(M) > 0$ or $c_1(M) = 0$ or $c_1(M) < 0$.

"necessary condition"

By the same argument, one can easily see that

$\exists$ KE metric $\rightleftarrows$ $(\omega + dd^c\varphi)^n = e^{-\lambda\varphi + f}\omega^n$

where $f \in C^\infty(M)$ s.t.

$$dd^c f = \mathrm{Ric}(\omega) - \lambda\omega.$$

$\lambda = -1$ if $c_1(M) < 0$.

$\lambda = 0$ if $c_1(M) = 0$.

$\lambda = 1$ if $c_1(M) > 0$.

Thm (Aubin / Yau)

$c_1 < 0$ (canonically polarized) $\Rightarrow \exists!$ KE.

$c_1 = 0$ (Calabi–Yau) $\Rightarrow \exists!$ Ricci flat $\in$ each Kähler class.

Remark

$c_1 > 0$ Fano. $\Rightarrow \nexists$ KE in general.

(e.g. one pt blow up of $\mathbb{P}^2$)

Thm (Chen–Donaldson–Sun, Tian, Berman 2015)

$\exists$ KE $\Leftrightarrow (X, -K_X)$ K-polystable.

Applications of Kähler-Einstein metrics

(1) KE metric is an invariant metric

Thm (Schwarz lemma for volume forms — Mok–Yau).

M a complete hermitian with scar. curv $\geq -K_1$

N a hermitian mfd with Ricci curv $\leq -K_2$.

$f : M \to N$ holomorphic.

$$\Longrightarrow \quad \sup \frac{f^* V_N}{V_M} \leq \left(\frac{K_1}{n K_2} \right)^n$$

- **Corollary.** M a complete Kähler metric with KE metric ω with $\mathrm{Ric}(\omega) = -\omega$.

  $\Rightarrow f^*\omega = \omega$ for $f: M \to M$ biholomorphic.

- $c_1(M) = 0$. KE metric is an invariant metric for $f: M \to M$ with

  $$f^*[\omega] = [\omega]$$

- $c_1(M) > 0$: M does NOT have a holomorphic v.f.

  $\Rightarrow$ Inv. metric.

(2) Uniformization theorem.

Thm M a Kähler surface with ample canonical bundle

$$\Rightarrow \quad 3c_2(M) \geq c_1(M)^2 \quad \&$$

"$=$" iff M is covered biholomorphically by $\mathbb{B}^2$.

Thm (M, g) KE mfd.

$$\Rightarrow (\tilde{M}, \omega) \cong \mathbb{P}^n, \mathbb{C}^n \text{ or } \mathbb{B}^n$$

iff

$$\left(\frac{2(n+1)}{n} c_2(M) - c_1(M)^2 \right) [\omega]^{n-2} = 0$$

(3) Domain of holomorphy in SCV.

$\Omega \subset \mathbb{C}^n$ a bounded domain.

Thm (Mok-Yau)

Ω is a domain of holomorphy

$\Updownarrow$

$\exists!$ complete KE metric with $Ric = -(n+1)$

(4) Deformation theory / Moduli space.

(5) Curvature of direct image sheaves.

(6)

.....

In this talk, we will solve

$$(\omega + dd^c\varphi)^n = e^{\varphi + f}\omega^n \qquad \text{(MA)}$$

$$\omega + dd^c\varphi > 0$$

$$C^{k,\alpha}(M) = \{\varphi : M \to \mathbb{R} \mid \|\varphi\|_{C^{k,\alpha}} < \infty\}.$$

for $k \in \mathbb{N}$. $\alpha \in (0,1)$.

- Continuity method. (Yau)

Consider

$$\begin{cases} \cdot\ (\omega + dd^c\varphi)^n = e^{\varphi + tf}\omega^n \\ \cdot\ \omega + dd^c\varphi > 0 \end{cases} \qquad (MA)_t$$

for $0 \le t \le 1$.

$\Rightarrow$ $t=0$: $(MA)_0$ is solvable.

$t=1$: What we want.

$$S = \{ t \in [0,1] \mid (MA)_t \text{ is solvable} \}.$$

$$\Rightarrow \quad 0 \in S.$$

Claim: S is open & closed.

(1) Openness

$$(MA) \Leftrightarrow \log \frac{(\omega + dd^c \varphi)^n}{\omega^n} - \varphi = f.$$

Define $\mathcal{F}_c : C^{k+2,\alpha}(M) \to C^{k,\alpha}(M)$ by

$$\mathcal{F}_c(\varphi) := \log \frac{(\omega + dd^c \varphi)^n}{\omega^n} - \varphi.$$

If $\mathcal{F}_c$ has a local inverse, then S is open.

So, we want apply the inverse function theorem.

$$D\mathcal{F}_c(\varphi) : C^{k+2,\alpha}(M) \to C^{k,\alpha}(M) \qquad \text{"Frechét derivative"}$$

$$\psi \longmapsto D\mathcal{F}_c(\varphi)(\psi)$$

$$D\mathcal{F}_c(\varphi)(\psi) = \frac{d}{dt}\Big|_{t=0} \mathcal{F}_c(\varphi + t\psi)$$

$$= \frac{d}{dt}\Big|_{t=0} \left\{ \log \frac{(\omega + dd^c(\varphi + t\psi))^n}{\omega^n} - (\varphi + t\psi) \right\}$$

$$= \frac{d}{dt}\Big|_{t=0} \log (\omega + dd^c(\varphi + t\psi))^n - \psi$$

$$= \frac{1}{(\omega + dd^c\varphi)^n} \cdot \frac{\partial}{\partial t}\Big|_{t=0} \left(\omega + dd^c(\varphi + t\psi)\right)^n - \psi$$

$$= \frac{1}{\hat{\omega}^n} \cdot n\, dd^c\psi \wedge \hat{\omega}^{n-1} - \psi$$

where $\hat{\omega} = \omega + dd^c\varphi$

$$\left(\underline{\omega + dd^c(\varphi + t\psi)}\right) \wedge \cdots$$

$$\wedge \cdots \wedge -$$

$$= tr_{\hat{\omega}}\, dd^c\psi - \psi$$

$$= \Delta_{\hat{\omega}}\psi - \psi = \left(\Delta_{\hat{\omega}} - id\right)\psi$$

Note that $\Delta_{\tilde{\omega}} - I : C^{k+2,\alpha} \longrightarrow C^{k,\alpha}$ is invertible.

$\Rightarrow$ IFT implies the openness.

(2) Closedness

Recall $(\omega + dd^c\varphi)^n = e^{\varphi + f}\omega^n$

$\omega + dd^c\varphi > 0$

Claim: $\|\varphi\|_{C^{2,\alpha}} < C$ "A priori estimate"

$\|$

$C(M, \omega, \sup|f|, \sup|\Delta f|)$

「 Note : Claim ⇒ S is closed !

$t_m \in S$ with $t_m \to t_\infty$.

$\Rightarrow \exists\, \varphi_m \in C^{k+2,\alpha}(M)$ s.t. φ_m is a solution of $(MA)_{t_m}$

$\Rightarrow$ Claim $\quad \|\varphi_m\|_{C^{2,\alpha}} < C$

$\Rightarrow$ Schauder estimate $\quad \|\varphi\|_{C^{k+2,\alpha}} < C$.

$\Rightarrow$ Arzela-Ascoli Theorem $\quad \varphi_m \to \varphi_\infty$ in $C^{k+2,\alpha}$

$\Rightarrow t_\infty \in S$. 」

Goal $\|\varphi\|_{C^{2,\alpha}} < C$.

- C^0-estimate.
- C^2-estimate.
- C^3-estimate. (or $C^{2,\alpha}$-estimate)

(1) C^0-estimate

Lemma. $\sup|\varphi| \leq \sup|f|$.

Proof $\exists p \in M$ s.t. $\varphi(p) = \sup_M \varphi$.

$\Rightarrow dd^c\varphi = i\partial\bar{\partial}\varphi \leq 0$ at p.

$\Rightarrow \quad \omega + dd^c \varphi \leq \omega .$

$\Rightarrow \quad 0 \geq \log \frac{(\omega + dd^c \varphi)^n}{\omega^n} = f + \varphi . \qquad$ at p.

$\Rightarrow \quad \varphi(p) \leq -f(p)$

$\Rightarrow \quad \sup \varphi \leq \sup |f| .$

Similarly, $\inf \varphi \geq -\sup |f| .$

Therefore, $|\varphi| \leq \sup |f| .$

(2) C^2-estimate ($\Delta\varphi$)

$$\omega' = \omega + dd^c\varphi > 0.$$

- $$0 < \mathrm{tr}_\omega \omega' = \mathrm{tr}_\omega(\omega + dd^c\varphi)$$
$$= n + \Delta_\omega \varphi.$$
$$\Rightarrow \Delta\varphi > -n$$

Want to find an upper bound of $\Delta\varphi$.

- Note that
$$\omega' \leq \underbrace{(\mathrm{tr}_\omega \omega')}\, \omega.$$

Lemma $\exists B = B(M, g)$ s.t.

$$\Delta' \log \mathrm{tr}_{\omega} \omega' \geq -B \mathrm{tr}_{\omega'} \omega - \frac{\mathrm{tr}_{\omega} R_{\omega'}}{\mathrm{tr}_{\omega} \omega'}$$

Lemma $\exists C = C(M, \omega, \sup|f| \sup|\Delta f|) > 0$ s.t.

$$|\Delta \varphi| < C$$

Moreover,

$$\frac{1}{C} \omega \leq \omega' \leq C \omega.$$

Proof. $\omega'^n = e^{\varphi+f}\omega^n$

$$\mathrm{Ric}(\omega') = \mathrm{Ric}(\omega) - dd^c f - \underbrace{dd^c\varphi}_{\omega'-\omega}.$$

$$\Rightarrow \Delta' \log \mathrm{tr}_\omega \omega' \geq -B\,\mathrm{tr}_{\omega'}\omega - \frac{\mathrm{tr}_\omega R_{\omega'}}{\mathrm{tr}_\omega \omega'}$$

$$\boxed{(\mathrm{tr}_\omega \omega')(\mathrm{tr}_{\omega'}\omega) \geq n^2} \quad = -B\,\mathrm{tr}_{\omega'}\omega + \frac{\Delta f + \mathrm{tr}_\omega \omega' - n - R}{\mathrm{tr}_\omega \omega'}$$

$$\Rightarrow \Delta' \log \mathrm{tr}_\omega \omega' \geq -B\,\mathrm{tr}_{\omega'}\omega - C\,\mathrm{tr}_{\omega'}\omega$$

$$= -(B+C)\,\mathrm{tr}_{\omega'}\omega.$$

On the other hand,

$$\Delta' \varphi = \mathrm{tr}_{\omega'}(\omega' - \omega) = n - \mathrm{tr}_{\omega'}\omega .$$

It follows that

$$\Delta'(\underbrace{\log \mathrm{tr}_{\omega}\omega' - A\varphi}_{\| \; F}) \geq \mathrm{tr}_{\omega'}\omega - An .$$

$$\sum g'^{j\bar{k}} \cdot g_{j\bar{k}}$$

$\exists p \in M$ s.t. $F(p) = \sup_M F$

$$\Rightarrow \quad 0 \geq \mathrm{tr}_{\omega'}\omega\,(p) - An . \qquad (*)$$

$$\Rightarrow \quad \mathrm{tr}_{\omega'}\omega \leq An .$$

Choose a normal coord. for ω at p w.r.t. ω' is diagonal at p, i.e.,

$$\omega = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \qquad \omega' = \begin{pmatrix} g'_{1\bar{1}} & & 0 \\ & \ddots & \\ 0 & & g'_{n\bar{n}} \end{pmatrix} \quad \text{at } p.$$

$$(*) \Rightarrow \frac{1}{g'_{1\bar{1}}} + \cdots + \frac{1}{g'_{n\bar{n}}} \leq An \quad \text{at } p.$$

On the other hand,

$$\omega'^n = e^{f+\varphi}\omega^n$$

$$\Rightarrow g'_{1\bar{1}} \cdots g'_{n\bar{n}} = e^{f(p)+\varphi(p)} \leq C.$$

$$\Rightarrow g'_{i\bar{i}} \leq C.$$

$$\Rightarrow \mathrm{tr}_\omega \omega' \leq C.$$

Hence,

$$\log \mathrm{tr}_\omega \omega' - A\varphi \leq \left(\log \mathrm{tr}_\omega \omega' - A\varphi\right)(\varphi) \leq C.$$

$$\Rightarrow \quad \log \mathrm{tr}_\omega \omega' \leq C. \quad (**)$$

$$\Rightarrow \quad |\Delta \varphi| < C.$$

Moreover, $(**)$ implies that for $x \in M$

$$g'_{1\bar{1}} + \cdots + g'_{n\bar{n}} \leq C \quad \text{in a normal coord.}$$

$$g'_{1\bar{1}} \cdots g'_{n\bar{n}} = e^{-f(x)+\varphi(x)} \geq c.$$

$$\Rightarrow \quad \frac{1}{C} \leq g'_{i\bar{i}} \leq C. \qquad \blacksquare$$

(3) C^3 - estimate

$\omega' = \omega + dd^c\varphi$.

$$|S|^2 = |\nabla \omega'|^2 = |\nabla(\omega + dd^c\varphi)|^2$$

Lemma $\Delta |S|^2 \geq -C|S|^2 - C$.

$\hookrightarrow$ $|S|^2 \leq C$.

Maximum Principle

Remark ① Evans – Krylov theory (interior regularity) 1982.

C^2-estimate $\Rightarrow$ $C^{2,\alpha}$-estimate

② $\lambda = 0$

$$(\omega + dd^c\varphi)^n = e^f \omega^n$$

All estimates are available except C^0-estimate.

(Maximum principle does NOT give the C^0-estimate).

- $\|\varphi\|_p < C$.
- $\|\varphi\|_{\frac{np}{n-1}} \leq (C_p)^{\frac{1}{p}} \|\varphi\|_p$

Moser iteration ✓

$$p_0 = 2 \qquad p_k = \frac{n p_{k-1} - 1}{n-1}$$

$$\Rightarrow \|\varphi\|_\infty = \lim_{k\to\infty} \|\varphi\|_{p_k} \leq \|\varphi\|_2 \prod_{j=0}^{\infty} (C_{p_j})^{1/p_j}$$

$$\leq C.$$

- $\lambda = 1$ (Fano)

$$(\omega + dd^c\varphi)^n = e^{-\varphi + f}\omega^n.$$

$$\omega + dd^c\varphi > 0$$

KE metric on cpt Kähler manifolds with $c_1 < 0$, $c_1 = 0$.

&

$c_1 > 0$

K-stable Fano.

$\Rightarrow$ KE metric

(1) KE metric on complete Kähler manifold.

- Cheng–Yau 1980

M a complete Kähler mfd with

- bounded geometry.

$|\nabla^m R| < C$ — $\mathrm{Ric}(\omega) < -C\omega$.

$\Rightarrow \exists!$ complete Kähler–Einstein metric

Mok–Yau 1983.

(2) Parabolic version : Kähler-Ricci flows.

M a cpt Kähler mfd.

$$(*)\quad \begin{cases} \dfrac{\partial}{\partial t}\omega = -\mathrm{Ric}(\omega) + \nu\omega. & (\nu = 0 \text{ or } -1). \\ \omega|_{t=0} = \omega_0. \end{cases}$$

Note that Ricci flow preserves Kählerness!

- H.-D. Cao 1985.

$c_1(M) = 0$ or < 0.

$\Rightarrow$ $(*)$ has a long time solution. Moreover,

$$\omega(t) \to \omega_{KE} \quad \text{as} \quad t \to \infty.$$

- Chen – Sun – Wang 2018

  K-stable Fano $\Rightarrow$ KRF $\to$ KE.

- Tian–Zhang 2006

(M, ω_0) a cpt Kähler mfd.

$$\frac{\partial}{\partial t}\omega = -\mathrm{Ric}(\omega)$$

$$\omega(0) = \omega_0 \qquad (*)$$

"necessary condition" $\quad \left[\frac{\partial}{\partial t}\omega\right] = \left[-\mathrm{Ric}(\omega)\right]$

$$\frac{\partial}{\partial t}[\omega] = -c_1(M)$$

$$[\omega(t)] = [\omega_0] - t c_1(M)$$

$$T := \sup\{t > 0 \mid [\omega_0] - t c_1(M)\}.$$

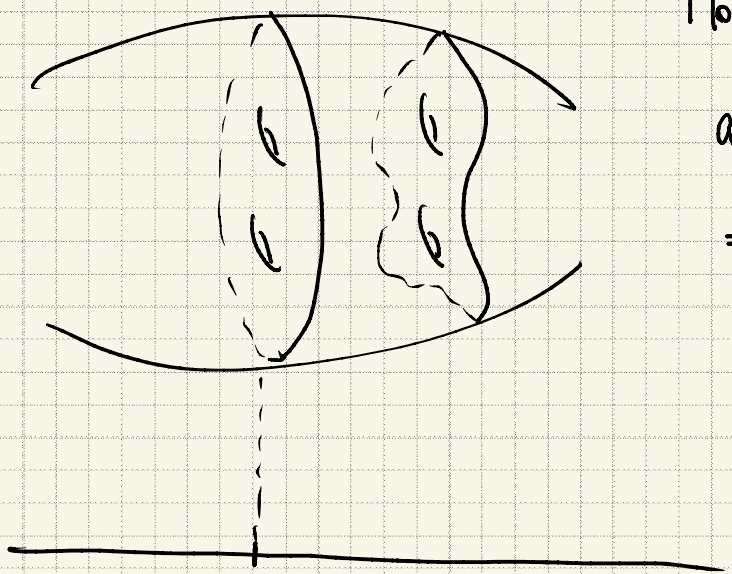
Thm $\exists!$ maximal solution of $(*)$ for $t \in [0, T)$.

⟿ Analytic Minimal Model Program.

(3) Deformation of complex str. / Moduli space.

On a cpt Kähler mfd with $c_1 < 0$,

KE metric $\rightleftarrows$ complex structure



How many complex str. on a cpt Kähler manifold ?

$\Rightarrow$ Kodaira-Spencer class

$\in H^{0,1}(X, TX)$.

$\Rightarrow$ The harmonic representative is given by the KE metric.

(4) Singular Kähler-Einstein metric

Pluripotential theory
Bedford-Taylor '76
Kołodziej '98.

Thm Eyssidieux-Guedj-Zeriahi 2008

X a cpt Kähler mfd

ω a semi-positive $(1,1)$-form s.t. $\int \omega^n > 0$

$f \in L^p(X)$ s.t.

$$\int f\omega^n = \int \omega^n$$

$$\Rightarrow (\omega + dd^c\varphi)^n = f\omega^n \text{ has a unique solution}$$

(*) $\omega + dd^c\varphi \geq 0$

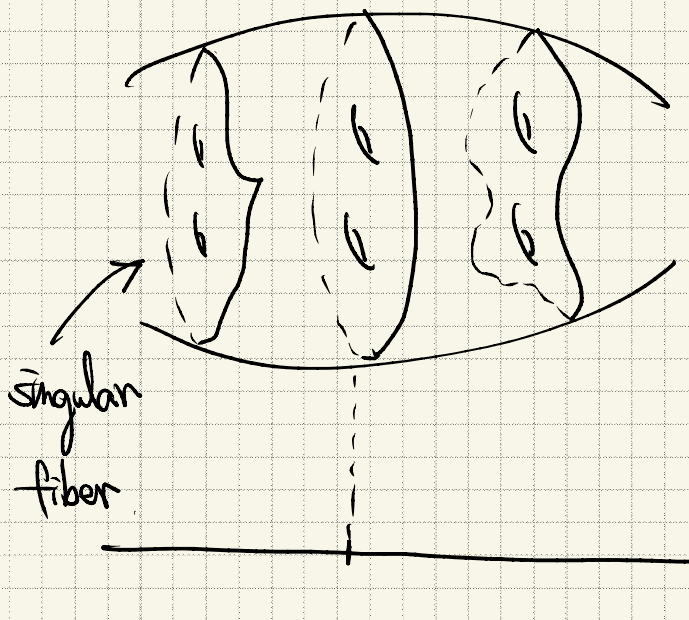
$\Rightarrow$ a singular Kähler-Einstein metric.

(& on $\begin{cases} \text{log canonical pairs} \\ \text{log Calabi-Yau pairs} \end{cases}$)

Thm (Boucksom-Eyssidieux-Guedj-Zeriahi 2010)

(*) is solvable when $[\omega]$ is big

(5) Variation of KE metrics



X

$\downarrow f$ proper surj hol. s.t (submersion)

Y

$X_y = f^{-1}(y)$ a cpt Kähler mfd with $c_1 < 0$, $c_1 = 0$

$\Rightarrow \exists!$ d-closed real $(1,1)$-form ρ s.t.

$$\rho|_{X_y} = \omega^{KE}_{X_y}$$

Q. $\rho > 0$ or ≥ 0 on X ?

$\textcircled{H}(K_{X/Y}) > 0$ or ≥ 0 on X ?

- 2013. Schumacher : $c_1(X_y) < 0 \quad \rho > 0$.
- 2021. Braun-C-Schumacher : $c_1(X_y) = 0$
  $f_* \rho > 0$
- 2021. C-Schumacher : $c_1(X_y) = 0 \quad \textcircled{H}(K_{X/Y}) > 0$.
  & extension across singular fibers.

- 2019 Ar Cao – Guenancia – Păun : $c_1(X_y) = 0$.

  $f_*(\omega_{X/Y})$ hermitian-flat w.r.t. Narashimhan-Simha metric

  $\Rightarrow \rho > 0$.

Q 1. On a families of K-stable Fano manifolds, $\rho < 0$ along the horizontal direction ?

Q 2. On a family of complete Kähler-Einstein mfds, with $\mathrm{Ric} < 0$.

$\Rightarrow \rho > 0$?

2015 C. str. pseudoconvex domains in $\mathbb{C}^n$

2020 C.-Yau " in Kähler mfds

Real Lagrangian tori in symplectic four-manifolds

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20–25 Jan 2022

Outline

Part I.

- Basics on symplectic geometry
- Lagrangian submanifolds
- J-holomorphic curves theory

Part II.

- Monotone symplectic four-manifolds
- Isotopy problems on real Lagrangian tori

References.

McDuff–Salamon

1. Introduction to Symplectic Topology
2. J-holomorphic Curves and Symplectic Topology

Part I. Lagrangian submanifolds in symplectic geometry

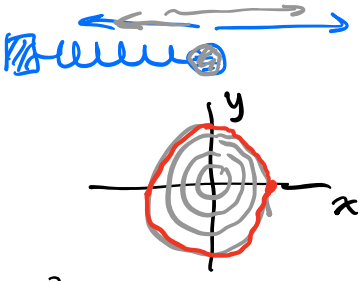
What is symplectic geometry (사교 기하학)?

The word 'symplectic' was coined by Hermann Weyl in his book '*The classical groups*', as the Greek-form of 'com-plex'. Literally, it means *twined together*.

- $(x,y) = (x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n}$
- Any smooth function $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ is called a *Hamiltonian*.
- For $(x(t), y(t))$ consider $\begin{cases} \dot{x} &= \frac{\partial H}{\partial y} \\ \dot{y} &= -\frac{\partial H}{\partial x} \end{cases}$ (*Hamiltonian equation*)
- $\phi_H^t\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ the flow of the Hamiltonian vector field $X_H = (\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x})$

$$\frac{d}{dt}\phi_H^t(x,y) = X_H(\phi_H^t(x,y)), \qquad \phi_H^0(x,y) = (x,y) \quad \text{for } (x,y) \in \mathbb{R}^{2n}.$$

- (Harmonic oscillator) $H(x,y) = \frac{1}{2}(x^2 + y^2)$ on $\mathbb{R}^2$



(Poincaré 1901)

$(\phi_H^t)^* \omega_0 = \omega_0$ where $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$ is the 2-form on $\mathbb{R}^{2n}$.

A pair (M, ω) is called a *symplectic manifold* if

- M is a smooth manifold (of even dimension).
- ω is a 2-form on M that is closed and non-degenerate.

The *Hamiltonian vector field* X_H of a Hamiltonian $H\colon M \to \mathbb{R}$ is defined by

$$\omega(X_H, \cdot) = dH.$$

Its flow $\phi_H^t\colon M \to M$ is called a *Hamiltonian diffeomorphism*.

Note. ϕ_H^t is a *symplectomorphism*, i.e. $(\phi_H^t)^* \omega = \omega$.

Example

- $(\mathbb{R}^{2n}, \omega_0)$
- T^*Q over any smooth manifold Q
- Orientable surfaces Σ_g endowed with area form
- $(\mathbb{C}P^n, \omega_{\mathrm{FS}})$, Kähler manifolds (M, J, ω)
- $S^4, S^6, \ldots$ are NOT symplectic as they are closed and $H^2_{\mathrm{dR}} = 0$.

Remarkable features

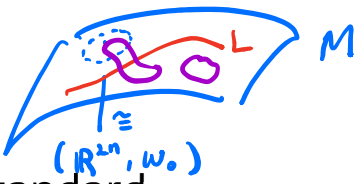
Symplectic linear algebra (i.e. $(V,\omega) = (T_xM, \omega_x)$ for $x \in M$)

Let (V,ω) be a symplectic vector space and $W \subset V$ a subspace.

- (Symplectic) $W \cap W^\omega = \{0\}$ ($\iff \omega|_W$ is nondeg)
- (Coisotropic) $W^\omega \subset W$ $\rightsquigarrow$ W/W^ω : sympl
- (Isotropic) $W \subset W^\omega$ ($\iff \omega|_W = 0$)
- (Lagrangian) $W = W^\omega$ ($\iff \omega|_W = 0$ and $\dim W = \frac{1}{2}\dim V$)
  "Maximally degenerate"

Cf. $W \subset (V,g) \Rightarrow W \cap W^\perp = \{0\}$





(Darboux) Every symplectic manifold locally looks standard.
Cf. $g \overset{\text{loc}}{=} \sum dx^i \otimes dx^i$ does not always hold.

