Lecture 1

Monday, January 17, 2022 12:55 PM

Asymptotic behaviour of convex curve shortening flow (CSF).

Consider 1-parameter family of closed convex curve $\gamma(\cdot, t)$ embedded in $\mathbb{R}^2$.

$\gamma(\theta,t)$
o
$\underline{n}$

Let κ = curvature of γ
$\underline{n}$ = inward unit normal vector
s = arclength parameter

Suppose that $\gamma : M^1 \times [0,T) \to \mathbb{R}^2$ is a smooth 1-parameter family of immersions. ($M^1 \cong S^1$)

Then we say $\Gamma(\cdot, t) = (\gamma^1(\cdot,t), \gamma^2(\cdot,t)) \in \mathbb{R}^2$ is a sol^n to CSF if

$$\frac{\partial \gamma}{\partial t} = \kappa \underline{n} \qquad \text{(CSF)}$$

Recall Frenet-Serret formulae:

$$\underline{T} = \underline{t} = \frac{d\gamma}{ds}, \quad \frac{d\underline{T}}{ds} = \kappa \underline{n}, \quad \frac{d\underline{n}}{ds} = -\kappa \underline{T}$$

Note $\kappa = -\left\langle \frac{d\underline{n}}{ds}, \underline{T} \right\rangle = \left\langle \underline{n}, \frac{d\underline{T}}{ds} \right\rangle$

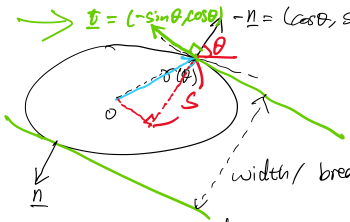
Then (CSF) becomes $\frac{\partial \gamma}{\partial t} = \frac{\partial^2 \gamma}{\partial s^2}$
"System of quasilinear parabolic eq^{n}s"

Ref: 1) X.P. Zhu "Lectures on mean curvature flow" AMS / IP.
2) Gage-Hamilton "The heat eq^n shrinking convex plane curves" JDG 86(?)

Def^n The support function $S(\cdot, t)$ of $\gamma(\cdot, t)$ is given by

$$S = \langle \gamma(\theta), -\underline{n} \rangle = \langle \gamma(\theta), (\cos\theta, \sin\theta) \rangle$$

→ $\underline{t} = (-\sin\theta, \cos\theta)$ $\quad -\underline{n} = (\cos\theta, \sin\theta)$.



Recall $\kappa = \left\langle \underline{n}, \frac{d\underline{T}}{ds} \right\rangle = \left\langle (\cos\theta, \sin\theta), \frac{d\theta}{ds}(-\cos\theta, -\sin\theta) \right\rangle$
$= \frac{d\theta}{ds}$

For $\gamma(\theta) = (\gamma^1(\theta), \gamma^2(\theta))$,

$$S(\theta) = \langle \gamma, -\underline{n} \rangle = \gamma^1 \cos\theta + \gamma^2 \sin\theta \quad \text{''} S$$

$$\Rightarrow S_\theta = -\gamma^1 \sin\theta + \gamma^2 \cos\theta + \cancel{\langle \gamma_\theta, -\underline{n} \rangle}$$

$\hookrightarrow \frac{ds}{d\theta}\frac{\partial \gamma}{\partial s} = \frac{1}{\kappa}\underline{t}$

$$\Rightarrow S_{\theta\theta} + S = -\gamma^1_\theta \sin\theta + \gamma^2_\theta \cos\theta \cancel{- \gamma^1 \cos\theta - \gamma^2 \sin\theta}$$
$$\cancel{+ \gamma^1 \cos\theta + \gamma^2 \sin\theta}$$

$$= \langle \underbrace{(\gamma^1_\theta, \gamma^2_\theta)}_{\frac{d\gamma}{d\theta}}, \underbrace{(-\sin\theta, \cos\theta)}_{\overset{\shortparallel}{\underline{t}}} \rangle$$

$$= \langle \underset{\overset{\shortparallel}{\underline{t}}}{\underbrace{\frac{ds}{d\theta}\frac{\partial \gamma}{\partial s}}}, \underline{t} \rangle = \frac{1}{\kappa} \quad \cdots (\dagger)$$

