Lecture 1. Dehn surgery along a framed link

Low dim. Topology

: Study on mfds of dim. ≤ 4

- 3/4-mfds
- Knots & links in 3/4-mfds
- Homeomorphisms of surfaces (MCG)
- Geometric structurse on them.

⋮

Main objects of this lecture

Y: a closed, oriented 3-mfd

E.g. · $S^3 = \{x \in \mathbb{R}^4 \mid |x|=1\}$

$= \mathbb{R}^3 \cup \{\infty\}$

$= B^3 \cup_{S^2} B^3$ (Alexander's lemma)

· $\mathbb{R}P^3 = S^3/_{x \sim -x}$

· $L(p,q) = \{(z_1,z_2) \in \mathbb{C}^2 \mid |z_1|^2+|z_2|^2=1\}$

$/\,(z_1,z_2) \sim (\xi_p z_1, \xi_p^q z_2)$

ξ_p: pth root of unity

· $\Sigma(p,q,r)$

$= \{(x,y,z) \in \mathbb{C}^3 \mid x^p+y^q+z^r=0\}$

$\cap\, S^5_\varepsilon$, the link of surface singularity

Def Y is a homology sphere

if $H_*(Y;\mathbb{Z}) \cong H_*(S^3;\mathbb{Z})$

More invariants of homology spheres

· $\pi_1(Y)$

· Rohlin / Casson inv'ts.

· Invariants using geometric structures...

Q How to represent (or produce) a closed, oriented 3-mfd?

Dehn surgery

Def · $L \subset Y$ is an **n-component link** if $\underbrace{S^1 \amalg \cdots \amalg S^1}_{n} \hookrightarrow Y$.

· $K \subset Y$ is a **knot** if K is an one-comp. link.

E.g. ·  (right handed) the trefoil knot.

·  the figure 8 knot

·  the Hopf link

Def Two links L_1 & L_2 are **equivalent** if $\exists$ an orientation preserving homeo. $h: S^3 \to S^3$ s.t. $h(L_1) = h(L_2)$.

Def [Dehn surgery along a knot]

K: a knot in S^3

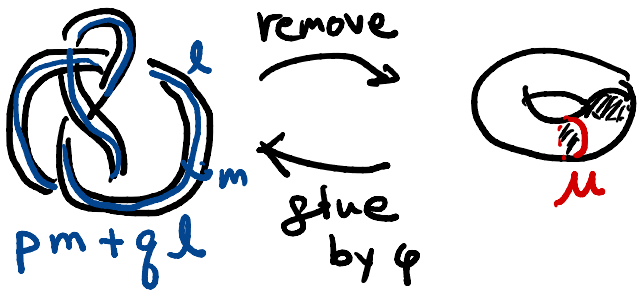
$S^3_{p/q}(K)$

= the 3-mfd obtained by p/q-surgery along K

$$= \left(\overline{S^3 - \underbrace{nb(K)}_{\| \; K \times D^3_\varepsilon}}\right) \cup_\varphi (S^1 \times D^2)$$

where $\varphi_*([\mu]) = p[m] + q[l]$

$[m]$: a meridian of K; $[l]$: canonical longitude



Rmk $S^3_{p/q}(K)$ is well-defined.

(∵ $MCG(T^2)$

& Alexander's lemma

Eg. $\overset{0}{\bigcirc} \cong S^1 \times S^2$, $\overset{\frac{1}{q}}{\bigcirc} \cong S^3$

- $\overset{-\frac{p}{q}}{\bigcirc} = L(p,q)$
- $\overset{-1}{\text{trefoil}} \cong \Sigma(2,3,5)$
- $\boxed{K}\ \overset{\frac{p}{0}=\infty}{\bigcirc} = S^3$

- p/q : surgery coefficient
- If $q = \pm 1$, $p/\pm 1$: integral framing

Surgery along link

$L = K_1 \sqcup \cdots \sqcup K_n$, a link in S^3

$(p_1/q_1, \cdots, p_n/q_n) \in (\mathbb{Q} \cup \{\infty\})^n$

$\rightsquigarrow S^3_{p_1/q_1, \cdots, p_n/q_n}(L)$: a closed, oriented 3-mfd.

Thm (Lickorish-Wallace Thm)

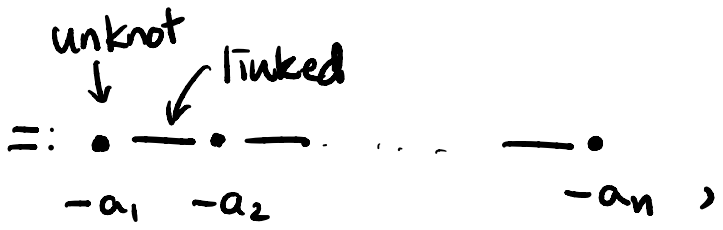
Any closed, oriented 3-mfd can be obtained by the surgery along a (integral) framed link in S^3

Pf Any 3-mfd is triangulable

$\rightsquigarrow$ ... □

Thm $L(p,q) =$ 

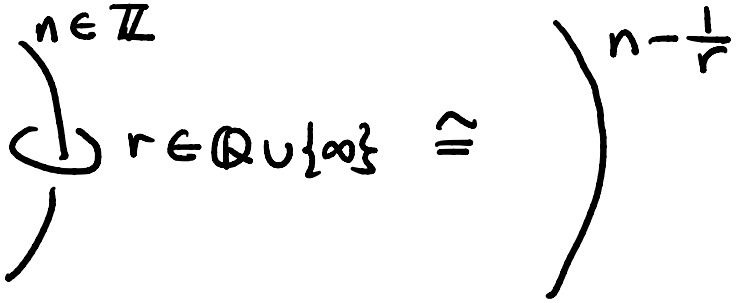
$\cong$ 

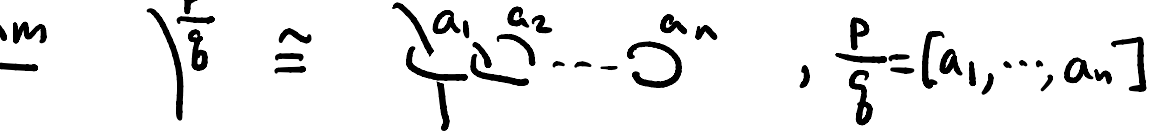


where

$$\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\cdots - \cfrac{1}{a_n}}} = [a_1, \cdots, a_n]$$

Lemma (Slam Dunk move)



Thm 

, $\frac{p}{q} = [a_1, \cdots, a_n]$

E.g. $L(7,3)$

$\frac{7}{3} = [3,2,2] = [2,-3]$