This motivates to study the global objects/invariants in symplectic manifolds.

Global invariant: $[\omega] \in H^2_{\text{dR}}(M)$, but what else? 1985 Gromov

Lagrangian submanifolds

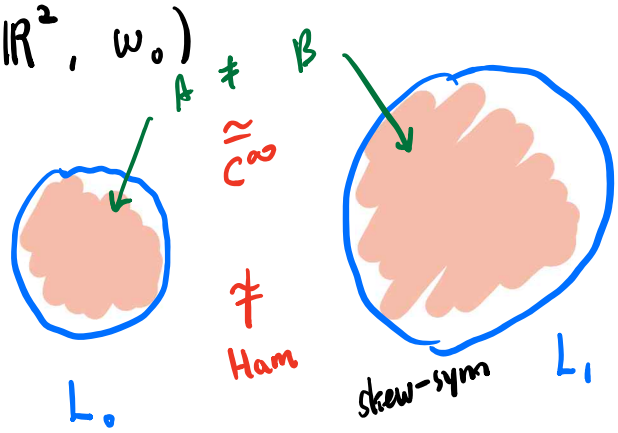
- A submanifold $L^{n} \subset (M^{2n}, \omega)$ is called *Lagrangian* if

$$\omega|_{TL} = 0 \quad \text{and} \quad \dim L = \tfrac{1}{2} \dim M$$

① $\forall X, Y \in TL : \omega(X, Y) = 0$

Example.

① $(\mathbb{R}^2, \omega_0)$



$H: \mathbb{R}^2 \to \mathbb{R} \rightsquigarrow \phi_H^1: \mathbb{R}^2 \to \mathbb{R}^2$ mechanical motion

? $\phi_H^1(L_0) = L_1$

$(\phi_H^1)^* \omega_0 = \omega_0$

↑ area ↑ area

$\omega(X, tX) = t\,\omega(X, X) = 0$

② $(\mathbb{R}^{2n}, \omega_0)$

$T^n = S^1 \times \cdots \times S^1$: Lagr (n)

Darboux

$(\mathbb{R}^{2n}, \omega_0) \hookrightarrow (M, \omega)$

$\cup$

$T^n \underset{\text{Lagr}}{\hookrightarrow} U$

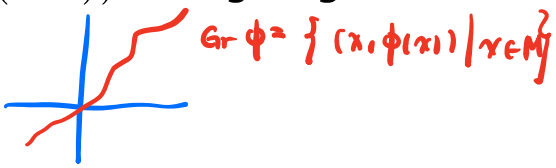
" Lagr T^n "

Why Lagrangians?

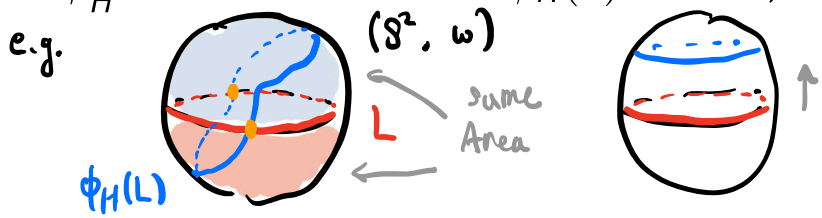
Symplectic properties can be described in terms of Lagrangian submanifolds.

- (Symplectomorphism) Let $\phi \in \mathrm{Diff}(M)$. Then

$$\phi^*\omega = \omega \iff \mathrm{Graph}(\phi) \subset (M \times M, \omega \oplus (-\omega)) \text{ is Lagrangian.}$$



- (Hamiltonian non-displaceability)
  Q. Consider a Lagrangian $L \subset (M, \omega)$.
  Can one find a *mechanical motion* (i.e. Hamiltonian diffeomorphism) $\phi_H^1 \colon M \to M$ such that $\phi_H^1(L) \cap L = \varnothing$?







- (Lagrangian embedding problem)
  Q. Which closed smooth manifolds admit a Lagrangian embedding into (M, ω)?

$\nexists \; S^n \xhookrightarrow{\text{Lagr}} (\mathbb{R}^{2n}, \omega_0)$

Theorem (Gromov 1985)

For $n \geqslant 2$ there exists no Lagrangian sphere S^n in $(\mathbb{R}^{2n}, \omega_0)$.

Remark.

- If n is even, then this follows from topological reasons, namely for orientable Lagrangian $L^n \subset (\mathbb{R}^{2n}, \omega_0)$ we have

$$\underset{0}{[L]} \bullet \underset{0}{[L]} = (-1)^{\frac{n(n-1)}{2}} \underset{2}{\chi(L)} \qquad L = S^n : \text{even}$$

- This is a *symplectic rigidity phenomenon* as there exist a smooth embedding of S^n into $\mathbb{R}^{2n}$.

Gromov's J-holomorphic curves theory (1985)

- Pick a Riemannian surface (Σ, j). (In this talk, $\Sigma = S^2$ or D^2)
- J is an *almost complex structure* on M if J is a fiberwise linear map of TM such that $J^2 = -\mathrm{id}$.
- A smooth map $u\colon \Sigma \to M$ is called *J-holomorphic* if

$$du \circ j = J \circ du,$$

where $du\colon T\Sigma \to TM$ is the derivative of u.

- Consider $(s,t) \in D^2 \subset \mathbb{C}$ with j standard complex structure. Then

$$\begin{aligned} u\colon D^2 \to \mathbb{C} \text{ is } J_0\text{-holomorphic} &\iff \partial_s u + J_0(u)\partial_t u = 0 \\ &\iff U_s = V_t, \quad U_t = -V_s \qquad \text{(CR-eqn)} \end{aligned}$$

where $J_0 = j$ and $u = (U, V)$.

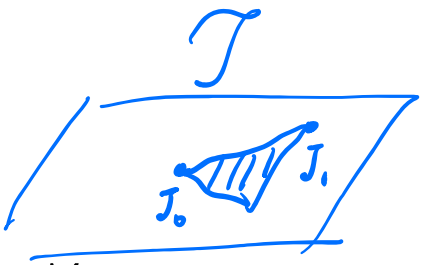
Some basics on J-curves

Gromov's key idea:

1. Use "compatible" almost complex structures on (M, ω).
2. Study the family of J-curves, i.e. the moduli space of J-curves.

- J is called ω-*compatible* if for $X, Y \in TM$

$$\omega(X, JX) > 0 \quad \text{for } X \neq 0$$
$$\omega(X, Y) = \omega(JX, JY)$$



In this case, J induces the metric $g_J(\cdot, \cdot) = \omega(\cdot, J\cdot)$ on M.

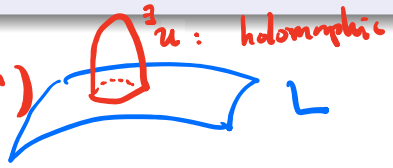
- The space $\mathcal{J}$ of ω-compatible almost complex structures on M is non-empty and contractible.
- Let $u \colon \Sigma \to M$ be J-holomorphic. Then the *energy* of u is given by

$$E(u) := \frac{1}{2} \int_\Sigma \|du\|^2 \, \mathrm{dvol}_\Sigma \overset{u: J\text{-hol}}{=} \int_\Sigma u^* \omega$$

In particular, $E(u)$ is a topological quantity.

Theorem (Gromov 1985)

Let $L \subset (\mathbb{R}^{2n}, \omega_0)$ be a closed Lagrangian submanifold. Then there exists a non-constant holomorphic disc $u\colon D^2 \to \mathbb{R}^{2n}$ with $u(\partial D^2) \subset L$.



Upshot. we have $H^1_{\mathrm{dR}}(L) \neq 0$. ($\because \nexists$ Lag $S^n \subset \mathbb{R}^{2n}$)

- To see this, we observe $E(u) = \frac{1}{2}\int_{D^2} \|du\|^2 \,\mathrm{dvol} > 0$. Moreover,

$$E(u) = \int_{D^2} u^*\omega_0 = \int_{D^2} u^* d\lambda_0 = \int_{\partial D^2} u^*\lambda_0, \quad = 0 \ (*)$$

where $\lambda_0 = \frac{1}{2}\sum(x_i dy_i - y_i dx_i)$ is a 1-form on $\mathbb{R}^{2n}$.
If the closed 1-form $\lambda_0|_L$ on L is exact, then $E(u) = 0$, yielding a contradiction.

Proof of Gromov's Theorem (i)

- Let $\{H^\lambda\}_{0\leqslant\lambda\leqslant 1}$ be a family of Hamiltonians on $\mathbb{R}^{2n}$ and $p_0 \in L$.
- For each $0 \leqslant \lambda \leqslant 1$ consider $u\colon D^2 \to \mathbb{R}^{2n}$ satisfying

$$\begin{aligned} &\partial_s u + J_0(u)\partial_t u = \nabla H^\lambda(u) \\ &u(\partial D^2) \subset L \\ &[u] = 0 \in \pi_2(\mathbb{R}^{2n}, L) \\ &p_0 \in \operatorname{Im} u \end{aligned} \tag{1}$$

Black Box. For suitable H and p_0, the moduli space

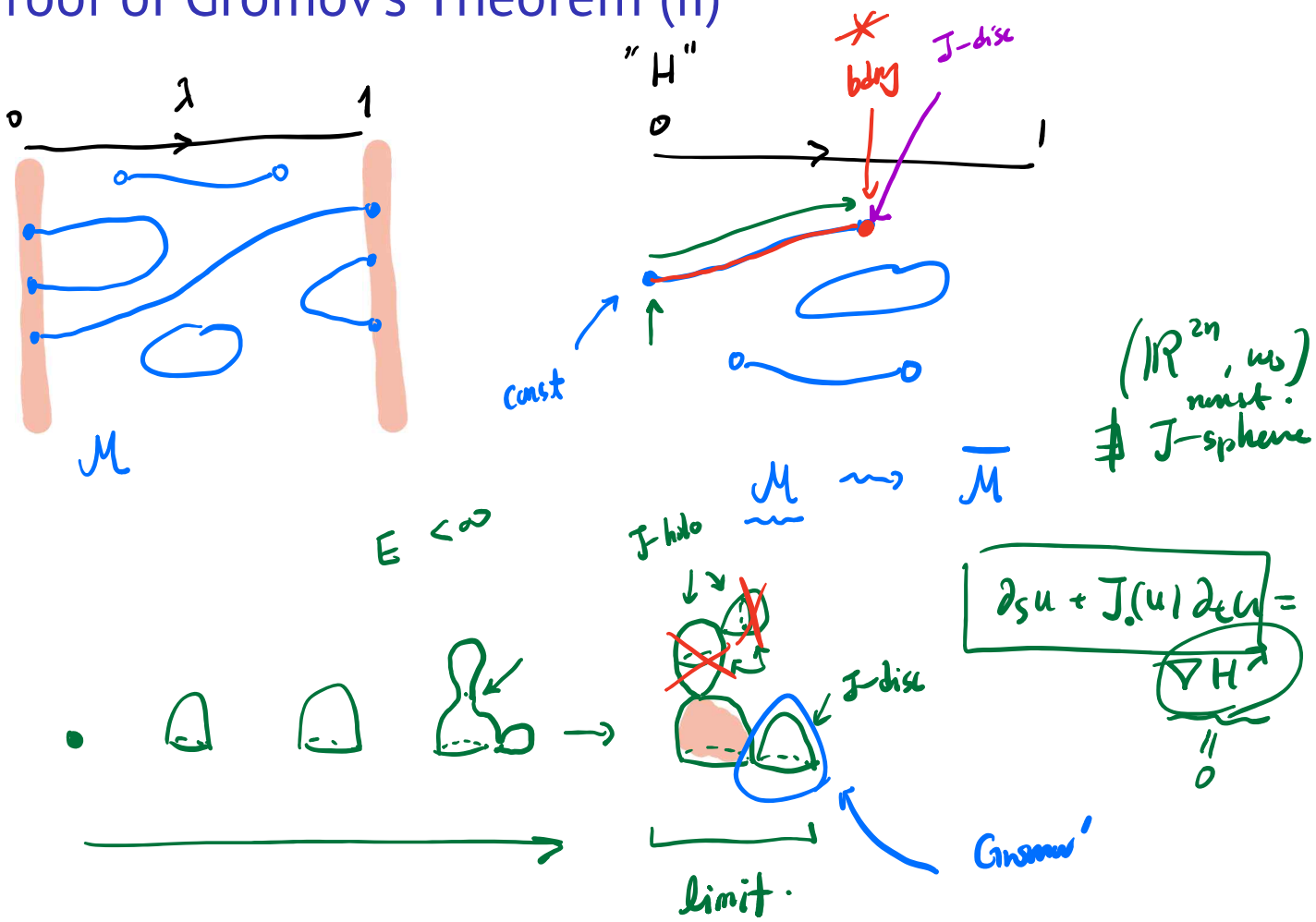
$$\begin{aligned} \mathscr{M} &:= \{(\lambda, u) \mid 0 \leqslant \lambda \leqslant 1,\ u\colon D^2 \to \mathbb{R}^{2n} \text{ satisfies (1)}\} \\ &= \bigcup_{0\leqslant\lambda\leqslant 1} \mathscr{M}_\lambda \end{aligned}$$

is a smooth manifold of dimension 1 with compact boundary

$$\partial\mathscr{M} = \mathscr{M}_0 \sqcup \mathscr{M}_1$$

and it admits a Gromov compactification $\overline{\mathscr{M}}$.

Proof of Gromov's Theorem (ii)



Conclusion

- J-holomorphic curves theory plays a key role in the geometry and topology of Lagrangian submanifolds.
- In particular, it gives an obstruction to Lagrangian embeddings.

Thank you for your attention.

Real Lagrangian tori in symplectic four-manifolds

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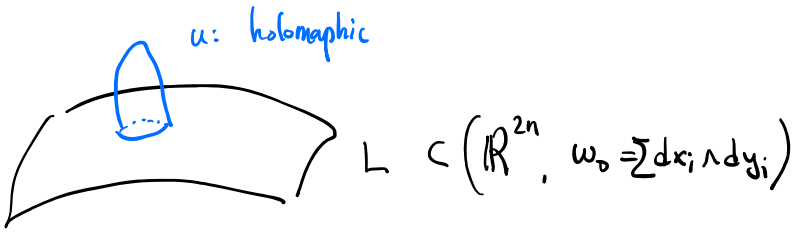
Part I.

- Basics on symplectic geometry
- Lagrangian submanifolds
- J-holomorphic curves theory

Part II.

- Lagrangian isotopy problem
- Invariant from J-holomorphic discs
- Real Lagrangian tori in symplectic del Pezzo surfaces

Review



- **(Gromov 1985)**
  Let $L \subset (\mathbb{R}^{2n}, \omega_0)$ be a closed Lagrangian submanifold. Then there exists a non-constant holomorphic disc $u\colon (D^2, \partial D^2) \to (\mathbb{R}^{2n}, L)$

- In particular, for $n \geqslant 2$ there exists no Lagrangian S^n in $(\mathbb{R}^{2n}, \omega_0)$.

Tasks.

1. Look at Lagrangian tori.
2. How to construct an invariant of L using J-holomorphic discs?

Lagrangian isotopy problem

$L_0 \overset{\text{h.e}}{\simeq} L_1$

Question. Let L_0, L_1 be Lagrangian submanifolds in (M, ω). Are they

- smoothly isotopic? $\exists\ \phi^t : M \to M$ ($0 \le t \le 1$) diffeo s.t. $\phi^0 = \mathrm{id}$, $\phi^1(L_0) = L_1$
- ⇑
- Lagrangian isotopic? $\exists\ L_t$, $0 \le t \le 1$: Lagr s.t. $L_0 \overset{L_t}{\sim} L_1$
- ⇑
- Hamiltonian isotopic? $\exists\ \phi_H^t : M \to M$: Hamiltonian diffeo s.t. $\phi_H^1(L_0) = L_1$

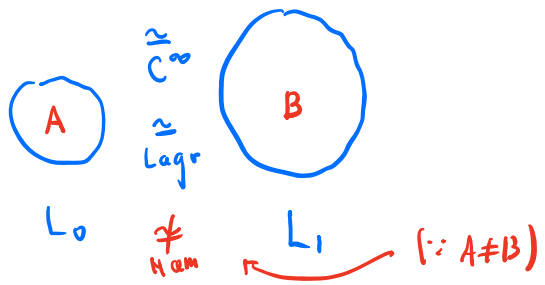
(ϕ_H^t: Hamiltonian, $H : M \to \mathbb{R}$)

$L_t := \phi_H^t(L_0)$

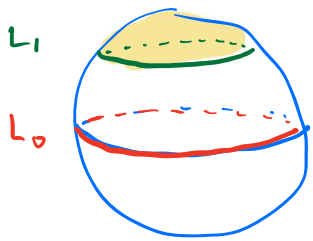
Example.

Loops in $\mathbb{R}^2$ or S^2 bounding discs with different areas

$(\mathbb{R}^2, \omega_0)$



(S^2, ω)



$\simeq C^\infty$
$\simeq$ Lagr
$\not\simeq$ Ham

Monotone and real Lagrangians

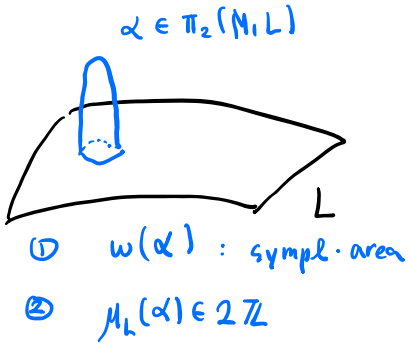
Let $L \subset (M, \omega)$ be an orientable Lagrangian in a symplectic manifold.

- L is called *monotone* if there exists $\lambda > 0$ such that

$$\mu_L(\alpha) = \lambda \cdot \omega(\alpha) \quad \forall \alpha \in \pi_2(M, L),$$

where $\mu_L : \pi_2(M, L) \to 2\mathbb{Z}$ is the *Maslov class* of L

(i.e. Lagrangian version of $c_1(M)$)



- $R : M \to M$ is called an *anti-symplectic involution* if

$$R^2 = \mathrm{id}_M \quad \text{and} \quad R^*\omega = -\omega$$

- L is called *real* if $\exists$ anti-symplectic involution $R : M \to M$ such that

$$L = \mathrm{Fix}(R) := \{x \in M \mid R(x) = x\}$$

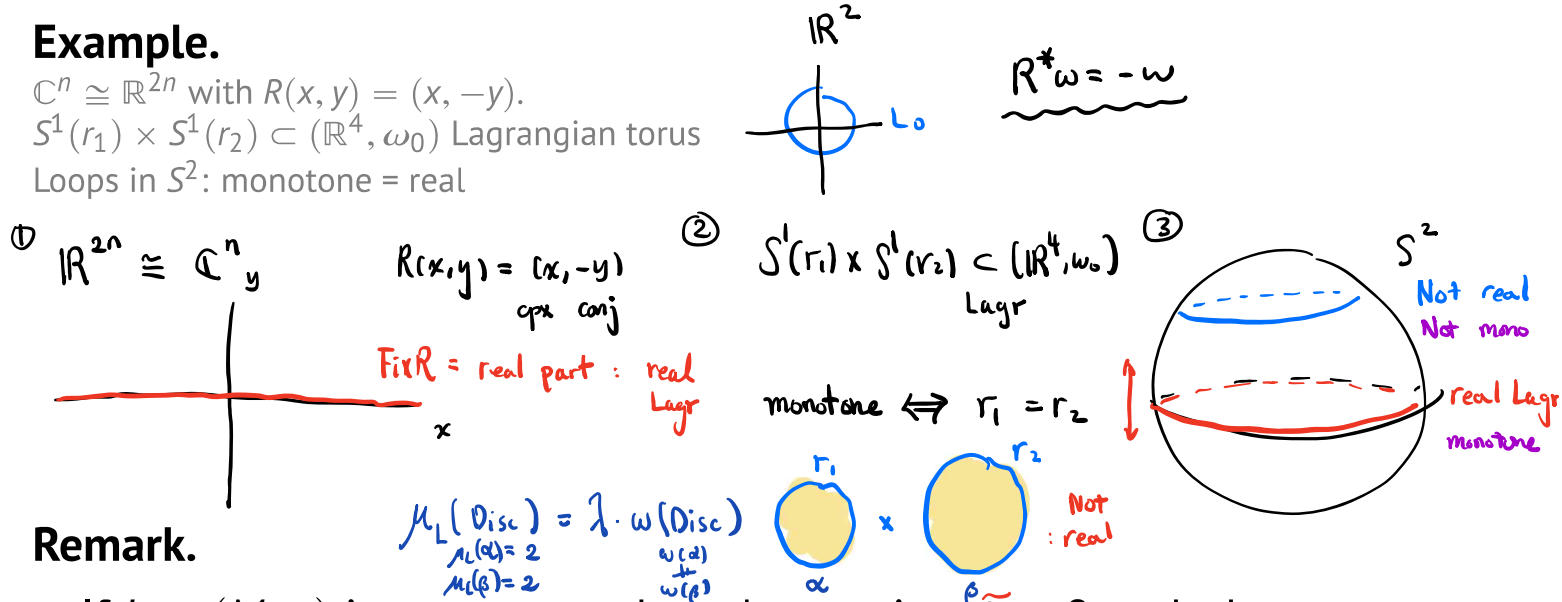
submfd

Example.

$\mathbb{C}^n \cong \mathbb{R}^{2n}$ with $R(x,y) = (x,-y)$.

$S^1(r_1) \times S^1(r_2) \subset (\mathbb{R}^4, \omega_0)$ Lagrangian torus

Loops in S^2: monotone = real



Remark.

- If $L \subset (M,\omega)$ is monotone, then there exists $\widetilde{\lambda} > 0$ such that

$$c_1(M) = \widetilde{\lambda} \cdot [\omega] \qquad \text{(Fano)}$$

  In this case, (M,ω) is called *monotone*.

- Suppose that (M,ω) is monotone. Then

  real Lagrangian $\Rightarrow$ monotone Lagrangian (?)

Lagrangian tori in dimension 4

- There is a monotone Lagrangian torus $\mathbb{T}_{\mathrm{Clif}} = S^1(r) \times S^1(r) \subset (\mathbb{R}^4, \omega_0)$.
- **(Chekanov 1996)**
  There is a monotone Lagrangian torus $\mathbb{T}_{\mathrm{Chek}} \subset \mathbb{R}^4$ such that

$$\mathbb{T}_{\mathrm{Clif}} \overset{\mathrm{Lag}}{\simeq} \mathbb{T}_{\mathrm{Chek}}, \qquad \text{while} \qquad \mathbb{T}_{\mathrm{Clif}} \overset{\mathrm{Ham}}{\not\simeq} \mathbb{T}_{\mathrm{Chek}}$$

- Let (M, ω) be a monotone closed symplectic 4-manifolds. Then
  (1) $M \overset{\mathrm{diff}}{\cong} S^2 \times S^2$ or $\mathbb{D}_k = \mathbb{C}P^2 \# k\overline{\mathbb{C}P}^2$ for $0 \leqslant k \leqslant 8$.
  (2) the uniqueness of monotone symplectic structures

  Such a (M, ω) is called a *symplectic del Pezzo surface*.

- **(Vianna 2014–2017)**
  Every symplectic del Pezzo surface has infinitely many monotone Lagrangian tori up to Hamiltonian isotopy.

Invariant from J-holomorphic discs

Let $L \subset (M, \omega)$ be a Lagrangian and J compatible almost complex str.

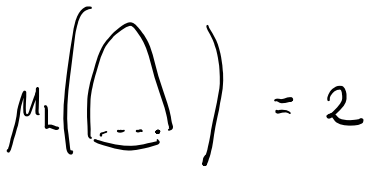
Goal. Construct an integer invariant of L

$$n(L,J) = \#\{u\colon (D^2, \partial D^2) \to (M,L) \mid u \text{ is } J\text{-holomorphic}\} \in \mathbb{Z}$$

Too big !

such that

- $n(L,J)$ is independent of the choice of J.
- if $L_0 \overset{\text{Ham}}{\simeq} L_1$, then $n(L_0) = n(L_1)$.



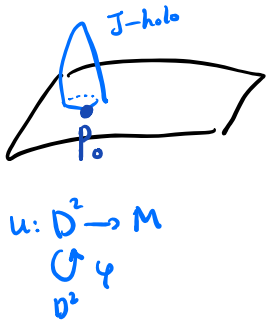
Assume:
L is monotone and orientable with minimal Maslov index $N_L = 2$

$\mu_L(\alpha) = \lambda \cdot \omega(\alpha)$

$\exists \alpha \in \pi_2(M, L)$ such that $\mu_L(\alpha) = 2$

- $\mathcal{M}(L,J) = \left\{ u:(D^2,\partial D^2)\to(M,L) \;\middle|\; \begin{array}{l} u:\text{J-holomorphic} \\ \mu_L(u)=2 \\ u(+1)=p_0 \end{array} \right\} / \mathrm{Aut}(D^2,+1)$

  Fix $p_0 \in L$.