Parametrise γ with normal angle θ

$$\tilde{\gamma}(\theta, t) = \gamma(u(\theta,t), t).$$

Then $\frac{\partial \tilde{\gamma}}{\partial t} = \frac{\partial \gamma}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial \gamma}{\partial t} = \gamma_u u_t + \kappa \underline{n}$

Since $S(\theta, t) = (\tilde{\gamma}(\theta,t), -\underline{n})$

$$\frac{\partial S}{\partial t} = (\tilde{\gamma}_t, -\underline{n}) + \cancel{(\tilde{\gamma}, -\underline{n}_t)}.$$
$$= -\kappa.$$

From $(\dagger)$, $\quad \frac{\partial S}{\partial t} = -\frac{1}{S_{\theta\theta} + S}, \quad (\theta, t) \in S^1 \times [0,T)$.

(Short time existence)
Let M be a closed 1-dim^l manifold and let $\gamma_0 : M \to \mathbb{R}^2$ be a smooth immersion. Then $\exists$ a family of smooth curves $\gamma(\cdot, t) : M \to \mathbb{R}^2$ for CSF for $t \in [0, \varepsilon)$ satisfying $\gamma(0,t) = \gamma_0$.

- can be proved using Nash-Moser inverse fn. thm.
- " " "De Turck trick"
- $\tilde{S} = S + rw \qquad r \in (-\delta, \delta)$
  $w : S^1 \times (0, \delta) \to \mathbb{R}$.

$$\tilde{S}_t = -\frac{1}{S_{\theta\theta} + rw_{\theta\theta} + S + rw}.$$

Diff. in r & put $r = 0$.

$$\frac{\partial w}{\partial t} = \frac{w_{\theta\theta} + w}{(S_{\theta\theta} + S)^2}$$

Eg 1/ (Round circle)

$-\underline{n} = \frac{\gamma}{|\gamma|}$

Let $\gamma(t)$ = round circle with radius $r(t)$.

$$S(\theta, t) = \langle \gamma, -\underline{n} \rangle = r(t).$$
$$\kappa = \frac{1}{r(t)}.$$

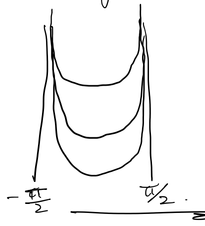
Since $S_t = -\kappa$, $\quad \frac{dr}{dt} = -\frac{1}{r(t)}$.

$$\Rightarrow \int r(t)\, dr = -\int dt.$$
$$\Rightarrow \frac{1}{2}(r^2(t) - r^2(0)) = -t.$$
$$\Rightarrow r(t) = \sqrt{r^2(0) - 2t}.$$

As $t \to \frac{1}{2}r^2(0)$, the round circle converges to a point.

Eg 2/ (Grim reaper).

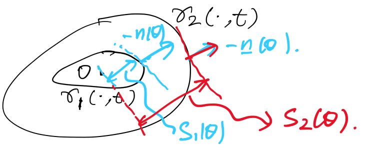
$y = t - \log \cos x,\quad x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.



$\hookrightarrow y_t = 1.\quad y_{xx} = \sec^2 x.$

$\gamma = (x, u(x,t))$.

$CSF \Rightarrow u_t = \frac{u_{xx}}{1+(u_x)^2}.$ ✓



(Containment principle)

Let $\gamma_1, \gamma_2 : S^1 \times [0,T) \to \mathbb{R}^2$ be convex sol^{n}s of CSF.

Let $E(\gamma_i)$ = closed domain with boundary $\gamma_i(\cdot,t)$

Suppose that $E(\gamma_1(\cdot,0)) \subset E(\gamma_2(\cdot,0))$.

Then $E(\gamma_1(\cdot,t)) \subset E(\gamma_2(\cdot,t))$.

Idea of proof. $S_t = -\frac{1}{S_{\theta\theta}+S}$ (✱)

Let S_i = support fn. of γ_i.
$\quad \overset{\shortparallel}{S_i(\theta,t)}$.