- $\text{vir-dim}_{\mathbb{R}}\mathcal{M}(L,J) = n\chi(D^2) + \ell - \dim_{\mathbb{R}} \mathrm{Aut}(D^2,\pm 1) - \dim L$
  $= n + \ell - 2 - n = \ell - 2 = 0 \quad (\because \ell = 2)$

Theorem.
For generic J the moduli space $\mathcal{M}(L,J)$ is a compact smooth 0-manifold.
($p_0 \in L$)

Note. $\forall u \in \mathcal{M}(L,J)$ the energy $E(u) > 0$ is minimal. ($\frac{1}{2}\int_{D^2} \|du\|^2 \, dvol$)

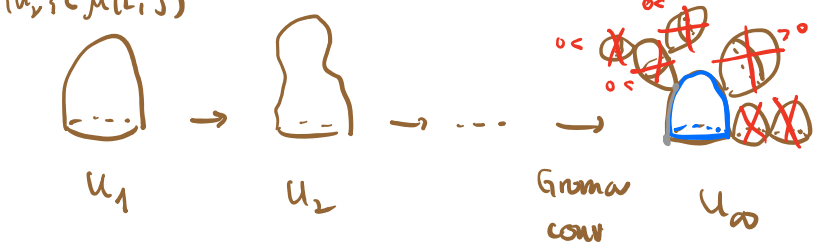
(Monotone, Maslov = 2)  "embedded"

⇝ Compactness and Transversality on $\mathcal{M}(L,J)$

① Uniform energy bound ② M: closed (cpt without ∂) ③ cptness thm

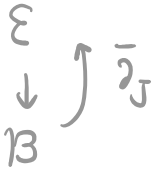
$E(u)$: minimal

$\{u_\nu\} \subset \mathcal{M}(L,J)$

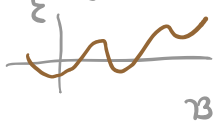


$\overline{\mathcal{M}(L,J)} = \mathcal{M}(L,J)$

§ Transversality



$\bar\partial_J \pitchfork$ zero section $= \mathcal{M}(L,J)$



For generic J we define the *count of J-holomorphic discs*

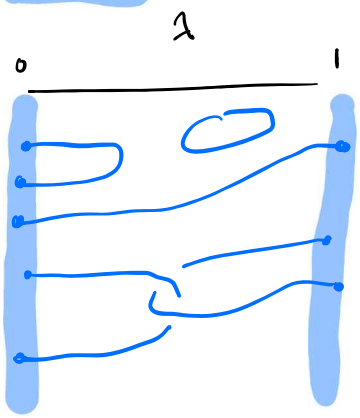
$$n(L,J) := \#\mathcal{M}(L,J) \in \mathbb{Z}_2$$

Claim. J_0 and J_1 generic $\implies$ $n(L,J_0) = n(L,J_1)$

$J_0 \xrightarrow{J_\lambda} J_1$, $0 \le \lambda \le 1$

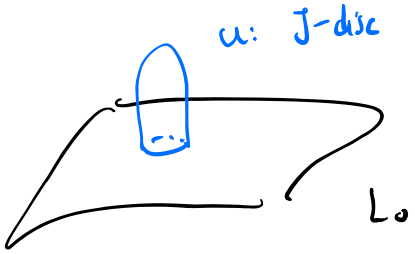
$\mathcal{M}(L,\{J_\lambda\}) = \{(\lambda,u) \mid 0 \le \lambda \le 1,\ u \in \mathcal{M}(L,J_\lambda)\}$

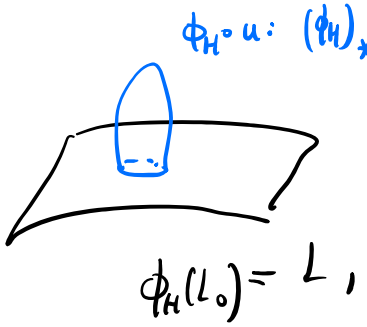
$= \bigcup_{0 \le \lambda \le 1} \mathcal{M}(L,J_\lambda)$: opt 1-dim'l smooth mfd



Claim. $L_0 \overset{\text{Ham}}{\simeq} L_1 \implies n(L_0) = n(L_1)$

$\exists\, \phi_H : M \to M$ s.t. $\phi_H(L_0) = L_1$





Applications

- **(Chekanov 1996)**
  There is a monotone Lagrangian torus $\mathbb{T}_{\mathsf{Chek}} \subset (\mathbb{R}^4, \omega_0)$ such that

$$\mathbb{T}_{\mathsf{Clif}} \overset{\mathsf{Lag}}{\simeq} \mathbb{T}_{\mathsf{Chek}}, \qquad \text{while} \qquad \mathbb{T}_{\mathsf{Clif}} \overset{\mathsf{Ham}}{\not\simeq} \mathbb{T}_{\mathsf{Chek}}$$

  ($\because n(\mathbb{T}_{\mathsf{Clif}}) = 2$ and $n(\mathbb{T}_{\mathsf{Chek}}) = 3$)

- $n(L)$ has a refined version, namely the *disc potential* $W_L \in \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$
  e.g. $W_L(1, \ldots, 1) = n(L)$

- Vianna tori have all different disc potentials, so they are not pairwise Hamiltonian isotopic.

Real Lagrangian tori in four-manifolds

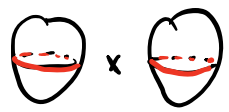
- There is no real Lagrangian torus in $\mathbb{R}^4$ at all.

  (P. Smith 1940)
  Let $\sigma : X \to X$ be a smooth involution with $F = \mathrm{Fix}(\sigma)$. Then

$$\begin{cases} \chi(X) = \chi(F) \mod 2 \\ \sum_{i=1}^{\dim X} \dim H_i(X; \mathbb{Z}_2) \geqslant \sum_{i=1}^{\dim F} \dim H_i(F; \mathbb{Z}_2) \end{cases}$$

- Among del Pezzo surfaces (M, ω), real Lagrangian tori can exist only if

$$(M, \omega) = \quad S^2 \times S^2, \qquad \mathbb{D}_3, \qquad \mathbb{D}_5, \qquad \mathbb{D}_7$$



Brendel – K. – Moon

monotone Lagr torus

Real ⇒ Mono

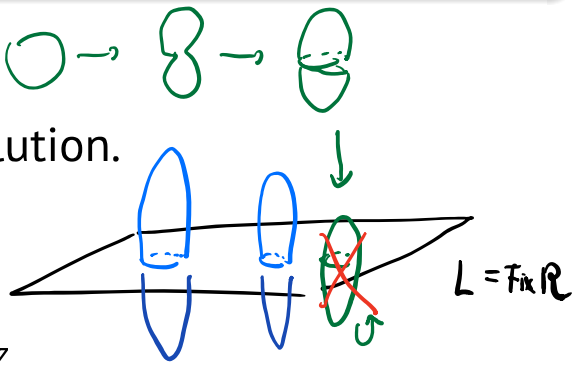
Theorem (K. 2021)

The Chekanov torus $\mathbb{T}_{\text{Chek}}$ in $(S^2 \times S^2, \omega \oplus \omega)$ is not real.

Remark.

- $\mathbb{T}_{\text{Chek}}$ is the fixed point set of a smooth involution.

  There is no smooth obstructions to $\mathbb{T}_{\text{Chek}}$ for being real.



- Indeed, we prove that

  L real Lagrangian torus $\implies$ $\begin{cases} n(L) \in 2\mathbb{Z} \\ W_L(x_1, \ldots, x_n) = W_L(x_1^{-1}, \ldots, x_n^{-1}) \end{cases}$

Theorem (K. 2020)

Every real Lagrangian torus in $S^2 \times S^2$ is Hamiltonian isotopic to $\mathbb{T}_{\text{Clif}}$.

$\mathbb{T}_{Clif}$ ∃! $\mathbb{T}_{Chek}$ $\mathbb{T}_{Vian}$ ∞

Real ⟷ Mono

∃! ∞

Thank you for your attention.



Symplectic geometry and Hamiltonian dynamics

KIAS winter school

Jungsoo Kang

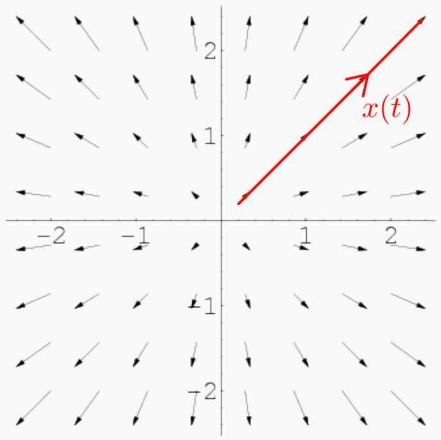
January 21, 2022

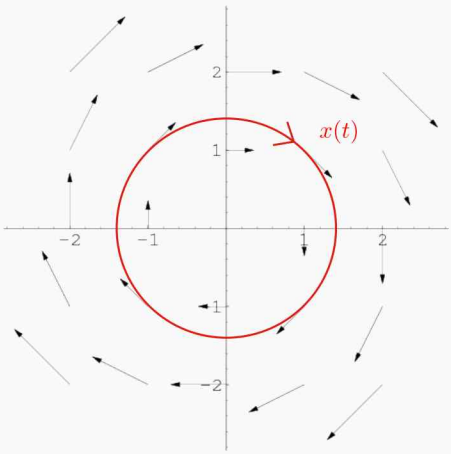
Seoul National University

Vector fields and Orbits

Vector field: $V : \mathbb{R}^n \to \mathbb{R}^n$ smooth.

Orbit: $x : (a, b) \to \mathbb{R}^n$ such that $\dot{x}(t) = V(x(t))$, $\forall t \in (a, b)$.





Hamiltonian vector field

- $H : \mathbb{R}^{2n} \to \mathbb{R}$ Hamiltonian or Energy
- $\mathbb{R}^{2n} = \{z = (q_1, \ldots, q_n, p_1, \ldots, p_n)\}$

$$\nabla H(z) = \left(\frac{\partial H}{\partial q_1}(z), \ldots, \frac{\partial H}{\partial q_n}(z), \frac{\partial H}{\partial p_1}(z), \ldots, \frac{\partial H}{\partial p_n}(z) \right)$$

Hamiltonian vector field:

$$X_H := -i\,\nabla H = \left(\frac{\partial H}{\partial p_1}(z), \ldots, \frac{\partial H}{\partial p_n}(z), -\frac{\partial H}{\partial q_1}(z), \ldots, -\frac{\partial H}{\partial q_n}(z) \right)$$

where i is the complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$.

Hamilton's equations

Let $x : (a, b) \to \mathbb{R}^{2n}$ be an **orbit** of X_H: $\dot{x}(t) = X_H(x(t))$.

If $x(t) = \big(q_1(t), \dots, q_n(t), p_1(t), \dots, p_n(t)\big)$,

$$\begin{cases} \dot{q}_i(t) = \dfrac{\partial H}{\partial p_i}(x(t)) \\ \dot{p}_i(t) = -\dfrac{\partial H}{\partial q_i}(x(t)) \end{cases} \qquad 1 \leq i \leq n \qquad \text{Hamilton's equations}$$

$x(t)$: orbit of X_H $\iff$ $x(t)$: solution of Hamilton's equations

Time-dependent Hamiltonian vector field

$H_t : \mathbb{R}^{2n} \to \mathbb{R}, \ t \in \mathbb{R}$ time-dependent Hamiltonian

Time-dependent Hamiltonian vector field:

$$X_{H_t} := -i\,\nabla H_t = \left(\frac{\partial H_t}{\partial p_1}(z), \ldots, \frac{\partial H_t}{\partial p_n}(z), -\frac{\partial H_t}{\partial q_1}(z), \ldots, -\frac{\partial H_t}{\partial q_n}(z)\right)$$

If $x(t) = \big(q_1(t), \ldots, q_n(t), p_1(t), \ldots, p_n(t)\big)$ is an orbit of X_{H_t},

$$\begin{cases} \dot{q}_i(t) = \dfrac{\partial H_t}{\partial p_i}(x(t)) \\ \dot{p}_i(t) = -\dfrac{\partial H_t}{\partial q_i}(x(t)) \end{cases} \qquad 1 \leq i \leq n \qquad \text{Hamilton's equations}$$

Symplectic structures

Symplectic form on $\mathbb{R}^{2n}$:

$$\begin{aligned} \omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} &\longrightarrow \mathbb{R} \\ (v, w) &\longmapsto v \cdot iw \end{aligned}$$

Hamiltonian vector field on $\mathbb{R}^{2n}$:

$$\omega(v, X_H) = dH(v) \quad \forall v \in \mathbb{R}^{2n}.$$

Symplectic form on smooth manifolds M^{2n}:

A closed and nondegenerate 2-form ω on M $\quad (\omega^n \neq 0)$

Hamiltonian vector field X_H of $H : M \to \mathbb{R}$:

$$\omega(\cdot, X_H) = dH \qquad \begin{aligned} TM &\xrightarrow{\cong} T^*M \\ v &\mapsto \omega(\cdot, v) \end{aligned}$$

Examples of symplectic manifolds

Example

- $(\mathbb{R}^{2n}, \omega = \sum_{i=1}^{n} dp_i \wedge dq_i)$, $(q_1, \ldots, q_n, p_1, \ldots, p_n) \in \mathbb{R}^{2n}$
- $(T^*Q, \omega = \sum_{i=1}^{n} dp_i \wedge dq_i)$, $(q_1, \ldots, q_n)$ local coord. on Q
  $(p_1, \ldots, p_n)$ fiber coord.

  - phase space of a classical Hamiltonian system
- $(\Sigma_g, \omega = \text{area})$, Kähler manifolds, ...

Darboux theorem (flexibility)

Every symplectic manifold (M, ω) is locally isomorphic to $(\mathbb{R}^{2n}, \sum dp_i \wedge dq_i)$.

$\Longrightarrow$ no local symplectic invariant

Conservation of energy

Theorem: For $H : M \to \mathbb{R}$,

$$\dot{x}(t) = X_H(x(t)) \quad \Longrightarrow \quad t \mapsto H(x(t)) \text{ is constant.}$$

Proof

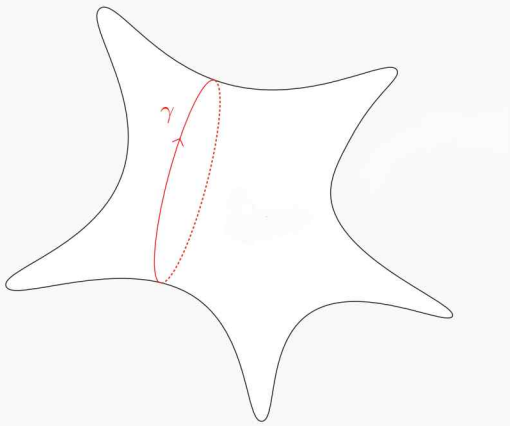
$$\begin{aligned} \frac{d}{dt} H(x(t)) &= dH(x(t))[\dot{x}(t)] \\ &= \omega\big(\dot{x}(t), X_H(x(t))\big) \\ &= \omega\big(X_H(x(t)),\, X_H(x(t))\big) = 0. \end{aligned}$$ □

It is natural to study **energy level sets** $\{H = E\}$, $E \in \mathbb{R}$.

Periodic orbits

- Central objects: periodic orbits of X_H

$$\begin{cases} x : \mathbb{R} \to \{H = E\} \\ \dot{x}(t) = X_H(x(t)) \\ x(t+T) = x(t),\ \forall t \in \mathbb{R},\ T > 0 \end{cases}$$



- Fundamental questions: existence?, how many?
- periodic orbits of X_H $\quad\longleftrightarrow\quad$ topology/geometry of (M, ω)
- $\ker \omega|_{\{H=E\}} = \langle X_H \rangle$.

Geodesics

Let (Q, g) be a Riemannian manifold.

$$H : T^*Q \to \mathbb{R}, \quad H(q,p) = \frac{1}{2}|p|_g^2$$

The flow of X_H is the cogeodesic flow of (Q, g).

- That is, the vector field G on TQ dual to X_H is the geodesic vector field w.r.t. g.
- $x(t) = (q(t), p(t))$ orbit of $X_H \Leftrightarrow q(t)$ geodesic of g
- Periodic orbits of $X_H \Leftrightarrow$ closed geodesics of g
- Conservation of energy $\Leftrightarrow$ speed is constant along a geodesic
- Orbits of X_H on $\{H = \frac{1}{2}\} \Leftrightarrow$ arc-length paramterized geodesic

Mechanical systems

(Q, g) Riemannian manifold, $V : Q \to \mathbb{R}$

$$H : T^*Q \to \mathbb{R}, \qquad H(q,p) := \frac{1}{2}|p|_g^2 + V(q)$$

$x(t) = (q(t), p(t))$ orbit of $X_H \;\Leftrightarrow\;$ $x(t)$: solution of Hamilton eq

$\Leftrightarrow$ $q(t)$ solution of Euler-Lagrange eq:

$$\ddot{q}(t) + \nabla V(q(t)) = 0$$

Conservation of energy: $\frac{1}{2}|\dot{q}(t)|_g^2 + V(q(t))$ constant along t

$\Rightarrow\;$ $x(t) \subset \{H = E\}$ for some $E \geq 0$

$\Rightarrow\;$ $q(t) \subset \{V \leq E\}$ potential well

Jacobi-Maupertuis principle

If $E > \max V$, then $\{V \leq E\} = Q$. Take

$$F : T^*Q \to \mathbb{R} \qquad F(q,p) = \frac{1}{E - V(q)} \frac{1}{2} |p|_g^2$$

Then

$$X_H \,/\!/\, X_F \quad \text{on} \quad \{H = E\} = \{F = 1\}\,.$$

Take a metric

$$\hat{g}_q(v,w) = \frac{1}{E - V(q)} g_q(v,w) \qquad v, w \in T_qQ\,.$$

Closed geodesics of $(Q, \hat{g}) \implies$ periodic orbits of X_F on $\{F = 1\}$
$\implies$ periodic orbits of X_H on $\{H = E\}$

Will consider the case $E < \max V$ later.

Contact structures

Question. Does $\{H = E\}$ always have a periodic orbit?

Answer. No, e.g. horocycle flow

Contact form on smooth manifolds N^{2n-1}:

A 1-form α on N s.t. $\alpha \wedge (d\alpha)^{n-1} \neq 0$.

Weinstein conjecture: If $\{H = E\}$ has a contact form α with $d\alpha = \omega$ on $\{H = E\}$, there is a periodic orbit.

Examples of contact manifolds:

- starshaped hypersurfaces in $\mathbb{R}^{2n}$
- fiberwise starshaped hypersurfaces in T^*Q
- level sets of $H(q,p) = \frac{1}{2}|p|_g^2 + V(q)$ on T^*Q

Note. Riemannian $\subset$ Finsler $\subset$ contact and symplectic

Variational method

For $x : S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{R}^{2n}$,

$$\Phi_H(x) = \frac{1}{2}\int_{S^1} i\dot{x} \cdot x \, dt - \int_{S^1} H(x) \, dt$$

Lemma: $\operatorname{Crit} \Phi_H = \{1\text{-periodic orbits of } X_H\}$

Proof:

$$\begin{aligned} d\Phi_H(x)[\xi] &= \frac{1}{2}\int_{S^1} i\dot{\xi} \cdot x \, dt + \frac{1}{2}\int_{S^1} i\dot{x} \cdot \xi \, dt - \int_{S^1} dH(x)[\xi] \, dt \\ &= \int_{S^1} i\dot{x} \cdot \xi \, dt - \int_{S^1} \nabla H(x) \cdot \xi \, dt \end{aligned}$$

Hence: $d\Phi_H(x)[\xi] = 0 \ \ \forall \xi \iff i\dot{x} = \nabla H(x)$

$\iff \dot{x} = X_H(x)$ $\square$

Hofer-Zehnder capacity

The Hofer-Zehnder capacity of (M, ω) is

$$c_{HZ}(M, \omega) := \sup_H \max H \ \in [0, \infty]$$

where the supremum ranges over all $H : M \to \mathbb{R}_{\geq 0}$ s.t.

- support of H is compact and $\neq M$,
- all nonconstant periodic orbits of H have period > 1.

Symplectic measurements

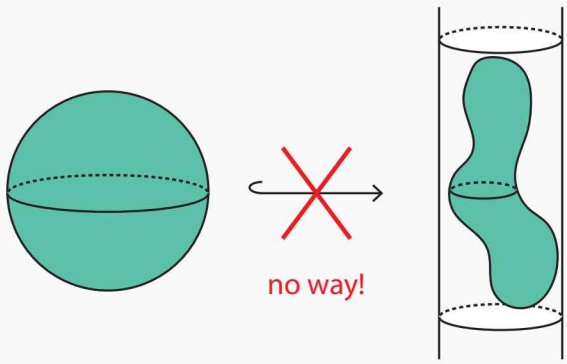
A symplectic capacity is a map $c : \{(M, \omega)\} \to [0, +\infty]$ such that

(i) $(M_1, \omega_1) \hookrightarrow (M_2, \omega_2) \implies c(M_1, \omega_1) \leq c(M_2, \omega_2)$

(ii) $c(M, s\omega) = |s| c(M, \omega) \quad \forall s \in \mathbb{R} \setminus \{0\}$

(iii) $c(B_1^{2n}, dp \wedge dq) = \pi = c(B_1^2 \times \mathbb{R}^{2n-2}, dp \wedge dq)$

Gromov non-squeezing theorem (rigidity)



$B_R^{2n} \hookrightarrow B_r^2 \times \mathbb{R}^{2n-2}$ symplectically $\iff$ $R \leq r$

$\implies$ volume preserving $\neq$ symplectic

Almost existence property (Struwe, Hofer-Zehnder)

Theorem: c_{HZ} is a symplectic capacity.

Theorem: Let $H : M \to \mathbb{R}$. Assume $c_{HZ}(M, \omega) < \infty$.

(i) For almost every $E \in \mathbb{R}$, $\{H = E\}$ has a periodic orbit (unless $\{H = E\} = \emptyset$).

(ii) If $\{H = E\}$ admits contact form, it has a periodic orbit.

Tools to show $c_{HZ}(M, \omega) < \infty$

Gromov-Witten invariants, Floer homology, etc.

Remarks and Open questions

- $c_{HZ}(\mathbb{T}^2, \omega) < \infty$ for all ω
- $c_{HZ}(\mathbb{T}^{2n}, dp \wedge dq) < \infty$?
- $\exists \omega$ s.t. $c_{HZ}(\mathbb{T}^4, \omega) = \infty$
- $c_{HZ}(T^*_{\leq 1} Q) < \infty$ for all closed (Q, g)?

Hofer geometry

ϕ_H^t Hamiltonian flow of $H : [0,1] \times M \to \mathbb{R}$, i.e. $\frac{d}{dt}\phi_H^t = X_{H_t}(\phi_H^t)$

$\mathcal{L}_{X_H}\omega = 0 \quad \Rightarrow \quad (\phi_H^t)^*\omega = \omega$

$\mathrm{Ham}(M,\omega) = \{\phi_H^t \mid H \in C_c^\infty([0,1] \times M)\}$

- ∞-dim'l Lie group
- simple group, i.e. $\nexists$ nontrivial normal subgroup (Banyaga)
- $T_{\mathrm{id}}\mathrm{Ham}(M,\omega) = C^\infty(M)/\mathbb{R}$

  Normalization (reason: $X_H = X_{H+c}$, $c \in \mathbb{R}$):

  $\mathrm{compact}\ M : \int_M H\,\omega^n = 0, \quad \mathrm{noncompact}\ M : \mathrm{supp}H\ \mathrm{compact}$

Hofer geometry

Bi-invariant Finsler metric on $\mathrm{Ham}(M,\omega)$

$$\mathrm{length}(\{\phi_H^t\}_{t\in[0,1]}) = \int_0^1 \max_{x\in M} H(t,x) - \min_{x\in M} H(t,x)\, dt$$

(equivalent to $\int_0^1 \|H(t,\cdot)\|_{L^\infty} dt$)

$$d_{Hofer}(\psi,\varphi) := \inf\{\mathrm{length}(\{\phi_H^t\}_{t\in[0,1]}) \mid \phi_H^0 = \psi,\ \phi_H^1 = \varphi\}$$

Hard part: $d_{Hofer}(\psi, \mathrm{id}) > 0\ \forall \psi \neq \mathrm{id}$.

If d bi-invariant pseudo-metric on Ham, either genuine or trivial:
$\{\psi \in \mathrm{Ham}(M,\omega) \mid d(\psi,\mathrm{id}) = 0\}$ normal subgroup
$\Longrightarrow$ it is either $\mathrm{Ham}(M,\omega)$ or $\{\mathrm{id}\}$

e.g. $\int_0^1 \|\cdot\|_{L^p} dt$, $p \neq \infty$ induces a trivial pseudo-metric

Displacement energy

For a subset U of (M, ω), its displacement energy is

$$e(U) := \inf\{d_{Hofer}(\psi, \mathrm{id}) \mid \psi \in \mathrm{Ham}(M, \omega)\,,\ \psi(U) \cap U = \emptyset\}.$$

It is also a symplectic capacity.

Energy-capacity inequality (Hofer, Zehnder, Usher ...)

$$c_{HZ}(U) \leq e(U)$$

Corollary: $d_{Hofer}(\psi, \mathrm{id}) > 0$ for $\forall \psi \neq \mathrm{id}$ (d_{Hofer} is a metric)

Proof: $\psi \neq \mathrm{id} \Rightarrow \psi(U) \cap U = \emptyset$ for some open subset $U \subset M$

$\exists \iota : B^{2n}_\epsilon \hookrightarrow U$ for small $\epsilon > 0$ (Darboux thm):

$$\pi\epsilon^2 = c_{HZ}(B^{2n}_\epsilon) \leq c_{HZ}(U) \leq e(U) \leq d_{Hofer}(\psi, \mathrm{id})$$

Note. Starshaped domains in $\mathbb{R}^{2n}$ are displaceable.

Fiberwise starshaped domains in T^*Q are not displaceable.

Revisited: Mechanical systems

(Q, g) Riemannian manifold, $H(q,p) = \frac{1}{2}|p|_g^2 + V(q)$

Lemma: For $E < \max V$, $\{H \leq E\}$ is displaceable.

Proof: Choose a function $f : Q \to \mathbb{R}$ with $\mathrm{Crit} f \subset \{V > E\}$. Take

$$F : T^*Q \to \mathbb{R} \quad F(q,p) = f(q) \ \text{ s.t. } \ X_F = -\sum_i \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}.$$

Note that X_F is nonsingular over $\{V \leq E\}$.
The flow of X_F eventually displaces $\{H \leq E\}$.

Corollary: $c_{HZ}(\{H \leq E\}) < \infty$ and $\{H = E\}$ has a periodic orbit.

Proof: By displaceability, $c_{HZ}(\{H \leq E\}) \leq e(\{H \leq E\}) < \infty$.

$\{H = E\}$ contact $\Rightarrow$ periodic orbit.

Floer homology

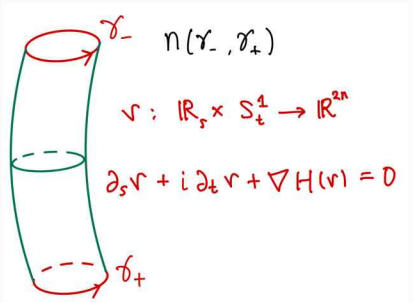
For $H : (M, \omega) \to \mathbb{R}$, suppose $D = \{H \leq E\}$ satisfies

$$\omega = d\lambda \text{ on } D \quad \text{and} \quad \lambda \wedge d\lambda^{n-1} > 0 \text{ on } \partial D.$$

E.g. starshaped domain in $\mathbb{R}^{2n}$, fiberwise starshaped domain in T^*Q, $\{H \leq E\} \subset T^*Q$ for $H = \frac{1}{2}|p|^2 + V(q)$

- Chain modules

$$CF(D) = \bigoplus_{\substack{\gamma \text{ per. orbit} \\ \text{in } \{H=E\}}} \mathbb{Z}\langle\gamma\rangle \oplus \bigoplus_{z \in \mathrm{Crit} f} \mathbb{Z}\langle z\rangle$$



$f : D \to \mathbb{R}$ small s.t $HM(f) \cong H(D, \partial D)$

- Boundary operators

$$\partial : CF(D) \to CF(D) \qquad \gamma_- \mapsto \sum_{\gamma_+} n(\gamma_-, \gamma_+)\, \gamma_+$$

- Floer homology of D: $HF(D) := H(CF(D), \partial)$

Displaceability $\Rightarrow$ $HF = 0$

$x \in \partial D = \{H = E\}$ is a leafwise intersection point of $\psi \in \mathrm{Ham}(M, \omega)$ if $\psi(x) = \phi_H^T(x)$ for some $T \in \mathbb{R}$.

Can define Floer homology $HF(D, \psi)$ such that

- "generators" are leafwise intersection points of ψ,
- $HF(D) \cong HF(D, \psi)$.

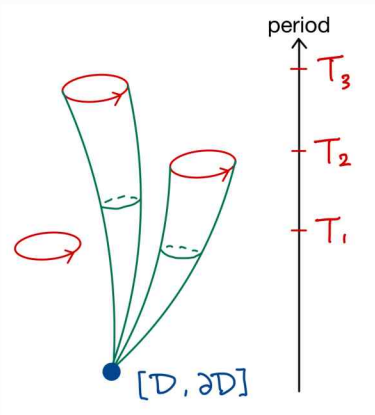
D displeaceable by ψ $\Rightarrow$ no leafwise intersection pt of ψ on ∂D
$\Rightarrow$ $HF(D) \cong HF(D, \psi) = 0$

Floer homology capacity

$c_{HF}(D) :=$ minimal period needed to kill the fundamental class $[D, \partial D]$

In this scenario, $c_{HF}(D) = T_2$.

$HF(D) = 0 \Leftrightarrow c_{HF}(D) < \infty$.



"$HF(D) = 0$ for displaceable D" can be rephrased as "$e(D) < \infty \ \Rightarrow \ c_{HF}(D) < \infty$".

Theorem: (Irie, K.) $c_{HZ}(D) \leq c_{HF}(D) \leq e(D)$

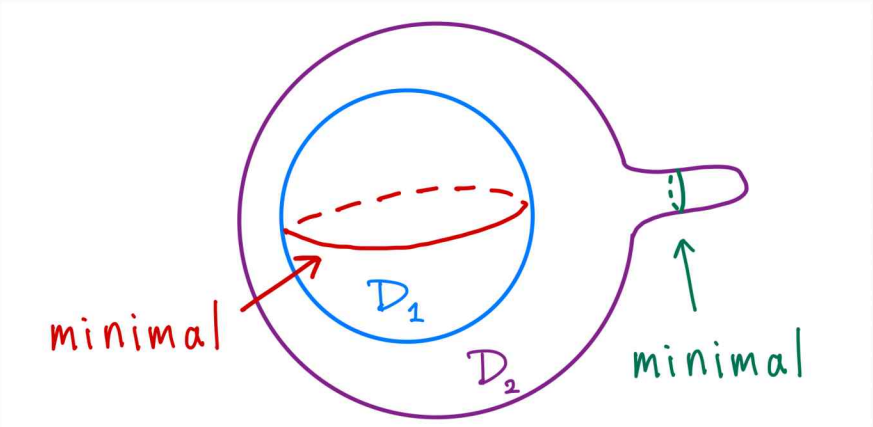
Note. Sometimes $e(D) = \infty$ while $c_{HF}(D) < \infty$.

$c_{HF}(D)$ is the period of a certain periodic orbit on D.

Minimal period is not a capacity

In general, minimal period is not a symplectic capacity.

$$T_{\min}(\partial D) := \min\{T > 0 \mid T : \text{period of a periodic orbit on } \partial D\}$$



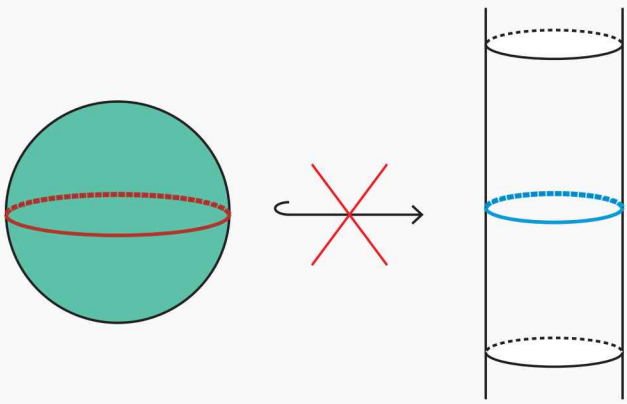
$$D_1 \subset D_2 \text{ but } T_{\min}(\partial D_1) > T_{min}(\partial D_2)$$

For convex domains in $\mathbb{R}^{2n}$, minimal period is a symplectic capacity:

$$D_1 \subset D_2 \text{ convex} \implies T_{\min}(\partial D_1) \leq T_{\min}(\partial D_2)$$

Revisited: Gromov non-sequeezing theorem

Why?



Answer:

red periodic orbit $>$ blue periodic orbit

$\Longrightarrow$ symplectic size of the ball $>$ symplectic size of the cylinder

Conjectures for convex domains

For convex domains D:

Conjecture 1. Every capacity c agrees with the minimal period:

$$c(D) = T_{\min}(\partial D)$$

$$\Downarrow$$

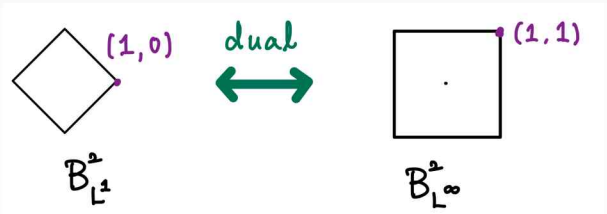
Conjecture 2. (Viterbo, 2000) A systolic inequality:

$$T_{\min}(\partial D)^n \leq n! \operatorname{volume}(D)$$

$$\Downarrow \text{ Artstein-Avidan-Ostrover-Karasev, 2014}$$

Conjecture 3. (Mahler, 1939)

$Q \subset \mathbb{R}^n$ centrally symmetric & convex, $Q^* \subset \mathbb{R}^n$ dual

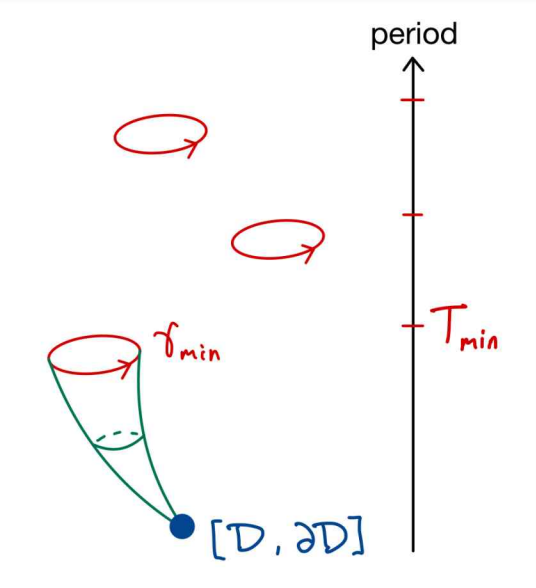


$$4^n \leq n! \underbrace{\operatorname{volume}(Q) \operatorname{volume}(Q^*)}_{=\operatorname{volume}(Q \times Q^*)}$$

Floer homology capacity = minimal period

Theorem (Abbondandolo - K., Irie)

For convex domains D in $\mathbb{R}^{2n}$, $c_{HF}(D) = T_{\min}(\partial D)$.



Applications to Billiard

$\Omega \subset \mathbb{R}^n$ bounded domain (billiard table)

Some results (Benci, Giannoni, Viterbo, Ghomi, Mazzucchelli, Albers, Irie, Artstein-Avidan, Ostrover...)

- $\exists$ closed billiard trajectory with at most $n+1$ bounce points with length $c_{FH}(T^*_{\leq 1}\Omega)$.
- Bounds on

  $\ell_{\min}(\Omega)$: length of a shortest closed Billiard trajectory on Ω

  in terms of volume, diameter, or inradius of Ω.
- For convex $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^n$, $\ell_{\min}(\Omega_1) \leq \ell_{\min}(\Omega_2)$

Systolic inequalities

For simplicity, consider Riemannian (S^2, g).

$\ell_{\min}(S^2, g) =$ minimal length of closed geodesics of (S^2, g)

Theorem (Abbondandolo, Bramham, Hryniewicz, Salomão)
If (S^2, g) is a surface of revolution or g is close to a Zoll metric,

$$\ell_{\min}(S^2, g)^2 \leq \pi \operatorname{area}(S^2, g)\,.$$

Equality holds iff g is Zoll.

In contact-symplectic language:

$\lambda = \sum p_i dq_i$ 1-form on T^*S^2 (primitive of $\omega = \sum dp_i \wedge dq_i$)

$\alpha = \lambda|_{T_1^*S^2}$ contact form on $T_1^*S^2$

$$\ell_{\min}(S^2, g) = T_{\min}(\alpha)\,, \qquad \operatorname{area}(S^2, g) = \frac{1}{2\pi}\int_{T_1^*S^2} \alpha \wedge d\alpha$$

How Viterbo conj is related to Riemannian (Finsler) systolic inequality?

Thank you!

Title: Cartan fubini txpe extension theorem

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2022.01.20.-25

IBS CCG 김호성

Contents

1. Motivation
2.minimal rational curve
3.CFE theorem
4. CF rigidity

1. Motivation

X, X' proj mfld with $\rho_X = \rho_{X'} = 1$

$U \subset X$, $U' \subset X'$ conn open subsets

Suppose $\exists\, \varphi : U \to U'$ biholo.

Q $\exists\, \psi : X \to X'$ biholo s.t. $\psi|_U = \varphi$.

ex $X = X' = \mathbb{P}^n$

$$\begin{array}{ccc} \mathbb{P}^n & & \mathbb{P}^n \\ \cup & & \cup \\ U & \xrightarrow[\text{biholo}]{\exists\, \varphi} & U' \end{array} \quad U' \text{ not extendable to } \mathbb{P}^n$$

Hence the answer to the above question is negative.

Q φ + ? $\Rightarrow$ $\exists \psi$.

Theorem (Jessen-Musso, 1994)

$X, X' \subset \mathbb{P}^{n+1}$ hypersurface of deg $d \leq n$

$U \subset X$, $U' \subset X'$ connected open subset.

$\varphi : U \longrightarrow U'$ biholo

Thms $\leftrightarrow$ Thms

$\Rightarrow \exists\, \psi : X \to X'$ bihol s.t $\psi|_U = \varphi$.

Rmk $X \subset \mathbb{P}^{n+1}$ hypersurf of deg d.

- $d \leq n \Rightarrow$ X is covered by lines
  (i.e; $\forall\, x \in X$, $\exists$ line $\ell \subset \mathbb{P}^{n+1}$ s.t $x \in \ell \subset X$)
- $d = n+1 \Rightarrow$ X is not covered by lines

2. Minimal rational curve

Minimal rational curve is a concept generalizing line 

Roughly speaking, if a morphism preserves minimal rational curves, then it is extendable.

An irred curve $C \subset X$ is rational if its normalization is of the form

$$\nu : \mathbb{P}^1 \longrightarrow C \subset X$$

Let X be a Fano mfld, $\rho_X = 1$, $d_{Tm} = n$.

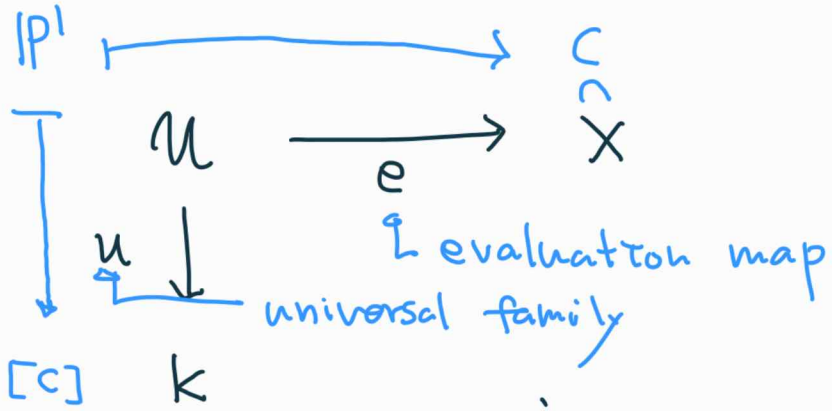
Fact X is uniruled

(ie $\forall\, x \in X$, $\exists$ rat curve $C \subset X$ with $x \in C$)

$\mathrm{RatCurves}^n(X) = \{\text{rat curves on } X\}^n$

$\cup$ ↑ countably many components

K an irreducible component.



$\forall\, x \in X$, let $K_x := \mathcal{U}(e^{-1}(x))^n$

↑ "normalized space of members of K through x"

Def K is a "minimal rational component" if for general $x \in X$,

① $K_x \neq \emptyset$

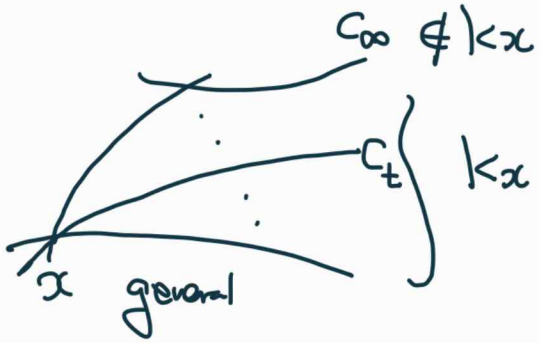
② K_x is projective.

A member of a minimal rational component K is called a "minimal rational curve".

Rmk

① $\Rightarrow$ open dense subset of X is covered by members of K.

② $\Rightarrow$ does not admits deformation of the form



($\Rightarrow$ has minimal deg among rational curves passing through general point $x \in X$)

Eg $X \subset \mathbb{P}^{n+1}$, hypersurface of deg d

$d \leq n \Rightarrow K = \{$ lines in $X\}$ minimal rational comp.

$d = n+1 \Rightarrow K = \{$ smooth conics in $X\}$ minimal rational comp.

From now on, we assume that K is a minimal rational component.

Properties

① General member $[C] \in K$ is standard.

(i.e., let $\nu : \mathbb{P}^1 \longrightarrow C \subset X$ be the normalization of C. Then

$$\nu^* T_X = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus p} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus q}$$

$\mathcal{O}_{\mathbb{P}^1}(2) \leftrightarrow T_C$, $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus p} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus q} \leftrightarrow N_{C/\mathbb{P}^1}$.

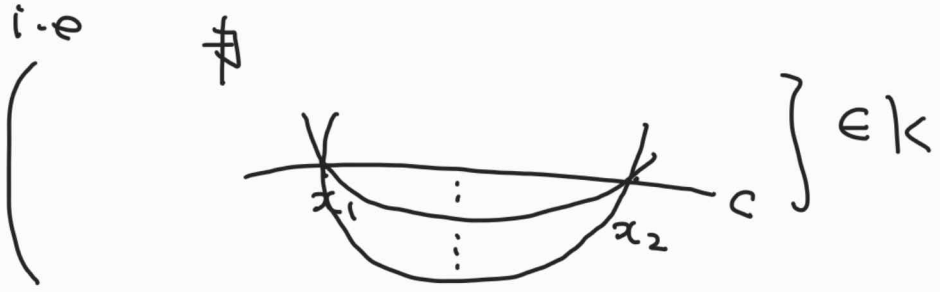
$(\ p+q = n-1 \ )$

Fact $x \in X$ general

$\Rightarrow K_x$ smooth of dimension p.

② Any standard $[C] \in K$ does not admit nontrivial deformation fixing two points $x_1 \neq x_2 \in C$

i.e.



$\} \in K$



dimension of def of C fixing $x_1 \& x_2$

$\leq h^0(C, N_{C/X} \otimes m_{x_1,x_2})$

$\leq h^0(\mathbb{P}^1, (\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus p} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus q}) \otimes \mathcal{O}_{\mathbb{P}^1}(-2))$

$= 0$

③ Let $Z \subset X$ be a closed subscheme with $\operatorname{codim}(Z, X) \geq 2$.

Then $Z \cap C = \phi$ for general $[C] \in K$.

Def For general $x \in X$, let

$$\begin{array}{ccc} \tau_x : K_x & \dashrightarrow & \mathbb{P}T_xX \\ \downarrow & & \downarrow \\ C & \longmapsto & \mathbb{P}T_xC \\ \text{smooth at } x & & \end{array}$$

$$C_x := \overline{\operatorname{Img}(\tau_x)} \subset \mathbb{P}T_xX$$

↳ VMRT of K at x

↳ variety of minimal rational tangents

$$\mathcal{C} := \overline{\bigcup_{x \in \text{general}} C_x} \subset \mathbb{P}TX$$

3. Cartan-Fubini type extension theorem

Notation

(X, k)

- X: Fano mfld, $\rho_X = 1$, $\dim = n$
- k: minimal rational component

Def (X, k) has CFE property if

$\forall (X', k')$ and

$\forall \varphi : U \longrightarrow U'$ ($U \subset X$, $U' \subset X'$ conn an. open)

s.t. $d\varphi(\mathcal{C}_x) = \mathcal{C}'_{\varphi(x)}$ for general $x \in U$,

$\exists \psi : X \dashrightarrow X'$ bihol s.t. $\psi|_U = \varphi$.

Thm (Hwang and Mok, 2001)

(X, k) has CFE property if

for general $x \in X$,

① $\mathcal{C}_x \subset \mathbb{P}T_xX$ not a union of linear spaces

② $\dim \mathcal{K}_x \geq 1$

Corollary (X, k), $\dim k_x \geq 1$

For any (X', k') and

$\varphi : U \longrightarrow U'$ biholo $(U \subset X,\ U' \subset X'$ conn. an. open$)$

min rat $\longleftrightarrow$ min. rat

$\exists\ \psi : X \longrightarrow X'$ biholo s.t $\psi|_U = \varphi$

Rmk $\exists$ generalized version of CFE-thm

for (X, k) and (X', k')

with $2 \leq \dim X \leq \dim X'$ and $1 \leq \dim k_x \leq \dim k'_{x'}$.

(Hong & Mok (2010), Huang (2014))

Thm (Huang, 2017)

$X, X' \subset \mathbb{P}^N$ covered by lines

$\forall\ \varphi : U \longrightarrow U'$ biholo

$(U \subset X,\ U' \subset X'$ conn. an. open$)$

s.t lines $\longleftrightarrow$ lines

$\exists\ \psi : X \longrightarrow X'$ biholo s.t $\psi|_U = \varphi$

(Note this includes the case $K_x = 0$ }

Where $K = \{$ lines th. X $\}$

CFE – property does not hold for general (X, k) with $\dim k_x = 0$

(eg) $X' \subset \mathbb{P}^4$ smooth cubic hypersurface

$k' =$ space of lines in X'

$(\Rightarrow \dim k'_x = 0)$

$f : X \longrightarrow X'$ double cover branched on $Y = X' \cap Q$

Q general quadric hypersurf in $\mathbb{P}^4$.

X Fano 3-fold, $\rho_X = 1$, $-K_X = H (= : f^*(\mathcal{O}_{X'}(1)))$

$k =$ space of inverse images of general lines in X'

Let $U' \subset X'$ be a conn. analytic open subset s.t. $U' \cap Y = \phi$.

Then $f^{-1}(U') = U_1 \cup U_2$

clearly $f|_{U_i} : U_i \longrightarrow U'$ biholo.

min. rat $\leftrightarrow$ min. rat

But $X \not\cong X'$.

Conj (X, k), (X', k') with $\dim k_x = \dim k'_{x'} = 0$

suppose $\exists\ \varphi : U \longrightarrow U'$ biholo s.t

$$\text{min rat} \longmapsto \text{min. rat}.$$

$(U \subset X,\ \bar{U} \subset X'$ conn. analytic open$)$

Then $\exists\ \psi : X \longrightarrow X'$ finite morphism

s.t. $\psi|_U = \varphi$.

4. Cartan-Fubini type rigidity

Def (X,k), (X',k'), $\dim X, \dim X' \geq 2$.

A morphism $f: X \to X'$ is CF-rigid if

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \cup & \nearrow_{f|_U} & \vdots \ \exists \psi \in \mathrm{Aut}(X') \\ U & \xrightarrow{\varphi} & X' \end{array} \quad \forall\, U \text{ biholomorphic}$$

min. rat $\longmapsto$ min. rat.

(es) Hwang & Mok CFE-thm.

$\Rightarrow (X,k)$, $\dim_{(X)} \geq 1$,

$\mathrm{Id}_X : X \to X$ is CF-rigid

(ie; $\begin{array}{ccc} U & & \\ & \searrow & \vdots \ \exists \psi \in \mathrm{Aut}(X) \\ U & \xrightarrow{\varphi} & X \end{array} \quad \forall\, U$ biholomorphic immersion

min. rat $\longmapsto$ min. rat)

Thm (Hwang and —, 2013)

(X, k)

Assume $\mathcal{C}_x \subset \mathbb{P}T_x X$ is not contained in any quadric hypersurface.

Let $\begin{array}{ccc} X & \xrightarrow{f} & \mathbb{P}^n \\ \cup & & \cup \\ \text{min. rat} & \longmapsto & \text{line} \end{array}$ finite surjection.

Then f is CF-rigid

(ex) $f : X \xrightarrow{2:1} \mathbb{P}^n$ double cover branched on hypersurface Y of degree $2m$, $m \le n+1$

$$k = \{ c \mid c \cdot H = 1 \}, \quad H = f^*(\mathcal{O}(1))$$

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathbb{P}^n \\ \cup & & \cup \\ C & \longmapsto & l, \text{ a line s.t } \operatorname{mult}_x l \cap Y = \text{even} \\ \uparrow & & \\ k & & \end{array}$$

$\forall\, x \in l \cap Y$.

$\mathcal{C}_x = C_{m+1, \ldots, 2m}$ (complete int of degree $m+1, \ldots, 2m$) $\subset \mathbb{P}T_x X$ smooth.

$\therefore$ If $m \ge 2$, then f is CF-rigid.

Thm (Application)

$Y_1, Y_2 \subset \mathbb{P}^n$, $n \geq 3$, hypersurf of deg $d = 2(n+1)$.

$f_i : X_i \longrightarrow \mathbb{P}^n$ double cover branched on Y_i.

$K_i = \{ 1 \subset \subset X_i \mid \subset \cdot H_i = 1, \ H_i = f_i^* \mathcal{O}_{\mathbb{P}^n}(1) \}$.

$(\Rightarrow \dim K_{i,X} = 0)$

If $\exists\, g : X_1 \longrightarrow X_2$ finite morphism, then g is an isomorphism.

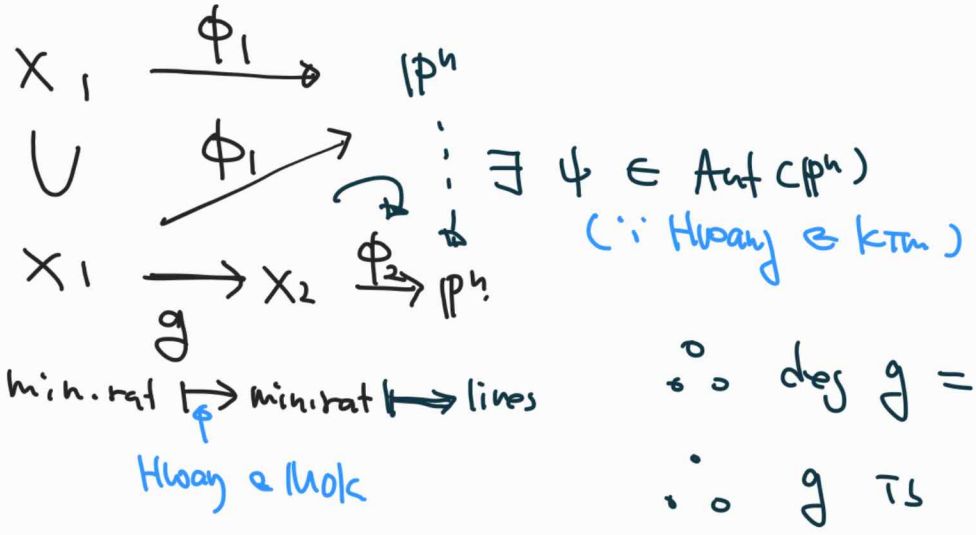
<proof>

Thm (Hwang & Mok)

X_1, X_2 Fano, $\rho_{X_i} = 1$

$f : X_1 \longrightarrow X_2$ finite morphism

$\Rightarrow$ g : min rat $\longmapsto$ min. rat



$\therefore \deg g = 1$.