Suppose $S_1(\theta,0) \le S_2(\theta,0)$.

By the max. principle for (✱),

$S_1(\theta,t) \le S_2(\theta,t)\quad \forall t \in [0,T)$. //

N.B. The image of γ is invariant under tangential diffeo^m.

Lecture 2

Convexity preserved.

(✱) $S_t = -\frac{1}{S_{\theta\theta}+S} \Rightarrow \kappa_t = \kappa^2(\kappa_{\theta\theta}+\kappa)$.
$= -\kappa.$

//

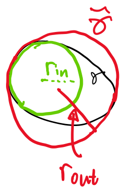
By the short time existence for CSF, $\gamma(\cdot,t)$ exists for some time interval $[0,\tilde{\omega})$.

$$S_t = -\kappa = -\frac{1}{S_{\theta\theta}+S} \quad \text{for } (\theta,t) \in S^1 \times [0,\tilde{\omega})$$

$$\kappa_t = -\frac{S_{t\theta\theta}+S_t}{(S_{\theta\theta}+S)^2} = -\kappa^2(-\kappa_{\theta\theta}-\kappa)$$
$$= \kappa^2(\kappa_{\theta\theta}+\kappa)$$

Since $\kappa > c \ge 0$ initially, by the strong max. principle,
$$\min_{\theta\in S^1} \kappa(\theta,t) \ge \min_{\theta \in S^1} \kappa(\theta,0) > 0$$
i.e. κ has a uniform positive lower bound. //

Finite time existence



Let $\tilde{\gamma}(\cdot,0)$ be a round circle with radius $\tilde{r}(0)$ enclosing $\gamma(\cdot,0)$.

Then by the containment principle, $\tilde{\gamma}(\cdot,t)$ encloses $\gamma(\cdot,t)$.

Since $\tilde{\gamma}(\cdot,t)$ converges to a point in a finite time $\frac{1}{2}\tilde{r}(0)^2$,

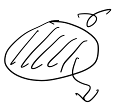
$\gamma(\cdot,t)$ exists for a finite time $T < \frac{1}{2}\tilde{r}(0)^2$.
and $\gamma(\cdot,T)$ is singular. //

Convergence in the Hausdorff metric.

Blaschke selection thm:
For a seq. $\{E_j\}$ of convex sets contained in a bounded set, $\exists$ subseq $\{E_{j_k}\}$ which converges to a convex set E in the Hausdorff metric.

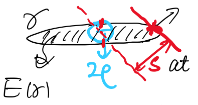
Applying this to $\{\gamma(\cdot,t)\}$, $\gamma(\cdot,t)$ converges to a convex curve $\gamma(\cdot,T)$ in the Hausdorff metric.

Enclosed area $A(t) \to 0$.



$A(t)$ = area enclosed by $\gamma(\cdot,t)$.

Suppose that $A(t) \not\to 0$ as $t \to T$.



Then $\exists$ circle of radius 2ε centered at the origin which lies inside $E(\gamma)$.

$$S - \varepsilon \ge \varepsilon$$

Let $\Phi(\theta,t) = \frac{\kappa}{S-\varepsilon} = \frac{-S_t}{S-\varepsilon}$.

"$t_0 < T$"

Suppose that Φ attains its max. at (θ_0, t_0).

At (θ_0, t_0), $0 = \Phi_\theta = -\frac{S_{t\theta}}{S-\rho} + \frac{S_t S_\theta}{(S-\rho)^2} \cdots (1)$

$$0 \geq \Phi_{\theta\theta} = -\frac{S_{t\theta\theta}}{S-\rho} + \cancel{\frac{S_{t\theta}S_\theta}{(S-\rho)^2}} + \cancel{\frac{S_{t\theta}S_\theta}{(S-\rho)^2}} + \frac{S_t S_{\theta\theta}}{(S-\rho)^2} - \cancel{\frac{2S_t S_\theta^2}{(S-\rho)^3}}$$

$\frac{2S_tS_{t\theta}}{(S-\rho)^2}$ $\|$ by (1)

$$\Rightarrow \frac{S_{t\theta\theta}}{S-\rho} \geq \frac{\overset{-k}{S_t} S_{\theta\theta}}{(S-\rho)^2} \cdots (2).$$

$$0 \leq \Phi_t = -\frac{S_{tt}}{(S-\rho)} + \frac{S_t^2}{(S-\rho)^2}$$

$S_t = -k$

$S_{tt} = -k_t = -k^2(k_{\theta\theta}+k)$

$-S_{t\theta\theta}$

$$= \frac{k^2(k_{\theta\theta}+k)}{S-\rho} + \frac{k^2}{(S-\rho)^2}$$

$$\leq -\frac{k^2 \overset{-k}{S_t} S_{\theta\theta}}{(S-\rho)^2} + \frac{k^3}{S-\rho} + \frac{k^2}{(S-\rho)^2}$$

$$= \frac{k^3(\underbrace{S_{\theta\theta}+S-\rho}_{\frac{1}{k}})}{(S-\rho)^2} + \frac{k^2}{(S-\rho)^2}$$

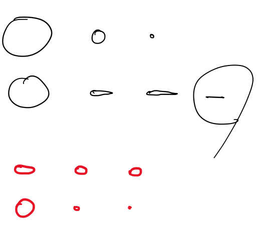
$$= \frac{2k^2}{(S-\rho)^2} - \frac{k^3\rho}{(S-\rho)^2}$$

$\Rightarrow k \leq \frac{2}{\rho} < C$ ※ to $k \to \infty$ as $t \to T$.

$\therefore \lim_{t\to T} A(t) = 0.$ □ //

Recall that $k > C_1$. Since $A(t) \to 0$ as $t \to T$,

$\therefore$ $\gamma(\cdot, t) \to$ point as $t \to T$.



Convergence to a "round point".

Ref : i) Gage 1983 Duke. "An isoperimetric ineq. with applications to curve shortening"

ii) Gage 1984 Inventiones "Curve shortening makes convex curves circular".

Lemma 1 (i) $A(t) = \frac{1}{2}\int_0^{2\pi} \frac{S}{k}\, d\theta$.

(ii) $\frac{dA}{dt} = -2\pi$

(iii) $\frac{d}{dt}L(t) = -\int_\gamma k^2\, ds$

(iv) $L(t) = \int_\gamma kS\, ds$.

Pf/ (i) By Green's thm

$$2A(t) = \iint_{E(\gamma)} \operatorname{div}\cdot\gamma\, dA = \int_\gamma \langle \gamma, -n\rangle ds$$

$$= \int_\gamma S\, ds = \int_0^{2\pi} \frac{S}{k}\, d\theta.$$

Recall $k = \langle n, \frac{dT}{ds}\rangle = \frac{d\theta}{ds} \Rightarrow \frac{d\theta}{k} = \cancel{k}\,ds.$ $\int d\theta = \int k\, ds$

(ii) $$\frac{dA}{dt} = \frac{1}{2}\int \frac{\partial}{\partial t}\left(\frac{S}{k}\right) d\theta$$

$$= \frac{1}{2}\int \left(\frac{\overset{-k}{S_t}}{k} - \frac{S}{\cancel{k^2}}\overset{\cancel{k^2}(k_{\theta\theta}+k)}{k_t}\right) d\theta$$

$$= \frac{1}{2}\int (-1 - S k_{\theta\theta} - kS)\, d\theta$$

int. by parts

$$= \frac{1}{2}\int (-1 - S_{\theta\theta}k - Sk)\, d\theta.$$

$$= \frac{1}{2}\int_0^{2\pi} (-1 - k\underbrace{(S_{\theta\theta} + S)}_{\frac{1}{k}})\, d\theta$$

$$= -2\pi.$$

(iii) Since $L(t) = \int_\gamma ds = \int_{S^1} \langle \gamma_\theta, \gamma_\theta\rangle^{1/2} d\theta.$

$$\frac{d}{dt}L(t) = \cancel{\frac{1}{2}}\int_{S^1} \frac{\cancel{2}\langle \gamma_{\theta t}, \gamma_\theta\rangle}{|\gamma_\theta|} d\theta = \int_{S^1} \langle \gamma_{t\theta}, T\rangle d\theta$$

$$= \int_{S^1} \langle \frac{\partial}{\partial\theta}(kn), T\rangle d\theta \quad \because \gamma_t = kn$$

$$= \int_{S^1} \left(\langle k\frac{\partial n}{\partial\theta}, T\rangle + \frac{\partial k}{\partial\theta}\cancel{\langle n, T\rangle}\right) d\theta$$