$\therefore$ g is an isom.

Calibrated geometry and Vector cross product

Lecture 1

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2022, 1, 24

이화여자대학교

이 재 혁

Lecture 1

- Geometry of $\mathbb{R}^3$
- Calibrations, Calibrated submfd
- Vector cross products

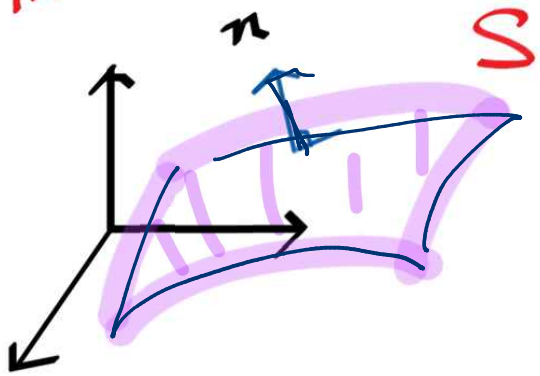
Lecture 2

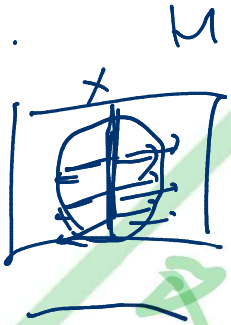
- Calibration v.s. Vector Cross product
- Holonomy and reduction
- G_2-mfd v.s Calabi-Yau 3-fold

Question

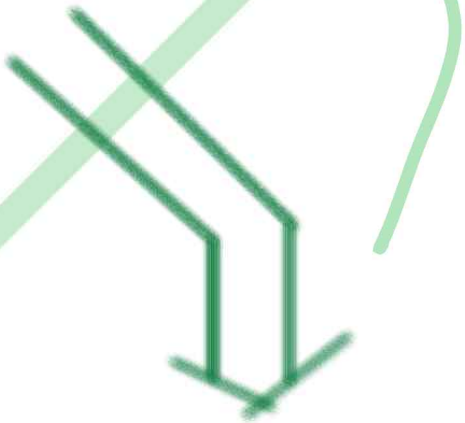
$S \subset \mathbb{R}^3$

(2-ed) surface $\begin{cases} \text{cpt w/o bdy} \\ \text{cpt w/ bdy} \\ \text{non cpt} \end{cases}$





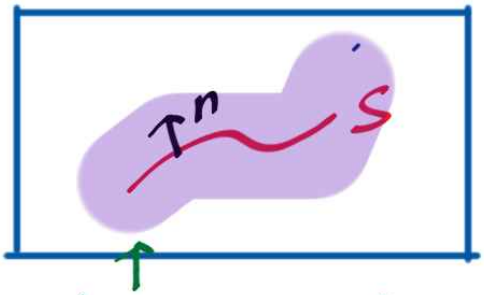
$\begin{cases} n : \text{normal vector field on } S \\ \langle \cdot, \cdot \rangle : \text{induced metric on } S \text{ from } \mathbb{R}^3 \end{cases}$



Existence of vectorfield N

$\begin{cases} \text{(i) on nbd } D \supset S \text{ with } N|_S = n \\ \text{(ii) } |N| = 1 \\ \text{(iii) } \operatorname{div} N = 0 \end{cases}$

(i) nbd $D \supset S$ issue



big enough ?

case D : 4pt 3-mfd

s.t $\partial D = S$

; Not good

divergence thm

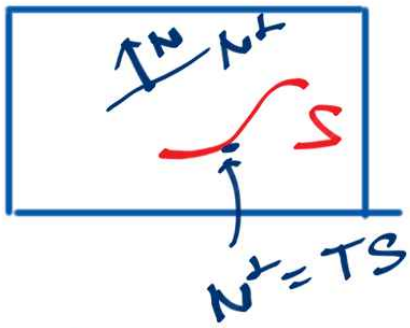
$$\int_S n \cdot d\sigma = \int_D \cancel{div\, N}^{\,0}\, dVol_{\mathbb{R}^3} = 0$$

surface integral

$$\int_S n \cdot n \, d\sigma = 1 \cdot Area(S) \neq 0$$

(II) $|N| = 1$

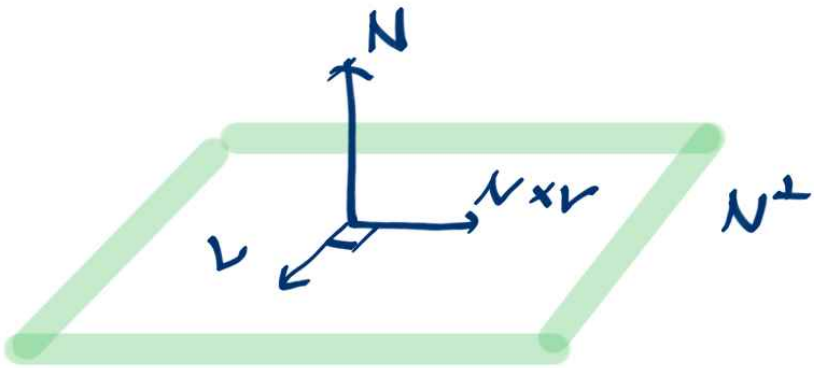
consider



$$J_N : N^\perp \to N^\perp$$

$$v \mapsto J_N(v) := N \times v$$

↑ cross product of $\mathbb{R}^3$



Claim J_N complex structure

$$J_N^2(v) = J_N(J_N(v))$$

$$= N \times (N \times v)$$

$$= \underbrace{\langle N, v \rangle}_{0} N - \underbrace{\langle N, N \rangle}_{"1"} v$$

$$= -v = -\mathrm{Id}(v)$$

Rmk

① on S

$$N^{\perp}|_S = n^2 = TS$$

$\Rightarrow (S, J_N)$ complex curve

②



(III) $\operatorname{div} N = 0$

consider

$$\omega_N :\hat{=}\ i_N \operatorname{Vol}_{\mathbb{R}^3} \quad \text{2-form}$$

observe

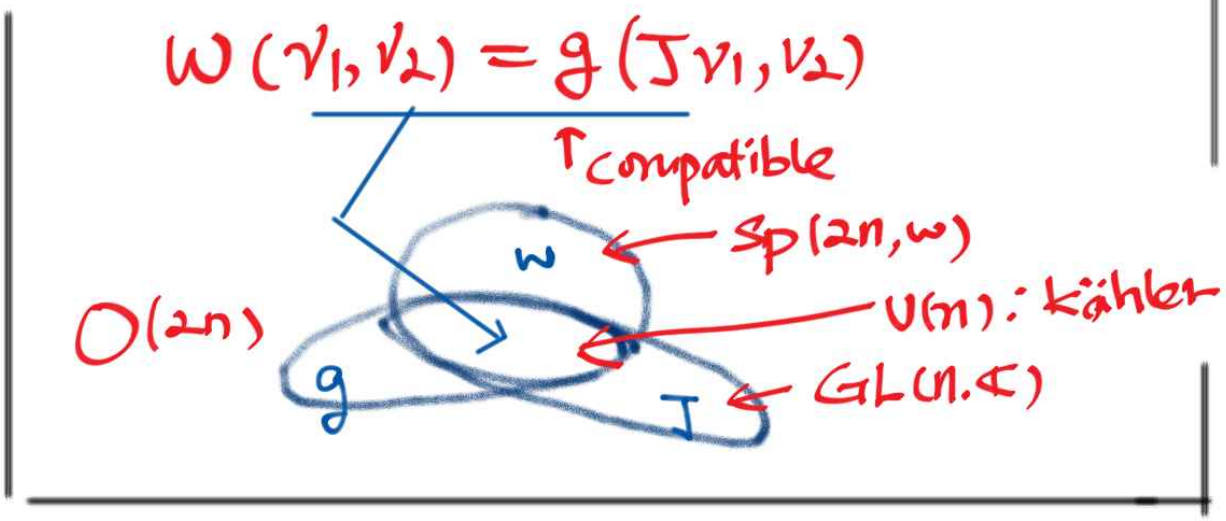
$$\begin{aligned}\omega_N(v_1, v_2) &= \operatorname{Vol}_{\mathbb{R}^3}(N, v_1, v_2)\\ &= \langle N \times v_1, v_2\rangle\\ &= \langle J_N(v_1), v_2\rangle\end{aligned}$$

$$\Rightarrow \quad \omega_N(v_1, v_2) = \langle J_N v_1, v_2\rangle$$

Note (M^{2n}, ω) symplectic mfd.

$$\omega(v_1, v_2) = g(Jv_1, v_2)$$

↑ compatible



Moreover

(i) $0 \neq d\omega_N = d i_N \mathrm{Vol}_{\mathbb{R}^3}$

$= \mathrm{div} N \, \mathrm{Vol}_{\mathbb{R}^3}$ $= 0$

$= (d i_N + i_N d)\, \mathrm{Vol}_{\mathbb{R}^3}$

$= \mathcal{L}_N \mathrm{Vol}_{\mathbb{R}^3}$ $\qquad d\mathrm{Vol}_{\mathbb{R}^3} = 0$

thus $N \rightsquigarrow \phi_N \in \mathrm{Diff}(D, \mathrm{Vol})$

(ii) for o.n. $e_1, e_2 \in N^\perp$

$|\omega_N(e_1, e_2)| = |\langle J_N(e_1), e_2 \rangle|$

$|\langle J_N(e_1), e_2 \rangle|$ $\leftarrow$ $= |\langle N \times e_1, e_2 \rangle|$

$\leq |J_N(e_1)|\,|e_2| = 1$

$= |e_1|$

$= 1$

$= \left| \det \begin{pmatrix} N \\ e_1 \\ e_2 \end{pmatrix} \right|$

Def (Calibration , Calibrated submfd)

(M, g) Riemannian mfd

(i) $\alpha \in \Omega^p(M)$ p-form is called a Calibration (p-calibration) if

① $d\alpha = 0$

② $|\alpha(e_1, \cdots, e_p)| \leq 1$

$e_1, \cdots, e_p$ O.N.

(ii) $X^p \subset M$ p-dim'l submfd is called a (α-) calibrated submfd if

$$\alpha|_X = \mathrm{Vol}_X$$

Note ① $\mathcal{G}$-ed

② α : p-calibration

$$\mathcal{C}ot(\alpha) := \left\{ \operatorname{span}\{e_1,\cdots,e_p\} \;\middle|\; |\alpha(e_1,\cdots,e_p)| = 1 \right\}$$

contact space.

e.g. ① ω_N : 2-calibration

$\mathcal{C}ot(\omega_N) = \{N^{\perp}\}$ $N^{\perp}$

② (M,ω) : symplectic ← $g(J\cdot, J\cdot) = g(\cdot,\cdot)$

$\Rightarrow \omega = g(J\cdot,\cdot)$

(i) $d\omega = 0$ ✓

(ii) ?

$$|\omega(e_1,e_2)| = |g(Je_1,e_2)|$$

e_1, e_2 o.N.

$$\leq \left(g(Je_1,Je_1)\, g(e_2,e_2) \right)^{\frac{1}{2}}$$

$$g(Je_1,Je_1) = g(e_1,e_1) = 1 \qquad g(e_2,e_2) = 1$$

$$= 1.$$

"$=$" $\Leftrightarrow$ $Je_1 = \pm e_2$

$$\mathcal{C}ot(\omega) = \{ \operatorname{span}\{e_1,e_2\} \mid \operatorname{span}\{e_1,e_2\} \subset J \}$$

Moreover

$S \subset (M,\omega)$ ω-calibrated

; $(J-)$ holomorphic curve

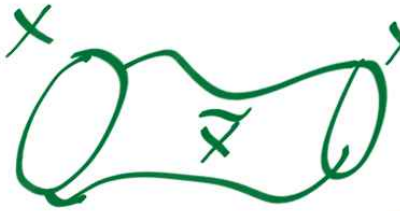
prop. Calibrated submfds are minimal

moreover volume minimizing

- cpt w/o bdy : minimal vol among homologous class
- cpt w/ bdy : minimal vol among ones with common bdy
- non cpt : local cpt nbd has volume minimizing property

pf) X α-calibrated, cpt w/o bdy

for each $X' \sim X$ homologous

(

)

α-calibrated

$$\mathrm{Vol}(X) = \int_X \mathrm{Vol}_X = \int_X \alpha$$

Stokes' thm

$$= \int_{\tilde{X}} d\alpha + \int_{X'} \alpha$$

$$\leq \int_{X'} \mathrm{Vol}_{X'} = \mathrm{Vol}(X')$$

□

back to question

$\begin{cases} \omega_N : 2\text{-calibration} \\ S \subset \mathbb{R}^3 \quad \omega_N\text{-calibration} \end{cases}$

$\Rightarrow$ S must be chosen "minimal"

Recall $S \subset \mathbb{R}^3$ cpt w/o

$\Rightarrow$ S can not be minimal

thus we remove

case S cpt w/o bdy

$X \subset \mathbb{R}^n$ cpt w/o can not be calibrated

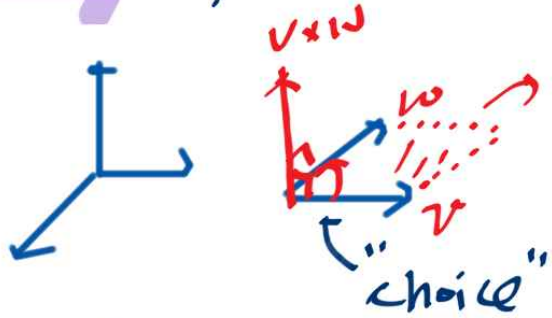
α calibration, $d\alpha = 0$

$\Rightarrow \alpha = d\eta$ Poincaré lemma $\mathbb{R}^n$

$$0 \neq Vol(X) = \int_X vol = \int_X \alpha = \int_X d\eta = \int_{\partial X} \eta = 0 .$$

$\partial X = \emptyset$

Key structure $(\mathbb{R}^3, \times)$ cross product.



Def (vector cross product, vector cross product form)

$(V, \langle\ \rangle)$ inner product space $(/\mathbb{R})$

(i) r-fold vector cross product (r-VCP)

$$\times : \underbrace{V \times \cdots \times V}_{r} \longrightarrow V$$

$$(v_1, \cdots, v_r) \longmapsto v_1 \times \cdots \times v_r$$

multilinear map

s.t. ① $v_1 \times \cdots \times v_r \perp \mathrm{span}\{v_1, \cdots, v_r\}$

② $|v_1 \times \cdots \times v_r| = |v_1 \wedge \cdots \wedge v_r|$

(ii) r-fold vector cross product form (VCP form)

$(r+1)$-form φ on V defined by

$$\varphi(v_1, \cdots, v_{r+1}) := \langle v_1 \times \cdots \times v_r, v_{r+1} \rangle$$

Note

①

$V \subset P$	$V \subset P$ form
$v_1 x \cdots x v_r \perp \mathrm{span}\{v_1, \cdots, v_r\}$	$\Leftrightarrow \varphi$: form
$\lvert v_1 x \cdots x v_r \rvert = \lvert v_1 \wedge \cdots \wedge v_r \rvert$	$\Leftrightarrow \lvert i_{v_1 \wedge \cdots \wedge v_r} \varphi \rvert = \lvert v_1 \wedge \cdots \wedge v_r \rvert$

e.g. ① $(\mathbb{R}^3, x)$ 2-fold VCP

$\Rightarrow \phi(v_1, v_2, v_3) = \langle v_1, x v_2 v_3 \rangle$

$= \mathrm{Vol}(v_1, v_2, v_3)$

↑ 2-fold VCP form

② $(\mathbb{C}^n, J)$ 1-fold VCP.

$$\begin{cases} Jv \perp v \\ \langle Jv, Jv \rangle = \langle v, v \rangle \end{cases}$$

$\Rightarrow \phi(v_1, v_2) = \langle Jv_1, v_2 \rangle$

↑ Kähler form

1-fold VCP form

classification of VCP (Gray)

V	$\mathbb{R}^n$	$\mathbb{C}^n$	$\mathrm{Im}\mathbb{O}$	$\mathbb{O}$
$\times$	$*$	J	$\times$	$\times$
VCP	Hodge star $(n-1)$-fold	complex str 1-fold	2-fold	3-fold
φ	Vol	ω	$\textcircled{H}$	Φ
VCP form		symplectic (kähler)	G_2-form	Cayley form

note $(\mathbb{R}^3, \times)$ 2-fold VCP

$3-1=2$

"$\times$" $= *$ Hodge star

$\mathrm{Im}\mathbb{H} \subset \mathrm{Im}\mathbb{O}$

2-fold

Calibrated geometry and Vector cross product

Lecture 2

17회 고등과학원 가하학겨울학교

2022, 1, 25

이화여자대학교

이 재 혁

Lecture 1

- Geometry of $\mathbb{R}^3$
- Calibrations, Calibrated submfd
- Vector cross products

Lecture 2

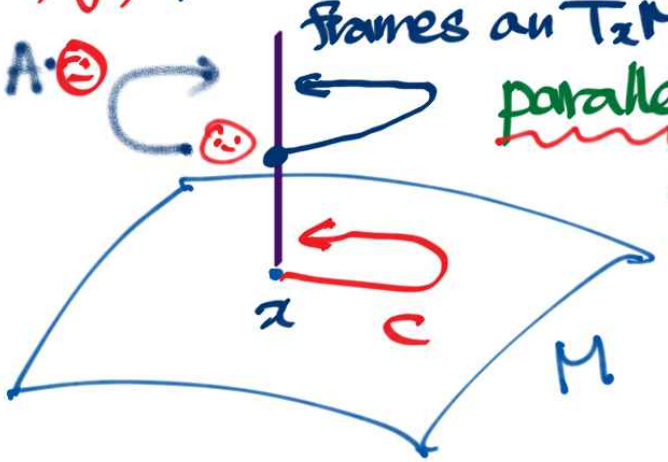
- Calibration v.s. Vector Cross product
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Riemannian holonomy group

(M,g) Riemannian mfd

frames on T_xM

for ∇ Levi-Civita



parallel lift of frames along c

frame $\rightsquigarrow$ A frame

$$Hol(M,g) := \{ A \mid \text{frame} \to A\,\text{frame} \}$$

via parallel transform

Classification (Berger)

(M,g) simply connected Irr.
not locally symmetry

$\Rightarrow$

$$Hol(M,g) \in \{ SO(n),\ U(m),\ SU(m),\ Sp(m),\ Sp(m)\cdot Sp(1),\ G_2,\ Spin(7) \}$$

Einstein

Kähler

Ricci-flat

Holonomy v.s. Geometry

$Hol(M,g) \subset$ Berger's List G

$\Leftrightarrow (M,g)$ with geometric str. determined by G

Holonomy group	$SO(n)$	$U(m)$	$SU(m)$	$Sp(m)$	G_2	$Spin(7)$
mfd ($\dim_{\mathbb{R}}$)	Q-able n	Kähler $2m$	Calabi-Yau $2m$	Hyper-kähler $4m$	G_2-mfd 7	Spin(7)-mfd 8
Structure	Volume form	kähler form	Kähler form, holo volume	Kähler forms, holo vol, holo symp.	G_2-form	Cayley form
VCP ( )-fold	$n-1$	$2m-1$, 1	$2m-1$, 1	$2m-1$ 1	6 2	7 3
$\mathbb{C}$-VCP			$m-1$	$m-1$ 1		

Complex VCP (form) (L-Leung) $=$ { holo volume form, holo sympl. form }

① octonions

consider

$$(a_1^2+a_2^2)(b_1^2+b_2^2)$$

$$\overset{?}{=} (\quad)^2_{1} + (\quad)^2_{2}$$

$a_1b_1-a_2b_2$ $\qquad$ $a_1b_2+a_2b_1$

$$\rightsquigarrow \begin{cases} z_1 = a_1 + \sqrt{-1}\,a_2 \\ z_2 = b_1 + \sqrt{-1}\,b_2 \end{cases} \in \mathbb{C}$$

$$|z_1|^2|z_2|^2 = |z_1 z_2|^2$$

; $|\ |$ and $\mathbb{C}$-alg. are compatible

$$(a_1^2+\cdots+a_n^2)(b_1^2+\cdots+b_n^2)$$

$$= (\quad)^2_{1} + \cdots + (\quad)^2_{n}$$

$\mathbb{H}$ $\mathbb{O}$

possible $n \overset{?}{=} 1, 2, \cdots ?$ → $4, 8$

Normed divison algebra

$(V, |\,|)$ normed space

$\cdot$: algebra

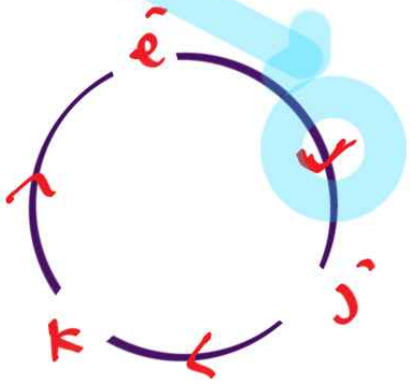
s.t $|v_1 \cdot v_2| = |v_1||v_2|$

$\mathbb{R}$, $\mathbb{C} = \langle 1, i \rangle$ $\quad i^2 = -1$

$\mathbb{H} = \langle 1, i, j, k \rangle$

$i^2 = j^2 = k^2 = -1$

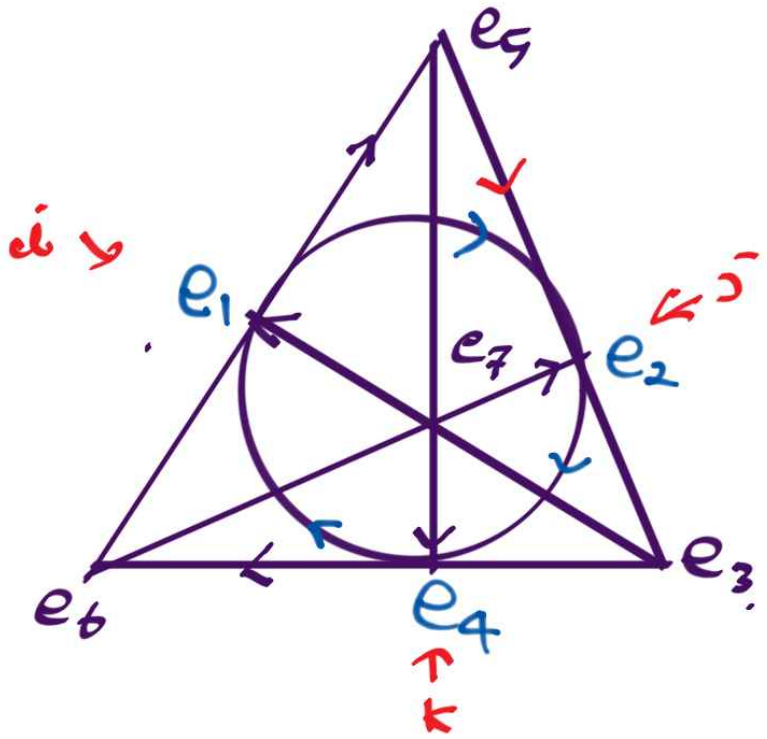
$ij = k = -ji$ ....



; non commutative

$\mathbb{O} = \langle 1, e_1, e_2, \cdots, e_7 \rangle$

$e_i^2 = +1 \quad i = 1, \cdots, 7$



- non commutative
- non associative

$$e_1(e_2\, e_2) \neq (e_1\, e_2)\, e_3$$
$$\| \qquad\qquad \|$$
$$e_1\, e_5 \qquad\qquad e_4 \cdot e_3$$
$$\| \qquad\qquad \|$$
$$e_6 \quad \neq \quad -e_6$$

note

① $\mathrm{Aut}(\mathbb{O}) \triangleq : G_2$

② $\mathrm{Im}\,\mathbb{O} \Rightarrow v_1, v_2$

$\mathcal{A}$ = alg generated $\subset \mathbb{O}$ by v_1, v_2

$\Rightarrow \mathcal{A}$ associative

③ 2-fold VCP

$\mathrm{Im}\,\mathbb{H} \subset \mathrm{Im}\,\mathbb{O}$

$\Downarrow$

v_1, v_2

$v_1 \times v_2 : \triangleq \mathrm{Im}\, v_1 \cdot v_2$

$\mathbb{H}$

$\Downarrow$

$\tilde{v}_1 \cdot \tilde{v}_2$

$\tilde{v}_1 \times \tilde{v}_2 := \mathrm{Im}\, \tilde{v}_1 \cdot \tilde{v}_2$

$\mathbb{O}$.