$\frac{\partial n}{\partial\theta} = -\frac{\partial s}{\partial\theta}kT$ $\because$ $\frac{\partial n}{\partial\theta} = \frac{\partial s}{\partial\theta}\frac{\partial n}{\partial s}$, $\frac{\partial n}{\partial s} = -kT$ $= -k\frac{ds}{d\theta}T$

$$= -\int_{S^1} k^2 \frac{ds}{d\theta}\underbrace{\langle T, T\rangle}_{1} d\theta.$$

$$= -\int_\gamma k^2\, ds.$$

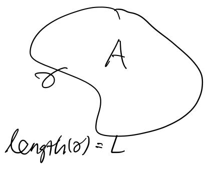
$$L(t) = \int \langle \gamma_s, \gamma_s\rangle ds = -\int \langle \gamma, \underset{kn}{\gamma_{ss}}\rangle ds$$

$$= -\int \kappa \langle \gamma, \underline{n} \rangle\, ds = \int \kappa S\, ds.$$ //

Lecture 3

Ref: • Gage 83 Duke
• Gage 84 Inventiones

Isoperimetric inequality



$L^2 \geq 4\pi A$ for a simple closed curve in $\mathbb{R}^2$
where the equality holds iff $\gamma = S^1(r)$

[Osserman 1979]

Bonnesen's inequality.

Let $\rho = r_{in}$ & $R = r_{out}$.

$$rL - A - \pi r^2 \geq 0 \text{ for } \rho \leq r \leq R.$$

Corollary B $\quad L^2 - 4\pi A \geq \pi^2 (R-\rho)^2 \geq 0.$

pf/ $\left.\begin{aligned} &4\pi(\rho L - A - \pi\rho^2) \geq 0 \\ &4\pi(RL - A - \pi R^2) \geq 0. \end{aligned}\right\} \Rightarrow$ //

Lemma 2 $\quad \left(\frac{L^2}{A}\right)_t = -\frac{2L}{A}\left(\int \kappa^2 ds - \frac{\pi L}{A}\right).$

pf/ Recall $L_t = -\int \kappa^2 ds$ & $A_t = -2\pi$

$$\left(\frac{L^2}{A}\right)_t = \frac{2L}{A} L_t - \frac{L^2}{A^2} A_t$$
$$= -\frac{2L}{A}\left(\int \kappa ds - \frac{L}{A}\pi\right).$$ ▨

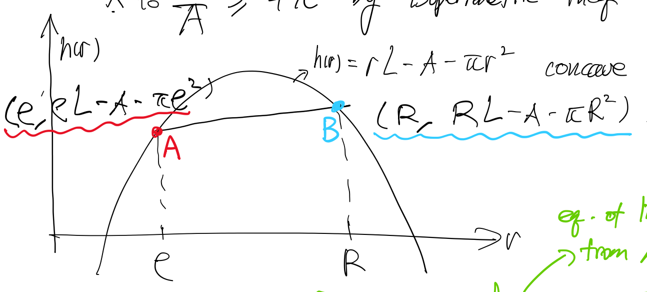
Prop^n 3 $\quad \liminf_{t \to T} L(t)\left(\int_\gamma \kappa^2 ds - \frac{L}{A}\pi\right) \leq 0.$

pf/ Suppose not. Then $L\left(\int \kappa^2 ds - \frac{L}{A}\pi\right) > \varepsilon$ in $[T-\delta, T)$.
for small δ and some $\varepsilon > 0$.

By Lemma 2, $\left(\frac{L^2}{A}\right)_t < -\frac{2\varepsilon}{A} = -\frac{2\varepsilon}{A(0) - 2\pi t}$

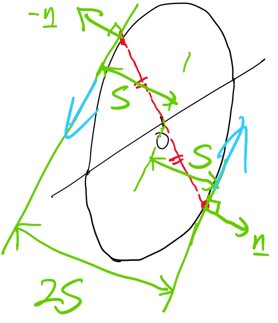
Integrating this, $\frac{L^2}{A} < \frac{\varepsilon}{\pi} \ln(A(0) - 2\pi t)$
$\to -\infty$ as $t \to T$

※ to $\frac{L^2}{A} \geq 4\pi$ by Isoperimetric ineq. ▨



$$\underline{rL - A} - \pi r^2 \geq \overbrace{\rho L - A - \pi\rho^2 + (r-\rho)\frac{(R-\rho)L - \pi(R^2-\rho^2)}{R-\rho}}^{\text{eq. of line segment from A to B}}$$
$$= \underline{rL - A} - \pi r(R+\rho) + \pi R\rho$$
$$\geq 0.$$

$$\Rightarrow \quad -\pi r^2 \geq -\pi r(R+\rho) + \pi\rho R \quad \cdots\cdots (*)$$



We can do this since the domain is symmetric through the origin

For a domain symmetric through the origin

$$\cancel{2}\rho \leq \cancel{2}S \leq \cancel{2}R.$$

$$(*) \Rightarrow -\pi\int S^2 \geq -\pi\underbrace{\int S}_{2A}(R+\rho) + \int \pi\rho R\, ds$$

$$\Rightarrow LA - \pi\int S^2 ds \geq LA\underbrace{\left(1 + \frac{\pi\rho R}{A} - \frac{2\pi(R+\rho)}{L}\right)}_{=:\, U(\gamma) \geq 0} \quad \because \int ds = L.$$
$$= LA \cdot U(\gamma) \geq 0 \quad \cdots\cdots (1)$$

Since $L^2 = \left(\int \kappa S\, ds\right)^2$
$$\leq \left(\int \kappa^2 ds\right)\left(\int S^2 ds\right) \quad \cdots (2)$$

(1) & (2) $\Rightarrow \quad LA\left(\int \kappa^2 ds\right) - \pi L^2 \geq LA\left(\int \kappa^2 ds\right) \cdot U(\gamma)$

$$\Rightarrow \overset{(✱)}{\boxed{L\left(\int \kappa^2 ds - \frac{\pi L}{A}\right)}} \geq L\left(\int \kappa^2 ds\right) U(\gamma)$$

$$\underbrace{\left(\int 1 \cdot ds\right)}_{L}\left(\int \kappa^2 ds\right) \geq \left(\int \kappa ds\right)^2 \longrightarrow \geq \left(\underbrace{\int \kappa\, ds}_{2\pi}\right)^2 U(\gamma)$$

Recall liminf (✱) ≤ 0 from Prop^{n}3. $\quad = 4\pi^2 U(\gamma) \geq 0$

[One can show (✱) $\longrightarrow 0$ so that $U(\gamma) \to 0$ as $t \to T$.]

Recall Lemma 2: $\left(\frac{L^2}{A}\right)_t = -\frac{2L}{A}\left(\int \kappa^2 ds - \pi\frac{L}{A}\right)$

$\Rightarrow \frac{L^2}{A}$ is decreasing in t.

Note that $U(\gamma)$ is invariant under scaling.

Also (1) & $L^2 \leq (\int S^2)(\int \kappa^2) \Rightarrow (\int \kappa^2)(1-U(\gamma)) \geq \pi\frac{L}{A}$.

let $\eta(t) = \sqrt{\frac{\pi}{A}}\,\gamma(t)$, Then $A(\eta(t)) = \pi$.

$\frac{L^2}{A} \geq 4\pi$.

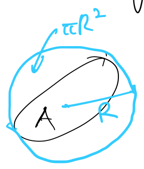
$U(\gamma)\to 0 \Rightarrow U(\eta) \to 0$.