④ G_2-form $\quad dx^{ijk} = dx^i \wedge dx^j \wedge dx^k$

$\Phi \triangleq dx^{124} + dx^{235} + dx^{346} + dx^{457}$

$+ dx^{561} + dx^{671} + dx^{713}$

Def (G_2-mfd)

(M^7, g) Riemannian mfd is called a G_2-mfd if $\exists$ exist G_2-form Φ on M s.t $\nabla\Phi = 0$

note ① $\nabla\Phi = 0$

$\Leftrightarrow d\Phi = 0 = d^*\Phi$

$\Leftrightarrow Hol(M, g) \subseteq G_2$

② $G_2 = Aut(\Phi)$

$= \{A \in GL(7) \mid$ preserving $\Phi\}$

$= \{A \in GL(7) \mid$ preserving Φ $\times$ vcp$\}$

prop. Each closed r-fold VCP form ϕ is a calibration.

moreover

$A \in \mathrm{Cal}(\phi) \Leftrightarrow A$ is preserved by VCP

pf) $e_1, \cdots, e_{r+1}$ O.N.

$$|\phi(e_1, \cdots, e_{r+1})|$$

$$= |\langle e_1 \times \cdots \times e_r, e_{r+1} \rangle|$$

$$\leq |e_1 \times \cdots \times e_r| \, |e_{r+1}|$$

$$= |e_1| \cdots |e_r| |e_{r+1}| = 1$$

"$=$" $\Leftrightarrow e_1 \times \cdots \times e_r = \pm e_{r+1}$ VCP.

$\Leftrightarrow \mathrm{span}\{e_1, \cdots, e_{r+1}\}$ ↺ under X

□

(M^7, g, Φ) G_2-mfd

↑ Calibration

or $\simeq \mathbb{H}$

① $\cot\Phi = \{W \cdot \mathrm{Im}\Phi \mid \mathrm{span}\langle r, w\rangle \subset \Phi\}$
$\dim W = 3$ associative

$\Rightarrow \Phi$ calibrated submfd A^3

; Associative submfd

② Φ calibrated

$*d*\Phi$ co

$\Rightarrow *\Phi$ also calibrated

$\cot\Phi = \{W \mid \mathrm{Im}\Phi \mid \omega^2 \subset \Phi\}$
$\dim W = 4$ asso

$*\Phi$ calibrated submfd C^4

; Coassociative submfd

note $N_{C/M} \simeq \Lambda^+_2 C$

Holonomy reduction v.s. Calibration

$$U(n) \supset SU(n) \supset Sp\left(\tfrac{n}{2}\right)$$

(if n even)

Kähler Calabi-Yau Hyperkähler

mfd holonomy	Calibrations	Calibrated submfd
Kähler mfd $U(n)$	ω Kähler form	holo curve
	$\frac{\omega^k}{k!}$ $k=1,\cdots,n$	complex submfd
Calabi-Yau $SU(n)$	$\mathrm{Re}(e^{i\theta}\Omega)$ θ phase Ω holo volume	special Lagr L of phase θ $\Leftrightarrow \mathrm{Im}(e^{i\theta}\Omega)\vert_L = 0$ $\omega\vert_L = 0$
Hyperkähler $Sp(\frac{n}{2})$	$\mathrm{Re}(e^{i\theta}\tilde{\omega})$ θ phase $\tilde{\omega}$ holo sympl. str	complex Lagrangian of phase θ

Holonomy reduction for G_2-mfd

$$G_2 \supset SU(3) \supset Sp(1) \supset \{e\}$$

$\downarrow$ $\mathrm{Hol}(M,g) \subsetneq$ (not $\subset$)

M	$S^1 \times X$	$T^3 \times C$	T^7
G_2-mfd	$\uparrow$ Calabi-Yau	$\uparrow$ K3-surface	

$\Rightarrow$ unit tangent vector field N on M

$\Rightarrow$
- $J_N : N^\perp \to N^\perp$ complex str.
  $v \mapsto N \times v$
- $\tilde{\iota}_N \Phi$: Kähler form on $N^\perp$.

$G_2 \supset SU(3)$ ⌐ Calabi-Yau 3-fold

$M = S^1 \times X$

"θ" coordinate ↗ ↓

↓ ω_X : Kähler form

↓ Ω_X : holomorphic volume

① $\hat{H} = \mathrm{Re}\,\Omega_X - \omega_X \wedge d\theta$

G_2-form

G_2	$\supset$	$SU(3)$
M		X
Associative submfd (3-mfd)		$\mathrm{Re}\,\Omega_X$ special Lagrangian (3-mfd)
		hol curve (ω-calibrated) (2-mfd)

② $*\hat{H} = \mathrm{Im}\,\Omega_X \wedge d\theta - \frac{\omega_X^2}{2}$

G_2-form

$G_2 \supset SU(3)$

M | X

Coassociative Submfd 4-mfd

$\mathrm{Im}\,\Omega_X$ Special Lagrangian 3-mfd

complex submfd 4-mfd

Mirror Conjecture on G_2-mfd

(L-, Leung)

Calabi-Yau

$X \longleftrightarrow \check{X}$

Complex geometry	Complex geometry
Symplectic geometry	Symplectic geometry

G_2-mfd

$M \longleftrightarrow \check{M}$?

Associative geometry	mixed	Associative geometry
Coassociative geometry		Coassociative geometry

Note

"objects"

Calibrated cycles

(mfd , unitary connection) F curvature

A associative submfd	flat connection $F=0$
C coassociative submfd	(Anti) Self dual connection $F^{-}=0$
M G_2-mfd	Deformed Donaldson–Thomas connection $F\wedge(*_\Phi)+\frac{1}{6}F^3=0$

Harmonic maps over nodal domain using Deligne-Mumford moduli space

Woongbae Park

University of Pittsburgh

The 17th KIAS Winter School on Geometry
January 25, 2022

Table of Contents

1 Harmonic maps

2 Deligne-Mumford moduli space

3 Capturing bubbles using DM moduli space

Motivation - harmonic functions

$u : \Omega \subset \mathbb{R}^m \to \mathbb{R}$ is a harmonic function if it satisfies

$$\Delta u = 0 \quad \text{on } \Omega. \tag{1}$$

Harmonic functions are critical points of the Dirichlet energy

$$E_\Omega(u) = \frac{1}{2}\int_\Omega |du|^2 dx^m. \tag{2}$$

Existence : find minimizer of E_Ω with fixed boundary data.
$\Rightarrow$ obtain a weak solution.
Regularity : by standard PDE theory
$\Rightarrow$ get smoothness.

Motivation - harmonic functions

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generalize to $u : (M^m, g) \to \mathbb{R}$ from a Riemannian manifold (M, g) with Laplace-Beltrami operator Δ_g satisfying $\Delta_g u = 0$.

Motivation - harmonic maps

If target is (N^n, h), another Riemannian manifold :

A $W^{1,2}(M,N)$ map $u : (M,g) \to (N,h)$ is a (weakly) harmonic map if it is a critical point of the Dirichlet energy

$$E(u) = \int_M e(u) = \frac{1}{2}\int_M |du|^2 dvol_g, \tag{3}$$

where

$$e(u) = |du|_g^2 dvol_g = \frac{1}{2} g^{\alpha\beta} h_{ij}(u(x)) \frac{\partial u^i}{\partial x_\alpha} \frac{\partial u^j}{\partial x_\beta} dvol_g \tag{4}$$

is the energy density.

M is a surface $\Rightarrow$ harmonic map : conformally invariant.

Motivation - harmonic maps

$u \in C^2(M, N)$ & local coordinates $\Rightarrow$

$$\Delta_g u^i + g^{\alpha\beta}\Gamma^i_{jk}(u)\frac{\partial u^j}{\partial x_\alpha}\frac{\partial u^k}{\partial x_\beta} = 0 \tag{5}$$

where Γ^i_{jk}: the Christoffel symbol of the metric h on N.

Motivation - harmonic maps

$u \in C^2(M, N)$ & local coordinates $\Rightarrow$

$$\Delta_g u^i + g^{\alpha\beta}\Gamma^i_{jk}(u)\frac{\partial u^j}{\partial x_\alpha}\frac{\partial u^k}{\partial x_\beta} = 0 \tag{5}$$

where Γ^i_{jk}: the Christoffel symbol of the metric h on N.
Embed $(N, h) \hookrightarrow \mathbb{R}^L$ isometrically $\Rightarrow$

$$\tau(u)(x) = \Delta_g u(x) + A_g(u(x))(du, du) = 0 \tag{6}$$

where $\tau(u) = \mathrm{tr}_g(\nabla du) \in \Gamma(u^*TN)$: the tension field of u,
A_g : the second fundamental form of $N \hookrightarrow \mathbb{R}^L$.

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A_g : the second fundamental form of $N \hookrightarrow \mathbb{R}^L$.

$\Rightarrow$ a semilinear second-order elliptic system of PDEs.

ex) If $N = S^n \subset \mathbb{R}^{n+1}$, then harmonic map equation becomes

$$\Delta u + |du|^2 u = 0. \tag{7}$$

Examples

- Any harmonic functions $u : (M, g) \to \mathbb{R}^L$.
- Any geodesics $u : \mathbb{R} \to N$ (or $u : S^1 \to N$) parametrized linearly.
- Any holomorphic maps $u : (M, g, J^M) \to (N, h, J^N)$ between Kähler manifolds.
- Especially, if M, N are surfaces, any holomorphic or antiholomorphic maps between them.
- For M, N surfaces, weakly conformal maps (i.e. differential preserves angles at regular points).
- For M a surface, weakly conformal and minimal branched immersions (i.e. image is minimal at regular points).

Existence of harmonic maps

Existence of harmonic maps are obtained by several different methods.

- If $\partial M \neq \emptyset$: using direct method (Morrey '48)
- If the action of $W^{1,2}(M,N)$ on $\pi_1(M)$ is prescribed : using direct method (Lemaire '79 and Schoen-Yau '79)
- If N has non-positive sectional curvature : using heat flow (Eells-Sampson '64)
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- If N has non-positive sectional curvature : using heat flow (Eells-Sampson '64)
- If $\pi_2(N) = \emptyset$: using Morse theory with α-perturbation of energy (Sacks-Uhlenbeck '81)
- There is NO harmonic map from T^2 to S^2 with degree 1 (Eells-Wood '76)
- There is NO non-trivial harmonic map $u : B^m \to N$ with $u|_{\partial B^m} = \{pt\}$ (Lemaire '79, Karcher-Wood '84)

Regularity of minimizing harmonic maps

A map $u \in W^{1,2}(M,N)$ is a minimizing harmonic map if for any $\Omega \subset M$,

$$E(u,\Omega) := \frac{1}{2}\int_{\Omega} e(u) dvol_g \leq E(v,\Omega) \tag{8}$$

for any $v \in W^{1,2}(\Omega,N)$ with $v|_{\partial\Omega} = u|_{\partial\Omega}$. Clearly,

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$$\{ \text{ minimizing harmonic maps} \} \subset \{ \text{ weakly harmonic maps } \}$$

- If $\dim M = 2$, any minimizing harmonic map is smooth. (Morrey '48)
- If $\dim M > 2$, $\exists$ a non-smooth minimizing harmonic map.
  (ex : $u : B^m \to S^{m-1}$ defined by $u(x) = \frac{x}{|x|}$.)
  Moreover, $\dim_{\mathcal{H}}(sing(u)) \leq m-3$. (Schoen-Uhlenbeck '82)

Regularity of weakly harmonic maps

For weakly harmonic maps, situation is more difficult.

- If $\dim M = 2$, any weakly harmonic map is smooth. (Helein '91)
- More generally, for $\dim M = 2$, any critical point of conformally invariant Lagrangian is smooth. (Riviere '07)
- If $\dim M > 2$, $\exists$ a weakly harmonic map $u : B^3 \to S^2$ which is discontinuous everywhere. (Riviere '95)

Compactness and bubbles

Theorem 1 (Schoen-Uhlenbeck '82 or Luckhaus '88)

Let $u_k \in W^{1,2}(M,N)$ be a sequence of minimizing harmonic maps. If $u_k \to u$ weakly in $W^{1,2}(M,N)$, then $u_k \to u$ strongly in $W^{1,2}_{loc}(M,N)$ and u is a minimizing harmonic map.

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For weakly harmonic maps with $m = 2$, we have

Theorem 2 (Sacks-Uhlenbeck '81)

(Compactness Theorem) Let $u_k \in W^{1,2}(M,N)$ be a sequence of harmonic maps with uniformly bounded energy. Then there are at mostly finitely many points $x_1, \cdots, x_l \in M$ such that a subsequence (still denoted by u_k) converges to u in C^1 away from $\{x_i\}$. Moreover, u is harmonic.

Those $\{x_i\}$ are called **bubble points** or energy concentration points.

Results about bubbles

- Bubble can occur due to conformal invariance of energy.
  ex) Let $u_\lambda : S^2 \to S^2$ be a composition of stereographic projection $\sigma : S^2 \to \mathbb{R}^2$ and rescaling $x \mapsto \lambda x$ and σ^{-1}.
  $E(u_\lambda) = 4\pi$ and $u_\lambda \to p$ weakly in $W^{1,2}$ but NOT strongly.
  $\lim_{\lambda \to 0} |\nabla u_\lambda|^2 dvol_g = 8\pi\delta_p$ as convergence of Radon measures.
- In general, we have limit of energy densities as measures:

$$\lim_{k\to\infty} e(u_k) = e(u) + \sum_{i=1}^{l} m_i \delta_{x_i}. \tag{9}$$

  Here there exists some $\varepsilon_0 > 0$ such that $m_i \geq \varepsilon_0$. This ε_0 is called energy gap.

Results about bubbles

Using renormalization process, we build **bubbles**, i.e. a harmonic map $\phi_i : S^2 \to N$ at each bubble point x_j with $\varepsilon_0 \leq E(\phi_i) \leq m_i$. More refined analysis is done by Parker.

Theorem 3 (Parker, '96)

(Bubble tree convergence theorem) Let $u_k : M \to N$ be a sequence of harmonic maps with uniformly bounded energy. Then there are finitely many harmonic maps $\phi_i : S^2 \to N$, $u : M \to N$ and a subsequence of u_k (still denoted by u_k) such that "bubbled map" $\tilde{u}_k : M \cup_i S^2 \to N$ converges to $u \cup_i \phi_i$ in "C^0 sense". Moreover, the following "energy identity" holds.

$$\lim_{k\to\infty} E(u_k) = E(u) + \sum_{i=1}^{l} E(\phi_i). \tag{10}$$

More results about bubbles

Similar theorem is valid in many situations, including:

- approxmiate harmonic maps with bounded $\|\tau\|_{L^2}$ (Ding-Tian, '95)
- approximate harmonic maps with bounded $\|\tau\|_{L^p}$ with $1 < p \leq 2$ (Wang-Wei-Zhang, '17)
- α-harmonic maps under entropy assumption (Lamm, '10)
- bi-harmonic maps under bounded L^p norm of bi-tension (Wang-Zheng, '12)
- half-harmonic maps (Da Lio, '15)
- harmonic maps with free boundary on $S^n \subset B^{n+1}$ (Laurain-Petrides, '17)

Generalize to changing domain

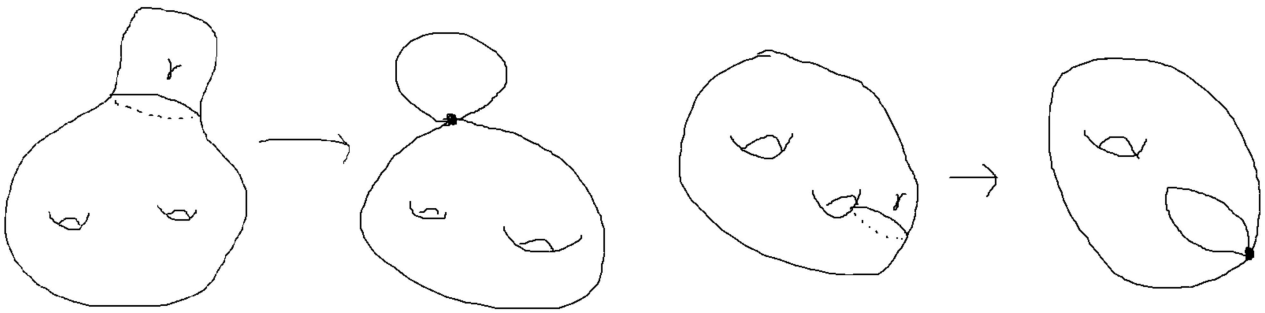
Can generalize bubble tree construction to "sequence of domains"??

Generalize to changing domain

Can generalize bubble tree construction to "sequence of domains"??
Warn!! The limit domain may be singular.

Generalize to changing domain

Can generalize bubble tree construction to "sequence of domains"??
Warn!! The limit domain may be singular.



Generalize to changing domain

Theorem 4 (Parker, '96)

(Counter example) For any positive integer p and real number $\alpha > 0$, there is a sequence of harmonic maps $u_k : T_k \to N$ with $E(u_k) \leq E_0 < \infty$ where $T_k = S^1 \times S^1$ with degenerating metric such that

$$\lim_{k\to\infty} E(u_k) = 8p\pi + \frac{\pi\alpha}{p}. \tag{11}$$

Counter example because α is arbitrary, while any nontrivial harmonic maps $\phi_i : S^2 \to N$ has $E(\phi_i) \geq \varepsilon_0$.

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3 Capturing bubbles using DM moduli space

Riemann moduli space

Conformal invariance $\Rightarrow$ see M as a complex curve $(M, j) = C$.
Let g be a genus of C.

$\mathcal{M}_g := \mathcal{J}(C)/\operatorname{Diff}(C)$, Riemann moduli space, where $\mathcal{J}(C)$: space of complex structures and $\operatorname{Diff}(C)$: diffeomorphism group of C.

Theorem 5 (Kodaira-Spencer '58)

For X a compact complex manifold, there exists a natural correspondence between

$$\{\textit{infinitesimal deformations of complex structure}\} \leftrightarrow H^1(X, TX).$$

Especially, complex dimension of $\mathcal{M}_g$ is $3g-3$ if $g \geq 2$ (such a curve is called stable).

Stable nodal curves with marked points

Definition 1

A nodal curve is a quotient $C = \tilde{C}/\pi$ where $\tilde{C}$ is a smooth complex curve with pairs of points, called nodal points or nodes, and $\pi : \tilde{C} \to C$ is identification of pairs. A genus of nodal curve C is (arithmetic) genus of smoothing of C.

Definition 2

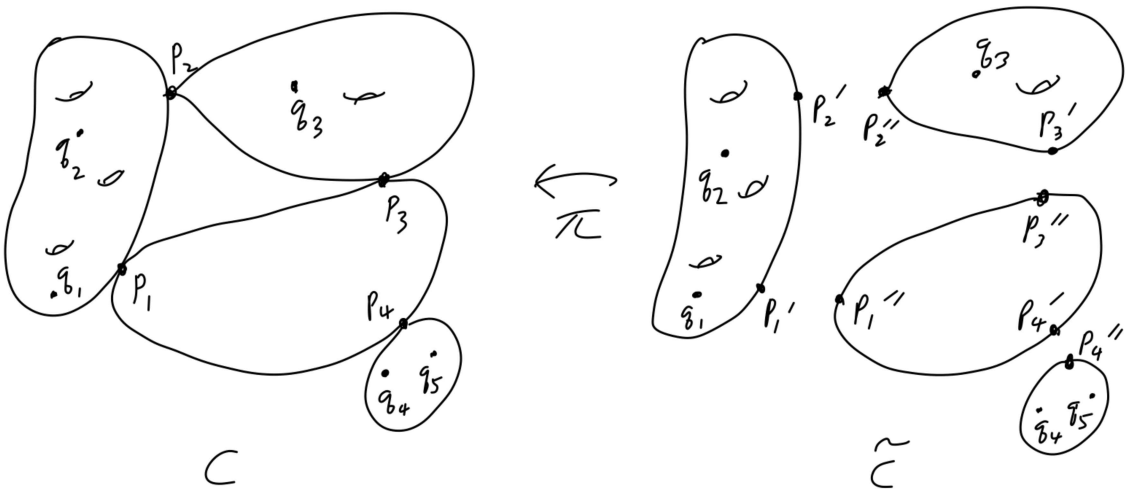
A marked nodal curve of type (g, n) (or just say (g, n) curve) is a nodal curve of genus g with n-marked non-nodal points.

Definition 3

A (g, n) curve C is said stable if $2g - 2 + n > 0$.

This is equivalent to saying that $Aut(C)$ is finite.

Stable nodal curves with marked points



Family of curves

Definition 4

A family of nodal curves is a proper surjective holomorphic map $X \xrightarrow{\pi} B$ between complex manifolds whose fibers are complex nodal curves.

Locally, $\pi : X \to B$ has two types of charts: One is regular chart given by $\pi(x, b) = b$. The other is nodal chart given by $\pi(x, y, b) = (xy, b)$.

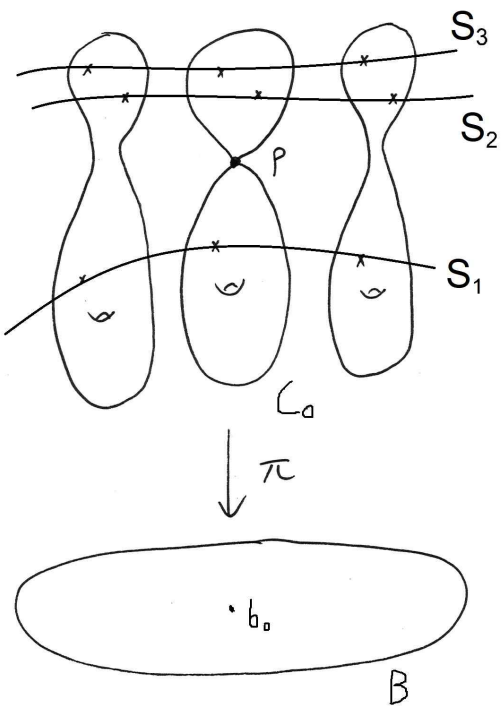
Definition 5

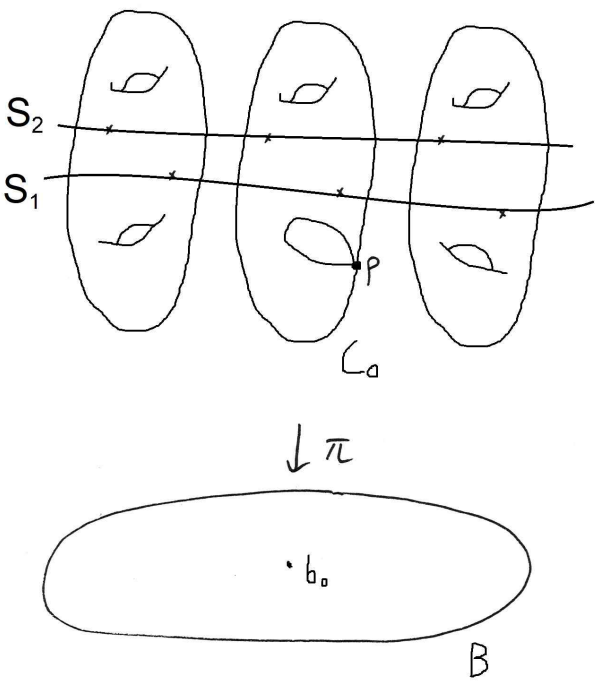
A family of (g, n) curves is a family of nodal curves together with n disjoint submanifolds S_i called marked sections.

Definition 6

A deformation of (g, n) curve C_0 is a family of (g, n) curves plus an isomorphism between C_0 and $\pi^{-1}(b_0)$ called central fiber.

Family of curves





Deligne-Mumford moduli space

Definition 7

Deligne-Mumford moduli space is a set of isomorphism classes

$$\overline{\mathcal{M}}_{g,n} = \left\{ \text{ isomorphism classes } [C] \text{ of stable } (g,n) \text{ curves } \right\}. \quad (12)$$

Deligne-Mumford moduli space

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$$\overline{\mathcal{M}}_{g,n} = \left\{ \text{ isomorphism classes } [C] \text{ of stable } (g,n) \text{ curves } \right\}. \tag{12}$$

Several properties of Deligne-Mumford moduli space:

- Let $\mathcal{M}_{g,n}$ be a moduli space of smooth ones. Then $\overline{\mathcal{M}}_{g,n}$ is its compactification.
- $\overline{\mathcal{M}}_{g,n}$ is a complex projective variety, and has orbifold structure.
- There is a projection map, called forgetful map, given by

$$\Phi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n} \tag{13}$$

  which forgets the last marked point and collapses unstable component to a point.

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Convergence of domains

Let C_k be a sequence of (g, n) curves.

Definition 8

We say $C_k \to C_0$ in a family $\pi : X \to B$ if it is a deformation of C_0 with isomorphisms $\phi_k : \pi^{-1}(b_k) \to C_k$ and $b_k \to 0$.

Since $\overline{\mathcal{M}}_{g,n}$ is compact, we can choose a subsequence $C_k \to C_0$ in a family, for some $C_0 \in \overline{\mathcal{M}}_{g,n}$.

Convergence of maps off the singular set

Definition 9

We say $u_k : C_k \to N$ *converges to* $u : C_0 \to N$ *off the singular set* $S \subset C_0$ *in* C^l *if*

1. $C_k \to C_0$ *in a family*
2. *for any nodal point* $p \notin S$, u_k *satisfies the zero neck property at* p
3. *in every regular chart* $p \notin S$, *the projected map* $\tilde{u}_k \to \tilde{u}$ *in* C^l *where* $\tilde{u}_k = \pi_k \circ u_k$, $\tilde{u} = \pi_0 \circ u_0$ *and* $\pi_k : C_k \cap B(p, r) \simeq U \times B \to U$ *are the projections* $\pi_k(x, b) = x$.

Roughly, zero neck property means, both energy of u_k and diameter of image of u_k on the set $C_k \cap B(p, r)$ converges to zero as first consider $\limsup_{k \to \infty}$ then consider $\lim_{r \to 0}$.

Results about singular set

- Singular set can be non-empty! (Parker '96)
- If images of u_k satisfies monotonicity condition (area of the neck grows in quadratic sense), then singular set is empty. (Parker-Wolfson '93)
- If $\alpha_n \frac{\pi^2}{l_n} \to 0$, then singular set is empty. (Zhu '10)
- If all nodes are regular node, then singular set is empty. (P submitted)

There are also several conjectures.

- If u_k are all in the same homotopy class, is singular set empty?
- If $\pi_1(N) = 0$, is singular set empty?