$\sqrt{\frac{\pi}{A}}\,\gamma(t) \to$ unit circle.

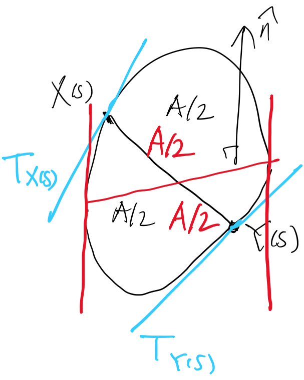
$\frac{L^2}{A}$ invariant under rescaling $\Rightarrow$ converges to 4π.

By Corollary, $\frac{L^2}{A} - 4\pi \geq \frac{\pi^2R^2}{A}\left(1-\frac{\rho}{R}\right)^2$

$\geq \left(1-\frac{\rho}{R}\right)^2$.



LHS $\to 0 \Rightarrow \frac{\rho}{R} \to 1 \Rightarrow$ Convergence to circle.



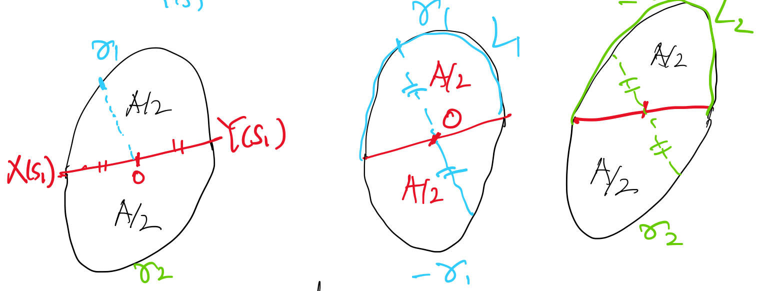
How to deal with an arbitrary convex domain?

let $f(X(s)) = \langle T_{X(s)} \times T_{Y(s)}, \hat{n}\rangle$

$f(Y(s)) = -f(X(s))$.

$\exists s_1$ s.t. $f(X(s_1)) = 0$.

by Intermediate Value Thm.



domains symmetric through O.

$2L_1A - 2\pi\int_{\gamma_1} S^2 ds \geq 2L_1 A\, U(\gamma_1 \cup -\gamma_1)$

Similarly for $\gamma_2 \cup -\gamma_2$.

$L = L_1 + L_2$.

$(L_1+L_2)A - \pi\int_\gamma S^2 ds$

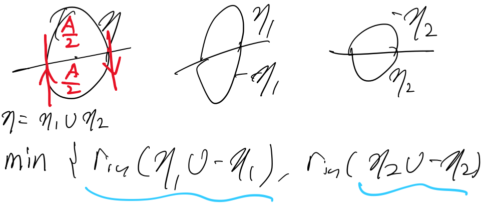
$\geq LA\left\{\frac{L_1}{L}U(\gamma_1\cup-\gamma_1) \cup \frac{L_2}{L}U(\gamma_2\cup-\gamma_2)\right\}$

$= LA\,G(\gamma)$

Take $F(\gamma)$ be supremum of $G(\gamma)$ over all line segments, which splits into two equally divided domains

For $\eta = \sqrt{\frac{\pi}{A}}\,\gamma$, $F(\gamma) = F(\eta)$.

As before, split the convex domain



$r_{in}(\eta) \geq \min\{r_{in}(\eta_1 \cup -\eta_1), r_{in}(\eta_2 \cup -\eta_2)\}$

as $t \to T$ $\to 1$

$r_{out}(\eta) \leq \max\{r_{out}(\eta_1 \cup -\eta_1), r_{out}(\eta_2\cup-\eta_2)\}$

$\to 1$

$\Rightarrow r_{in}(\eta) \to 1$ $r_{out}(\eta) \to 1$

$\Rightarrow \eta \to S^1(1)$. //

i.e. the normalized curve η converges to the unit circle in the Hausdorff metric.

To show the smooth convergence, see Gage-Hamilton 1986 JDG "The Heat Equation Shrinking Convex Plane Curves".

Also for non-convex case, see Grayson 1987 JDG.