More generally, we can require only partial zero neck property (requiring zero energy limit but NOT zero distance limit). This direction is still open.

Regular nodes

Consider a sequence of (g, n) curves C_k with $C_k \to C_0$ in a family $\mathcal{C}$.

Definition 10 (Regular node)

We say a node $p \in C_0$ is regular if there exist a family of (g, l) curves $\overline{\pi} : \overline{\mathcal{C}} \to \overline{B}$ with $l < n$ and a forgetful map $\Phi : \mathcal{C} \to \overline{\mathcal{C}}$ such that $\Phi(p)$ is a regular point.

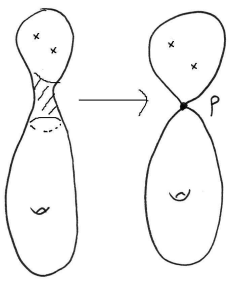
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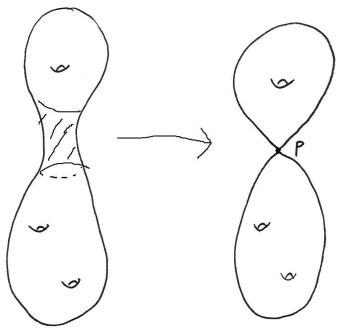
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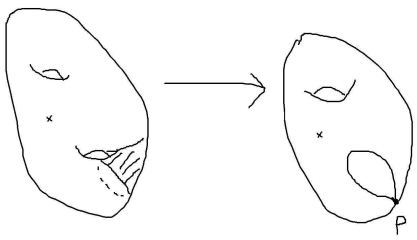
Not all nodes are regular!!



P : regular node



P : non-regular node



P : non-regular node

Main theorem and its corollary

Theorem 6

Let $u_k : C_k \to N$ be a sequence of harmonic maps with uniformly bounded energy. Then there is a way of marking points P_k on C_k such that $u_k : C_k' = (C_k, P_k) \to N$ converges to $u : C_0 \to N$ off the singular set S in C^1 where all points in S are non-regular nodal points. Moreover, u is harmonic on closure of each component $C_0 \setminus S$ separately.

Corollary 1

If C_k are $(0, n)$ curves, i.e. all homeomorphic to S^2, then we can add more marked points to get convergence without singular set.

Thank You!

The L_p Minkowski problem for the electrostatic $\mathfrak{p}$-capacity

Minhyun Kim

The 17th KIAS Winter School on Geometry
January 25, 2022



Overview

The Minkowski problem

The L_p Minkowski problem

The Minkowski problem for electrostatic $\mathfrak{p}$-capacity

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Minkowski problem

- The Minkowski problem (1903) asks for necessary and sufficient conditions on a non-negative finite Borel measure on $\mathbb{S}^{n-1}$ to be the surface area measure of a convex body in $\mathbb{R}^n$.

Notation

- *Convex body*: a compact convex set in $\mathbb{R}^n$ with nonempty interior.
- *Minkowski addition*: $A + B = \{a + b : a \in A, b \in B\}$.
- *Support function* of a closed convex set L:

$$\begin{aligned} h_L : \mathbb{S}^{n-1} &\to \mathbb{R} \\ \xi &\mapsto \max\{x \cdot \xi : x \in L\}. \end{aligned}$$

- Given a convex body K, there exists a non-negative finite Borel measure $S(K, \cdot)$ on $\mathbb{S}^{n-1}$ (*surface area measure of* K) such that

$$\left.\frac{dV(K + tL)}{dt}\right|_{t=0^+} = \int_{\mathbb{S}^{n-1}} h_L(\xi) dS(K, \xi), \quad \forall \text{compact convex set } L.$$

- Volume formula:

$$V(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K(\xi) dS(K, \xi).$$

Minkowski problem

Necessary conditions

- $S(K,\cdot)$ is not concentrated on any great subsphere of $\mathbb{S}^{n-1}$.
- K and $K+x$ induce the same surface area measures $S(K,\cdot)$. Thus,

$$0 = V(K+x) - V(K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} (h_{K+x}(\xi) - h_K(\xi))\, dS(K,\xi)$$
$$= \frac{1}{n}\int_{\mathbb{S}^{n-1}} (x\cdot\xi) dS(K,\xi) \quad \forall x \in \mathbb{R}^n.$$

This implies that the centroid is at the origin:

$$\int_{\mathbb{S}^{n-1}} \xi\, dS(K,\xi) = o.$$

Sufficient conditions

- Surprisingly, necessary conditions are also sufficient.

Discrete Minkowski problem

Theorem

Let $\xi_1, \dots, \xi_N \in \mathbb{S}^{n-1}$ *and* $f_1, \dots, f_N > 0$ *be such that*

$$\mathrm{span}(\xi_1, \dots, \xi_N) = \mathbb{R}^n \quad \text{and} \quad \sum_{i=1}^{N} f_i \xi_i = o.$$

Then $\exists$ *a polytope* P *having* $\xi_1, \dots, \xi_N$ *as its unit normal vectors and satisfying*

$$V_{n-1}(F(P, \xi_i)) = f_i,$$

where $F(P, \xi_i)$ *is the facet of* P *with the normal* ξ_i.

Idea of proof.

(i) For an N-tuple $A = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$ with $\alpha_i \geq 0$, define

$$P(A) = \bigcap_{i=1}^{N} \{x \in \mathbb{R}^n : x \cdot \xi_i \leq \alpha_i\}.$$

If $V(P(A)) > 0$, then $P(A)$ is a polytope by the assumptions.

(ii) Find a minimizer A^* of $\Phi(A) = \frac{1}{n} \sum_{i=1}^{N} \alpha_i f_i$ on $\{A : \alpha_i \geq 0, V(P(A)) \geq 1\}$.

(iii) Scale the polytope $P(A^*)$.

Approximation

Theorem

Let μ be a non-negative finite Borel measure on $\mathbb{S}^{n-1}$ which is not concentrated on any great subsphere of $\mathbb{S}^{n-1}$ and satisfies

$$\int_{\mathbb{S}^{n-1}} \xi \, d\mu(\xi) = o.$$

Then there is a convex body K for which $dS(K, \cdot) = d\mu$.

Idea of proof.

(i) Approximate μ by discrete measures $\{\mu_k\}_{k\in\mathbb{N}}$ with the same properties.

(ii) Find a sequence of polytopes $\{P_k\}_{k\in\mathbb{N}}$.

(iii) Uniform diameter control of P_k: for large k,

$$C \geq S(P_k, \mathbb{S}^{n-1})^{\frac{n}{n-1}} \gtrsim V(P_k) \geq \frac{|x|}{n} \int_{\mathbb{S}^{n-1}} \left\langle \frac{x}{|x|}, \xi \right\rangle_+ dS(P_k, \xi) \gtrsim |x|, \quad \forall x \in P_k.$$

(iv) Use the Blaschke selection theorem.

□

Uniqueness and regularity

Uniqueness (up to translations)

- Aleksandrov–Fenchel–Jessen theorem (or Brunn–Minkowski theorem):

$$S(K,\cdot) = S(L,\cdot) \implies K \text{ and } L \text{ are translations of each other.}$$

- We may assume that $o \in \operatorname{int}(K)$.

Regularity

- If $\mu = f\, d\sigma$, then $h = h_K$ satisfies the Monge–Ampére-type equation:

$$\det(h_{ij} + h\delta_{ij}) = f.$$

- If $f \in C^{\alpha}(\mathbb{S}^{n-1})$ is positive, then K is of $C^{2,\alpha}(\mathbb{S}^{n-1})$.
- See Nirenberg '53, Cheng–Yau '76, Pogorelov '78, and Caffarelli '90.

Overview

The Minkowski problem

The L_p Minkowski problem

The Minkowski problem for electrostatic $\mathfrak{p}$-capacity

The L_p Minkowski problem for electrostatic $\mathfrak{p}$-capacity

L_p Minkowski problem

- Recall the variational formula for the Minkowski addition:

$$\left.\frac{dV(K+tL)}{dt}\right|_{t=0^+} = \int_{\mathbb{S}^{n-1}} h_L(\xi)\, dS(K,\xi).$$

- p-addition and p-multiplication: for $p \geq 1$ and $K, L \ni o$,

$$h_{K+_p L} = (h_K^p + h_L^p)^{1/p} \quad \text{and} \quad h_{t\cdot_p K} = (t h_K^p)^{1/p} \quad \left(t\cdot_p K = t^{1/p} K\right).$$

- The variational formula for the p-addition:

$$\left.\frac{dV(K +_p t \cdot_p L)}{dt}\right|_{t=0^+} = \frac{1}{p}\int_{\mathbb{S}^{n-1}} h_L(\xi)^p\, dS_p(K,\xi)$$

  for some Borel measure $S_p(K,\cdot)$ on $\mathbb{S}^{n-1}$. $S_p(K,\cdot)$ is called the *L_p surface area measure*

- $dS_p(K,\cdot) = h_K^{1-p} dS(K,\cdot)$. This allows p to be any real number.

L_p Minkowski problem

L_p Minkowski problem

- Find necessary and sufficient conditions on a Borel measure μ on $\mathbb{S}^{n-1}$ s.t.

$$h_K^{1-p} dS(K, \cdot) = d\mu$$

for some convex body K with $o \in K$.

- Since L_p Minkowski problem is not translation-invariant (when $p \neq 1$), one must find a convex body which contains the origin.

Even L_p Minkowski problem

- Find necessary and sufficient conditions on an even Borel measure μ on $\mathbb{S}^{n-1}$ s.t.

$$h_K^{1-p} dS(K, \cdot) = d\mu$$

for some origin-symmetric convex body K.

Even L_p Minkowski problem

Review

- $p = 1$ (classical Minkowski problem): Necessary and sufficient conditions are NCG (Not Concentrated on Great subsphere of $\mathbb{S}^{n-1}$) & centroid $= o$. The convex body is unique up to translations.

Even L_p Minkowski problem

- $p > 1, p \neq n$: NCG is a necessary and sufficient condition. The convex body is unique.
- $0 < p < 1$: NCG is a necessary and sufficient condition. Uniqueness is known only when $n = 2$.
- $p < 0$: ??

Volume-normalized even L_n Minkowski problem

- Volume-normalized problem: $h_K^{1-n} dS(K, \cdot) = V(K) d\mu$.
- NCG is a necessary and sufficient condition. The convex body is unique.

Even logarithmic Minkowski problem

Even logarithmic Minkowski problem ($p = 0$)

- $S_0(K, \cdot) = h_K S(K, \cdot)$ is called the *cone-volume measure* since

$$V(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K(\xi) dS(K, \xi).$$

- A necessary and sufficient condition is the subspace concentration condition: $\forall$ subspace L of $\mathbb{R}^n$ with $0 < \dim(L) < n$,

$$\mu(L \cap \mathbb{S}^{n-1}) \leq \frac{1}{n} \mu(\mathbb{S}^{n-1}) \dim(L).$$

  Moreover, if the equality holds, then $\exists$ a complementary subspace L' s.t.

$$\mu(L' \cap \mathbb{S}^{n-1}) = \frac{1}{n} \mu(\mathbb{S}^{n-1}) \dim(L').$$

- Uniqueness is known only when $n = 2$.

L_p Minkowski problem

Position of the origin

- $\mathcal{K}_{(o)}$: collection of all convex bodies K with $o \in \text{int}(K)$.
- $\mathcal{K}_o$: collection of all convex bodies K with $o \in K$.

The case $p > n$

- Existence and uniqueness of $K \in \mathcal{K}_{(o)}$ under NCH (Not Concentrated on any Hemisphere of $\mathbb{S}^{n-1}$).

The case $p = n$ (normalized Minkowski problem)

- Existence and uniqueness of $K \in \mathcal{K}_{(o)}$ under NCH.

The case $1 < p < n$

- Existence and uniqueness of $K \in \mathcal{K}_o$ under NCH.

L_p Minkowski problem

The case $0 < p < 1$

- Existence of $K \in \mathcal{K}_o$ under NCG.
- Existence of $K \in \mathcal{K}_o$ under the following condition: either

$$L := \operatorname{span} \operatorname{supp} \mu = \mathbb{R}^n$$

or

$$\dim(L) \leq n-1 \quad \text{and} \quad \operatorname{pos} \operatorname{supp} \mu \neq L.$$

- A necessary and sufficient condition in $n = 2$: $\operatorname{supp}(\mu) \neq \{\xi, -\xi\}$.

The case $p = 0$ (Logarithmic Minkowski problem)

- Existence of $K \in \mathcal{K}_o$ under the subspace concentration condition.
- Non-uniqueness.

The case $p < 0$

- There are few results for discrete measures and smooth measures.
- Non-uniqueness.

Overview

The Minkowski problem

The L_p Minkowski problem

The Minkowski problem for electrostatic $\mathfrak{p}$-capacity

The L_p Minkowski problem for electrostatic $\mathfrak{p}$-capacity

Minkowski problem for electrostatic $\mathfrak{p}$-capacity

Electrostatic capacity

- A set's ability to hold electrical charge.
- Electrostatic capacity:

$$\mathrm{Cap}(K) = \int_{\mathbb{R}^n \setminus K} |\nabla U|^2 dx,$$

where U is the equilibrium potential of K:

$$\Delta U = 0 \text{ in } \mathbb{R}^n \setminus K, \quad U = 1 \text{ on } \partial K, \quad \text{and} \quad \lim_{|x| \to \infty} U(x) = 0.$$

Electrostatic $\mathfrak{p}$-capacity

- For $1 < \mathfrak{p} < n$,

$$\mathrm{Cap}_{\mathfrak{p}}(K) = \int_{\mathbb{R}^n \setminus K} |\nabla U|^{\mathfrak{p}} dx.$$

Minkowski problem for electrostatic $\mathfrak{p}$-capacity

Review: formulas for volume

- Variational formula for volume:

$$\left.\frac{dV(K+tL)}{dt}\right|_{t=0^+} = \int_{\mathbb{S}^{n-1}} h_L(\xi) dS(K,\xi).$$

- Volume formula:

$$V(K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_K(\xi) dS(K,\xi).$$

Formulas for $\mathfrak{p}$-capacity

- Hadamard variational formula for $\mathfrak{p}$-capacity:

$$\left.\frac{d\mathrm{Cap}_{\mathfrak{p}}(K+tL)}{dt}\right|_{t=0^+} = (\mathfrak{p}-1)\int_{\mathbb{S}^{n-1}} h_L(\xi) d\mu_{\mathfrak{p}}(K,\xi),$$

  where $\mu_{\mathfrak{p}}(K,\cdot) = |\nabla(U\circ g^{-1})|^{\mathfrak{p}} S(K,\cdot)$ is called the *electrostatic $\mathfrak{p}$-capacitary measure*.

- Poincaré $\mathfrak{p}$-capacity formula:

$$\mathrm{Cap}_{\mathfrak{p}}(K) = \frac{\mathfrak{p}-1}{n-\mathfrak{p}}\int_{\mathbb{S}^{n-1}} h_K(\xi) d\mu_{\mathfrak{p}}(K,\xi).$$

Minkowski problem for electrostatic $\mathfrak{p}$-capacity

- Find necessary and sufficient conditions on a Borel measure μ on $\mathbb{S}^{n-1}$ s.t. $d\mu_{\mathfrak{p}}(K,\cdot) = d\mu$ for some convex body K.

Minkowski problem for electrostatic capacity

Theorem (Jerison '96)

Let $\mathfrak{p} = 2$.

- *(Existence) Necessary and sufficient conditions are NCG & centroid* $= o$.
- *(Uniqueness) The convex body is unique up to translations.*
- *(Regularity) If* $\mu = f\, d\sigma$ *for some* $f \in C^{\alpha}$, $f > 0$, *then* $K \in C^{2,\alpha}$.

Minkowski problem for electrostatic $\mathfrak{p}$-capacity

- Existence and uniqueness have been extended to the case $1 < \mathfrak{p} < n$ by Colesanti–Nyström–Salani–Xiao–Yang–Zhang '15 and Akman–Gong–Hineman–Lewis–Andrew '17.
- $\mathfrak{p}$-capcity-normalized Minkowski problem when $1 + \mathfrak{p} = n$.

Overview

The Minkowski problem

The L_p Minkowski problem

The Minkowski problem for electrostatic $\mathfrak{p}$-capacity

The L_p Minkowski problem for electrostatic $\mathfrak{p}$-capacity

L_p Minkowski problem for electrostatic $\mathfrak{p}$-capacity

Variational formulas

- Variational formulas for volume:

$$\left.\frac{dV(K+tL)}{dt}\right|_{t=0^+} = \int_{\mathbb{S}^{n-1}} h_L(\xi)\, dS(K,\xi),$$
$$\left.\frac{dV(K+_p t\cdot_p L)}{dt}\right|_{t=0^+} = \frac{1}{p}\int_{\mathbb{S}^{n-1}} h_L(\xi)^p\, dS_p(K,\xi).$$

- Variational formulas for $\mathrm{Cap}_{\mathfrak{p}}$:

$$\left.\frac{d\mathrm{Cap}_{\mathfrak{p}}(K+tL)}{dt}\right|_{t=0^+} = (\mathfrak{p}-1)\int_{\mathbb{S}^{n-1}} h_L(\xi)\, d\mu_{\mathfrak{p}}(K,\xi),$$
$$\left.\frac{d\mathrm{Cap}_{\mathfrak{p}}(K+_p t\cdot_p L)}{dt}\right|_{t=0^+} = \frac{\mathfrak{p}-1}{p}\int_{\mathbb{S}^{n-1}} h_L(\xi)^p\, d\mu_{p,\mathfrak{p}}(K,\xi),$$

where the L_p $\mathfrak{p}$-*capacitary measure* $\mu_{p,\mathfrak{p}}(K,\cdot)$ is given by

$$\begin{aligned}\mu_{p,\mathfrak{p}}(K,\cdot) &= h_K^{1-p}\mu_{\mathfrak{p}}(K,\cdot)\\ &= h_K^{1-p}|\nabla(U\circ g^{-1})(\xi)|^{\mathfrak{p}} S(K,\cdot).\end{aligned}$$

L_p Minkowski problem for electrostatic $\mathfrak{p}$-capacity

- Find necessary and sufficient conditions on a Borel measure μ on $\mathbb{S}^{n-1}$ s.t. $d\mu_{p,\mathfrak{p}}(K,\cdot) = d\mu$ for some convex body K.

The case $p \geq n$

- Existence and uniqueness of $K \in \mathcal{K}_{(o)}$ under NCH.

The case $1 < p < n$

- Existence and uniqueness of $K \in \mathcal{K}_o$ under NCH (for the capacity-normalized problem when $p + \mathfrak{p} = n$).

The case $0 < p < 1$ and $1 < \mathfrak{p} < 2$

- Existence for the (normalized, when $p + \mathfrak{p} = n$) discrete problem under NCH and

$$\mu(\{-\xi\}) = 0 \quad \text{whenever} \quad \mu(\{\xi\}) > 0.$$

The case $p = 0$

- Existence for the discrete problem under NCH and the following condition:

any n elements of $\operatorname{supp}\mu$ are linearly independent.

Thank you for your attention!

Transverse foliations in Hamiltonian systems

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Korea Institute for Advanced Study

January 25, 2022

Recall

- $(\mathbb{R}^4, z = (x_1, x_2, y_1, y_2))$
- $\omega_0 = dy_1 \wedge dx_1 + dy_2 \wedge dx_2$ standard symplectic form
- $H\colon \mathbb{R}^4 \to \mathbb{R}$ time-independent Hamiltonian (total energy)
- $X_H(z) = (H_{y_1}(z), H_{y_2}(z), -H_{x_1}(z), -H_{x_2}(z))$ Hamiltonian vector field
- $\phi_H^t\colon \mathbb{R}^4 \to \mathbb{R}^4$ the flow of X_H Hamiltonian flow
- $\dot{z}(t) = X_H(z(t))$ if and only if $z(t) = (x_1(t), x_2(t), y_1(t), y_2(t))$ satisfies Hamilton's equations:

$$\begin{cases} \dot{x}_j(t) = H_{y_j}(z(t)) \\ \dot{y}_j(t) = -H_{x_j}(z(t)) \end{cases} \quad j = 1, 2.$$

Preservation of energy

Lemma

For every $z \in \mathbb{R}^4$ it holds that $H(\phi_H^t(z))$ is constant, namely, it is independent of $t \in \mathbb{R}$.

Hence, it suffices to study the dynamics of X_H restricted to the energy level $H^{-1}(c)$, where $c \in \mathbb{R}$ is a prescribed energy.

In other words, preservation of energy reduces the 4-dimensional study to the 3-dimensional study.

We would like to reduce one more dimension by looking at a certain geometric structure of $H^{-1}(c)$, called a transverse foliation.

Some history



Figure: taken from Wikipedia

Poincaré gave a strong push toward a qualitative point of view for studying dynamical systems.

Some history



Volume 7, issue 1, December 1885

16 articles in this issue

Mittheilung, einen von König Oscar II gestifteten mathematischen Preis betreffend
OriginalPaper | Published: 01 December 1885 | Pages: I - VI

Sur un théorème de M. Fuchs
H. Poincaré
OriginalPaper | Published: 01 December 1885 | Pages: 1 - 32

Sur un théorème concernant les fonctions elliptiques
Edvard Phragmen
OriginalPaper | Published: 01 December 1885 | Pages: 33 - 42

Über die Begrenzungen von Continua
Edvard Phragmén
OriginalPaper | Published: 01 December 1885 | Pages: 43 - 48

Über Systeme von Plancurven
H. Krey
OriginalPaper | Published: 01 December 1885 | Pages: 49 - 94

Déduction arithmétique d'une relation due à Jacobi
R. Lipschitz
OriginalPaper | Published: 01 December 1885 | Pages: 95 - 100

Über verschiedene Theoreme aus der Theorie der Punctmengen in einem n-fach ausgedehnten stetigen Raume G_n
Georg Cantor
OriginalPaper | Published: 01 December 1885 | Pages: 105 - 124

Die intermediäre Bahn des Mondes
Hugo Gyldén
OriginalPaper | Published: 01 December 1885 | Pages: 125 - 172

Über die auflösbaren Gleichungen von der Form $x^5+ux+v=0$
C. Runge
OriginalPaper | Published: 01 December 1885 | Pages: 173 - 186

Über $\int_0^\infty \frac{\sin ax}{\sin bx}\frac{dx}{1+x^2}$ und Verwandte Integrale
L. Schläfli
OriginalPaper | Published: 01 December 1885 | Pages: 187 - 196

Beweis eines Satzes aus der Theorie der elliptischen Functionen
M. Falk
OriginalPaper | Published: 01 December 1885 | Pages: 197 - 200

Untersuchungen über quadratische Formen
Hermann Minkowski
OriginalPaper | Published: 01 December 1885 | Pages: 201 - 258

Sur l'équilibre d'une masse fluide animée d'un mouvement de rotation
H. Poincaré
OriginalPaper | Published: 01 December 1885 | Pages: 259 - 380

taken from the journal website

Some history

Seine Majestät König Oscar II haben, in dem Wunsche einen neuen Beweis des Interesses zu geben, welches Allerhöchstdieselbe für die Entwicklung der mathematischen Wissenschaften hegen und durch die Förderung der Herausgabe der Acta Mathematica, die sich Allerhöchstderselben gnädigster Protection erfreuen, bereits bezeugt haben, beschlossen, am 21. Januar 1889, dem sechzigsten Jahrestage Allerhöchstderselben Geburt, einer wichtigen Entdeckung auf dem Gebiete der höheren Analysis einen Preis zuzuerkennen. Dieser Preis wird in einer Medaille mit dem Bilde Seiner Majestät, im Goldwerthe von tausend Francs, und ausserdem in einer Summe von zwei tausend fünf hundert Kronen in Gold bestehen (1 Krone = 1 Franc 40 Centimes circa).

Seine Majestät geruhten die Ausführung dieses Planes einer Commission von drei Mitgliedern anzuvertrauen: Herrn Carl Weierstrass zu Berlin, Herrn Charles Hermite zu Paris und Herrn Gösta Mittag-Leffler zu Stockholm, Chefredacteur dieses Journals. Die Bemühungen der Commission waren Gegenstand eines Berichtes, dessen nach-

Sa Majesté Oscar II désireux de donner une nouvelle preuve de l'intérêt qu'Elle porte à l'avancement des sciences mathématiques, intérêt qu'Elle a déjà témoigné, en encourageant la publication du journal Acta Mathematica, qui se trouve sous Son auguste protection, a résolu de décerner le 21 Janvier 1889 soixantième anniversaire de Sa naissance un prix à une découverte importante dans le domaine de l'analyse mathématique supérieure. Ce prix consistera en une médaille, du dix-huitième module, portant l'effigie de Sa Majesté et ayant une valeur en or de mille francs, ainsi qu'en une somme de deux mille cinq cents Kronor en or (1 Krona = 1 franc 40 centimes environ).

Sa Majesté a daigné confier le soin de réaliser Ses intentions à une commission de trois membres: M. Carl Weierstrass à Berlin, M. Charles Hermite à Paris, et le Rédacteur en chef de ce Journal, M. Gösta Mittag-Leffler à Stockholm. Le travail des commissaires a été l'objet d'un rapport dont Sa Majesté a pris connaissance, et voici leurs

Figure: Acta Math. 7 (1885), no. 1, I–VI.

Some history

"For a system of arbitrarily many mass points that attract each other according to Newton's law, assuming that no two points ever collide, find a series expansion of the coordinates of each point in known functions of time converging uniformly for any period of time."

(taken from the website of the Mittag-Leffler institute)

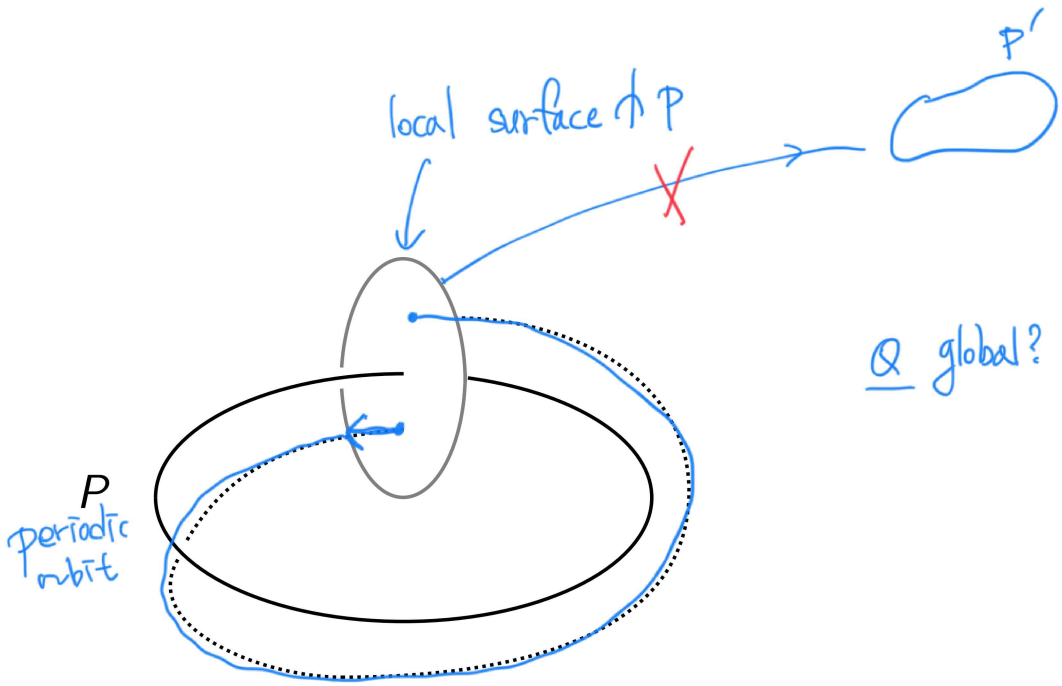
1 Poincaré
2 Appell

Some history

- November 30, 1889. Mittag-Leffler received a telegram from Poincaré: "I have found an error. Please stop the presses."
- December 1, 1889. A letter explaining the error reached Mittag-Leffler.
  - Mittag-Leffler let Poincaré work out a revised version, without mentioning about the mistake in order to avoid attention.
  - Mittag-Leffler tried to retrieve all the distributed copies, without explaning why (today there are only two known copies left, both at the Mittag-Leffler institute).
  - Poincaré had to pay for the first printing. The expense was more than the prize money.
- January 5, 1890. Poincaré finished a revised version.

1901 $(\phi_H^t)^* \omega_0 = \omega_0$.

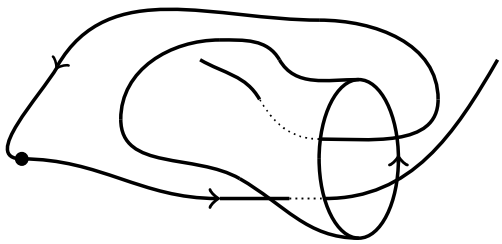
Surface of section



Global surface of section

Let M be a three-manifold equipped with a smooth flow ψ_t.

A **global surface of section** is an embedded compact surface Σ satisfying the following properties:



(i) The boundary components of Σ are periodic orbits for ψ_t.

(ii) The flow ψ_t is transverse to $\mathrm{int}(\Sigma)$.

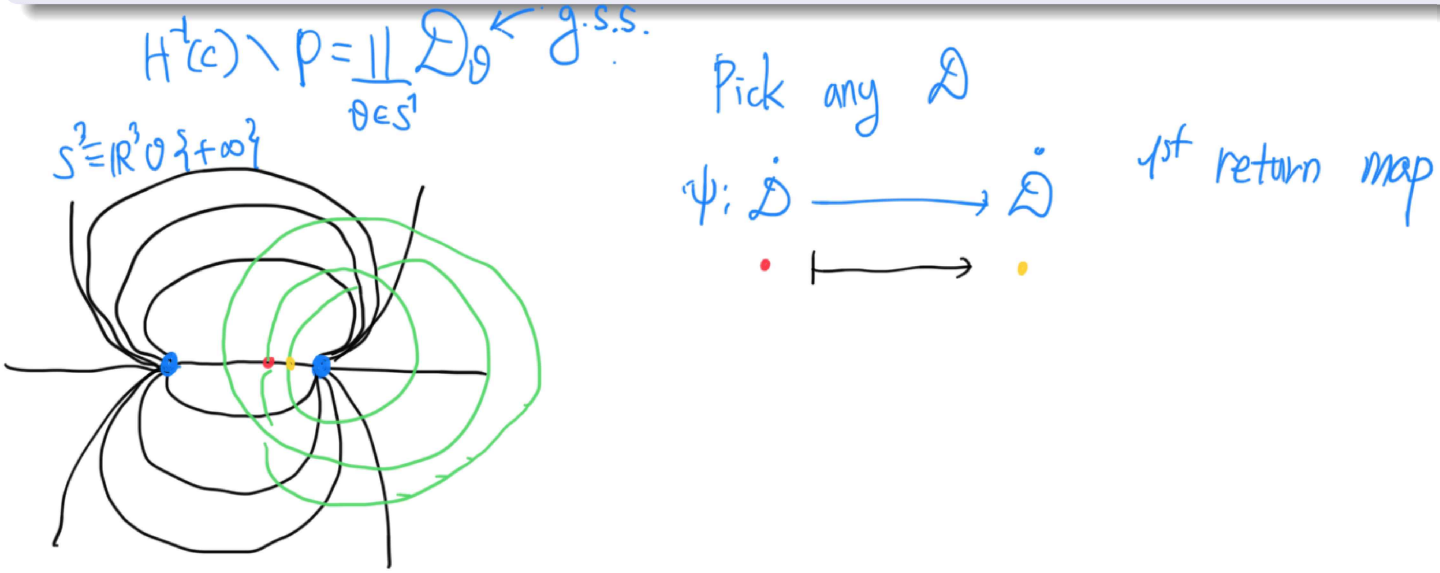
(iii) Every orbit in $M \setminus \partial\Sigma$ hits $\mathrm{int}(\Sigma)$ infinitely often forward and backward in time.

Open book decompositions

Let $H\colon \mathbb{R}^4 \to \mathbb{R}$ be a Hamiltonian and let $c \in \mathbb{R}$ be a regular value.

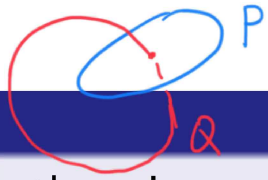
Theorem (Hofer-Wysocki-Zehnder, 1998)

If $H^{-1}(c) \cong S^3$ bounds a strictly convex domain in $\mathbb{R}^4$, then there is a periodic orbit $P \subset H^{-1}(c)$ that binds an open book decomposition of $H^{-1}(c)$ whose pages are disc-like global surfaces of section.



Properties of the first return map

- There is a one-to-one correspondence between periodic orbits different from the binding P and periodic points of the first return map ψ.
- $\psi\colon \dot{\mathcal{D}} \to \dot{\mathcal{D}}$ is conjugated to an area preserving and orientation preserving diffeomorphism $\phi\colon \mathbb{D} \to \mathbb{D}$, where $\mathbb{D} \subset \mathbb{C}$ is the open unit disc.



Theorem

The diffeomorphism ϕ admits a fixed point, i.e. there is a periodic orbit $Q \subset H^{-1}(c)$ different from P.

Proof. Assume by contradiction that there is no fixed point. Then by Brouwer's translation theorem we find a non-empty open set $U \subset \mathbb{D}$ such that $\phi^n(U), n \in \mathbb{N}$, are pairwise disjoint. Since U has a positive area that is preserved by ϕ, we derive a contradiction.

$$\infty > \mathrm{Area}(D) \geq \mathrm{Area}\left(\cup_{n\in\mathbb{N}}\phi^n(U)\right) = \sum_{n\in\mathbb{N}} \mathrm{Area}(\phi^n(U)) = \infty.$$

Two or infinity

Denote by $z_0 \in \mathbb{D}$ a fixed point of ϕ. We restrict ϕ to the open annulus $\mathbb{D} \setminus \{z_0\}$, still denoted by ϕ.

$\phi(z_0) = z_0 \rightsquigarrow \phi|_{\mathbb{D}\setminus\{z_0\}}$

Theorem (Franks)

Every area preserving homeomorphism of an open annulus admits either no or infinitely many periodic points.

Corollary (Hofer-Wysocki-Zehnder)

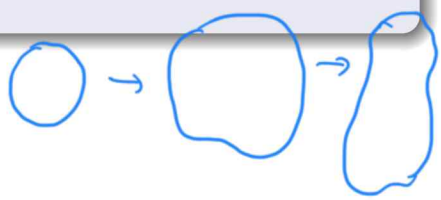
If $H^{-1}(c) \cong S^3$ bounds a strictly convex domain in $\mathbb{R}^4$, then there are either two or infinitely many periodic orbits.

$H : \mathbb{R}^4 \to \mathbb{R}$

c_{min} global minimum

$H^{-1}(c_{min}+\varepsilon) \cong S^3$ strictly convex

ε small



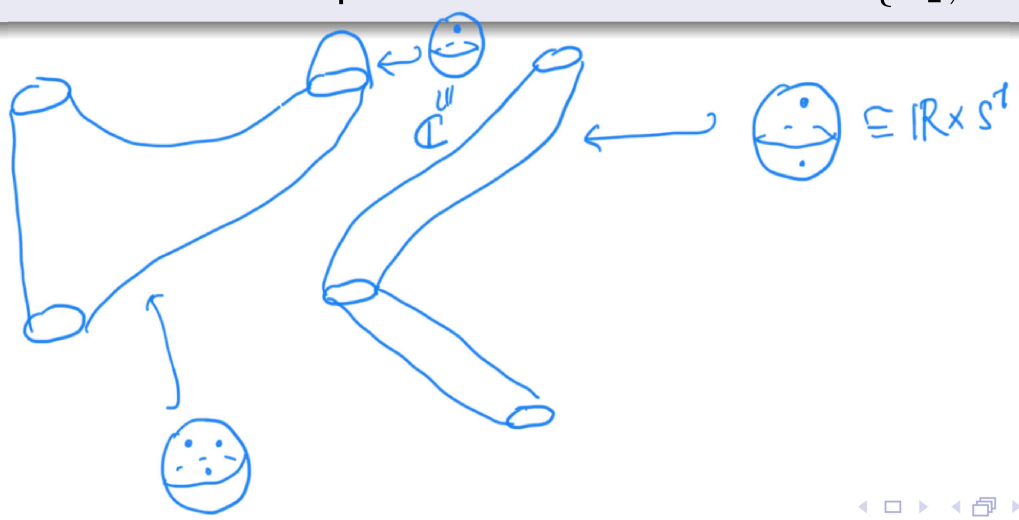
$S^3 \setminus S^1 = \coprod_{\theta \in S^1} D_\theta$

Transverse foliations

Definition

A transverse foliation adapted to $(H^{-1}(c), \phi_H^t)$ is a singular foliation $\mathcal{F}$ of $H^{-1}(c)$ having the following properties:

- the singular set consists of finitely many periodic orbits $P_1, \ldots, P_\ell$.
- every regular leaf of $\mathcal{F}$ is a properly embedded punctured sphere $\dot{\Sigma} \subset H^{-1}(0) \setminus \cup_{j=1}^{\ell} P_j$, which is transverse to the flow. Its closure is an embedded compact surface Σ with $\partial\Sigma \subset \{P_1, \ldots, P_\ell\}$.



Idea of a construction

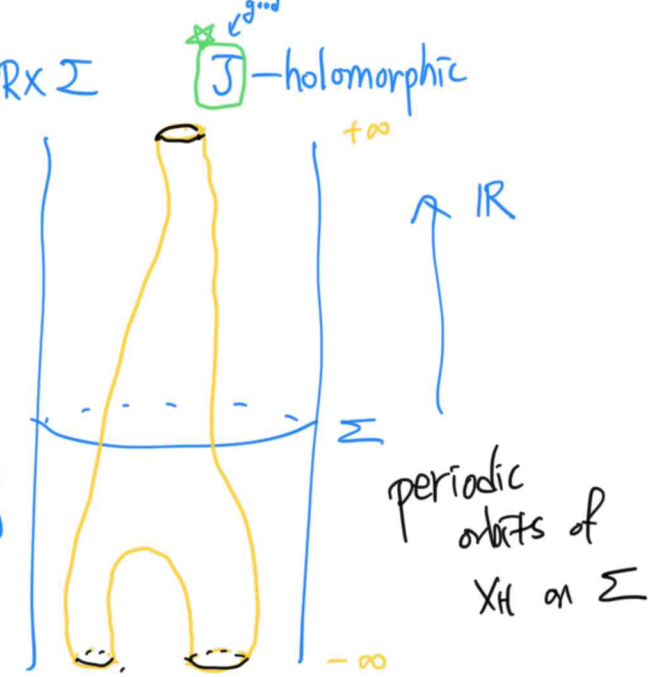
$\Sigma = H^{-1}(c)$ In a certain case, $\exists$ a contact form $\alpha \in \Omega^1(\Sigma)$ i.e. $\alpha \wedge d\alpha > 0$

$(\mathbb{R} \times \Sigma, d(e^r \alpha))$ symplectisation of (Σ, α).

(S^2, j) Riemann sphere $\Gamma \subset S^2$ finite set (of punctures)

$\tilde{u} = (a, u) : S^2 \setminus \Gamma \longrightarrow \mathbb{R} \times \Sigma$ J-holomorphic





periodic orbits of X_H on Σ

* Construct a SMOOTH foliation of $\mathbb{R} \times \Sigma$ by holomorphic curves $\tilde{u}$. (a, u)

$\Rightarrow$ a transverse foliation of Σ by u's.

Idea of a construction

Suppose $\Sigma \cong S^3 \subset \mathbb{C}^2 \equiv \mathbb{CP}^2 \setminus \mathbb{CP}^1_\infty \subset (\mathbb{CP}^2, \omega_{FS})$ → TF

Σ: star-shaped

↑ J-holomorphic sphere

☹ $\Sigma \not\cong S^3$ ⇒ NO general existence theorem.

have to attack case by case.

☺ . In many dynamical systems of interest,

$\Sigma \cong S^3, (\mathbb{RP}^3), S^1 \times S^2, (\mathbb{RP}^3 \# \mathbb{RP}^3)$

$S^3 \to \mathbb{RP}^3$: 2:1, $S^1 \times S^2 \to \mathbb{RP}^3 \# \mathbb{RP}^3$: 2:1

. In many systems with $\Sigma \cong S^1 \times S^2$, double!

The Hénon-Heiles system

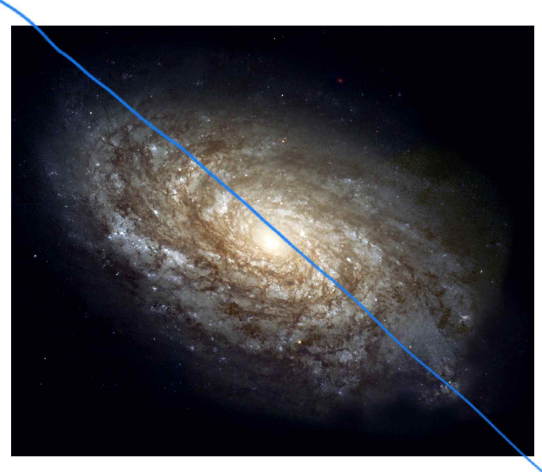


Figure: taken from Wikipedia

The Hénon-Heiles system describes the motion of a star in a galaxy:

- the galaxy has an axis of symmetry.
- the system preserves the total energy and the angular momentum.

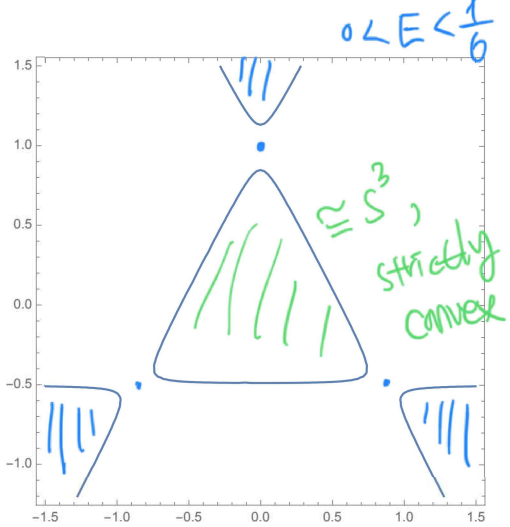
Then one can reduce the system with three degrees of freedom to the system with two degrees of freedom.

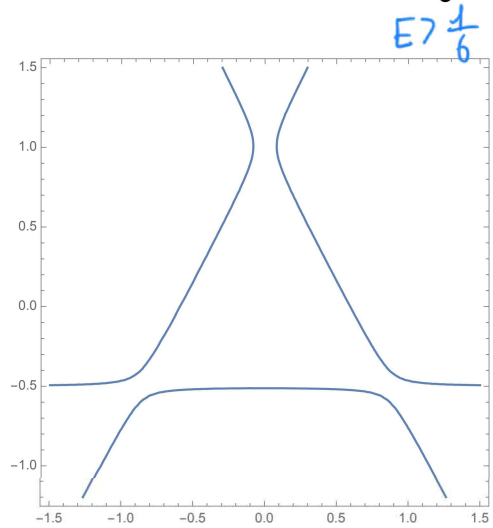
The Hénon-Heiles system

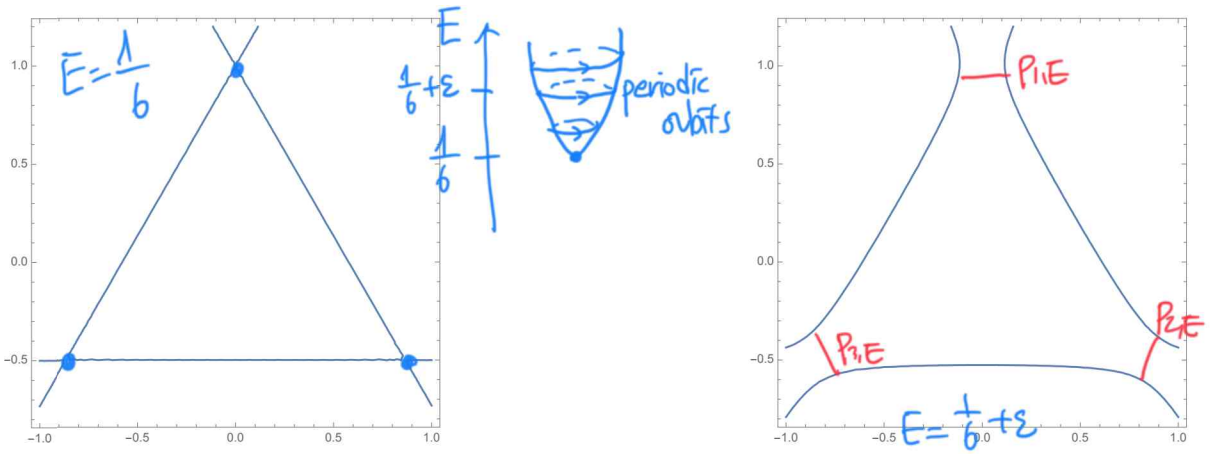
The Hamiltonian is

$$H\colon \mathbb{R}^4 \to \mathbb{R}, \quad (x_1, x_2, y_1, y_2) \mapsto \underbrace{\frac{1}{2}|y|^2}_{\text{kinetic}} + \overbrace{\frac{1}{2}|x|^2 + x_1^2 x_2 - \frac{x_2^3}{3}}^{\text{potential}}.$$

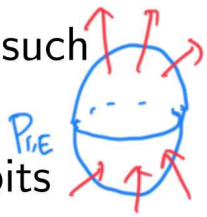
There are two critical energy values 0 (global minimum) and $\frac{1}{6}$ (saddle).





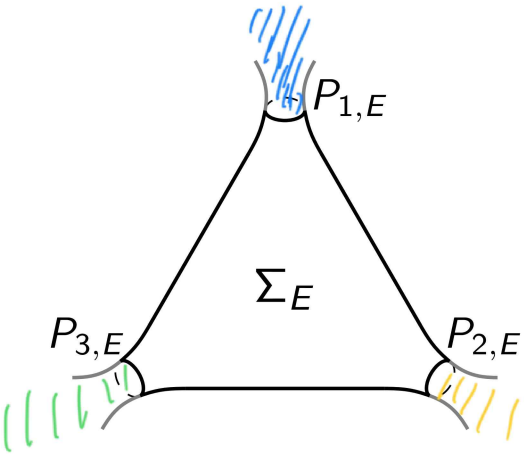


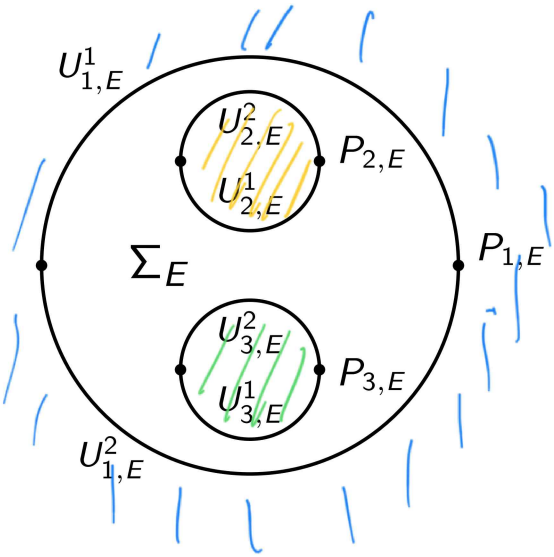
- On $H^{-1}(\frac{1}{6})$: $\exists$ three critical points $(x^i_{1c}, x^i_{2c}, 0, 0)$, $i = 1, 2, 3$, such that (x^i_{1c}, x^i_{2c})'s are saddles of the potential.
- Lyapunoff's theorem: $\exists$ local families of unstable periodic orbits emanating from these critical points, parametrised by energy.
- If $E = \frac{1}{6} + \varepsilon$ for $\varepsilon > 0$ small enough, then the energy level $H^{-1}(E)$ carries three Lyapunoff orbits $P_{1,E}, P_{2,E}, P_{3,E}$.
- Each $P_{i,E}$ bounds discs $U^1_{i,E}, U^2_{i,E}$ such that $H^{-1}(E)$ is separated by a two-sphere $\mathcal{S}_{i,E} = U^1_{i,E} \cup P_{i,E} \cup U^2_{i,E}$ into two components. They are transverse to the Hamiltonian flow.



The Hénon-Heiles system

Denote by Σ_E the closure of the bounded component of $H^{-1}(E) \setminus \cup_{i=1}^{3} \mathcal{S}_{i,E}$.





Question. Does Σ_E admit a transverse foliation having the Lyapunoff orbits $P_{1,E}, P_{2,E}, P_{3,E}$ as binding orbits?

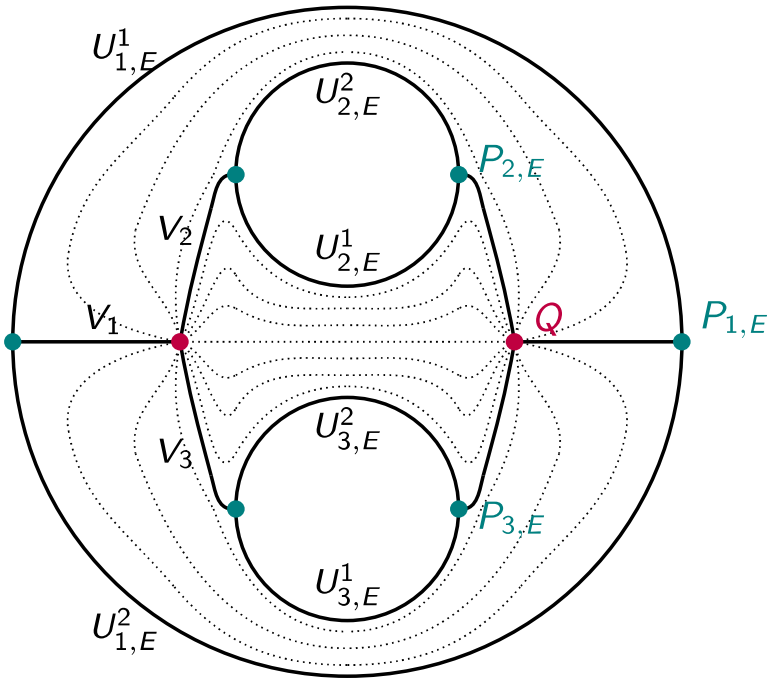
The Hénon-Heiles system

Theorem (de Paulo, K., Salomão, Schneider and Vanderlinde)

Yes! More precisely, there is $\varepsilon_0 > 0$ such that for every $E = \frac{1}{6} + \varepsilon$ with $0 < \varepsilon \leq \varepsilon_0$, the subset Σ_E admits a transverse foliation $\mathcal{F}_E$ with the following properties:

- The singular set of $\mathcal{F}_E$ is formed by the Lyapunoff orbits $P_{1,E}, P_{2,E}, P_{3,E}$ and another periodic orbit Q.
- For every i, the rigid discs $U^1_{i,E}, U^2_{i,E}$ bounded by $P_{i,E}$ are regular leaves.
- For every i there is a rigid cylinder V_i connecting Q and $P_{i,E}$.
- There exist 3 one-parameter families of discs, parametrised by $(0, 1)$, such that each such disc is bounded by Q. At each end, such family of discs breaks onto the union of some rigid cylinder and some rigid disc.

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Thank you!