Lecture 3. Intersection form of 4-mfds

Last time

Thm [Lickorish–Wallace]

$$Y \cong S^3(L)$$

Thm [Kirby]

$$S^3(L) \cong S^3(L')$$

$\Leftrightarrow$ L' can be obtained from L by a sequence of moves (K1) & (K2)

(K1) Blow up / down

(K2) Handle slide

Thm Any closed, oriented 3-mfd can be obtained from S^3 by surgery along an "even" link

↓ all framings are even integers

Pf Kirby moves

Ref. Saveliev, Lectures on the topology of 3-mfds (Lecture 4)

□

Surgery diagram & 4-mfd

Def W: a cobordism between Y_1 & Y_2

$\Leftrightarrow$ W: a cpt, oriented 4-mfd

s.t. $\partial W = -Y_1 \sqcup Y_2$



Thm Y_1, $L \subset Y_1$: a framed link in Y_1

$\Rightarrow \exists W$: a cobordism between Y_1

& $Y_2 = Y_1(L)$

↑ the 3-mfd obtained by surgery of Y_1 along L

pf

$$W := Y_1 \times [0,1] \cup_h \left(\underbrace{\coprod_{i=1}^{n} D^2 \times D^2}_{n \text{ 2-handles}} \right)$$

$\exists!$ (up to isotopy)

diffeo $h : S^1 \times D^2 \to nb(K)$

s.t. $h(S^1 \times \{0\}) = K$

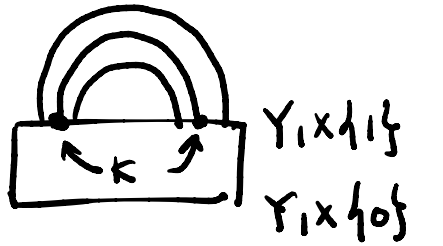
& $h(S^1 \times \{a\}) = K'$ ↑ the longitude of the framing of K

a pt on ∂D^2

Glue $D^2 \times D^2$

by $h : \partial D^2 \times D^2$

$\to nb(K) \subset M \times \{1\}$



□

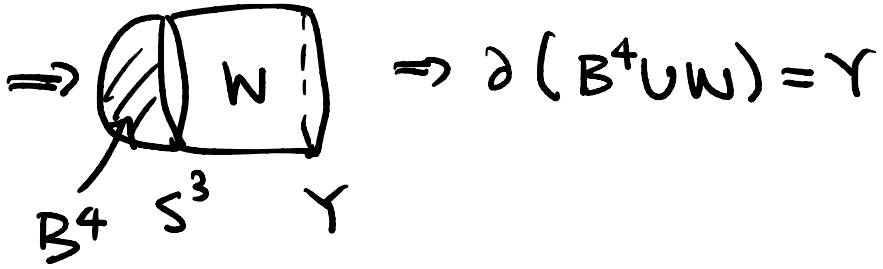
Def W : the trace of surgery on L.

Note W : the trace of $L \subset S^3$

$\Rightarrow$ W is simply-connected.

Cor Any closed, oriented 3-mfd is a bdry of (simply-connected) 4-mfd.

Pf $Y = S^3(L)$ (Lickorish-Wallace)



□

Rmk Not true in general in higher dim.

(Obstruction: Stiefel-Whitney & Pontrjagin numbers)

Intersection form of 4-mfds

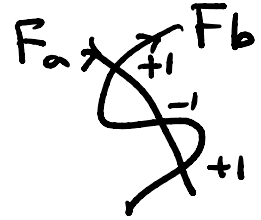
X : a cpt, oriented 4-mfd

$Q_X : H_2(X;\mathbb{Z})/_{Tors} \otimes H_2(X;\mathbb{Z})/_{Tors} \longrightarrow \mathbb{Z}$

defined by

$$Q_X(a,b) = \sum_{Alg} \#(F_a \cap F_b)$$

↑ an emb. ori. surface representing $a \in H_2(X;\mathbb{Z})/_{tors}$

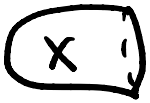


$\Rightarrow Q_X(a,b) = +1$

※ If X : closed (by Poincaré duality),

$$Q_X : H^2(X;\mathbb{Z}) \otimes H^2(X;\mathbb{Z}) \to \mathbb{Z}$$
$$(\alpha \ , \ \beta) \longmapsto \langle \alpha \cup \beta, [X] \rangle$$

Rmks

- Q_X : symmetric, bilinear form over $\mathbb{Z}^{b_2(X)}$
- By fixing a basis of $H_2/_{Tors}$, Q_X is represented by an $n \times n$ symmetric matrix with entries in $\mathbb{Z}$.
- 

$Y = \partial X$

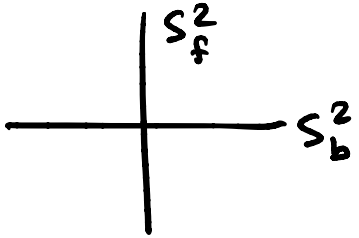
If ∂X is homology sphere or $\partial X = \phi$ (closed),

$\det Q_X = \pm 1$

(ie. Q_X is unimodular)

Eg. $X = S^2 \times S^2$

$\Rightarrow H_2(S^2 \times S^2) = \mathbb{Z}\langle [S^2_b] \rangle \oplus \mathbb{Z}\langle [S^2_f] \rangle$

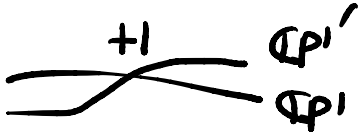


$Q_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} =: H$

Eg. $X = \mathbb{C}P^2$

$H_2(\mathbb{C}P^2; \mathbb{Z}) = \mathbb{Z}\{[\mathbb{C}P^1]\}$

- $Q_{\mathbb{C}P^2} = [1]$



- $Q_{\overline{\mathbb{C}P^2}} = [-1]$

In general, $Q_{-X} = -Q_X$

Eg. $H_2(X_1 \# X_2) = H_2(X_1) \oplus H_2(X_2)$

$Q_{X_1 \# X_2} = Q_{X_1} \oplus Q_{X_2}$

Eg

L : a framed link in S^3

$W = B^4 \cup$ trace 4-mfd

$(\Rightarrow \partial W = S^3(L))$

$\Rightarrow Q_W$ is represented by the linking matrix of L

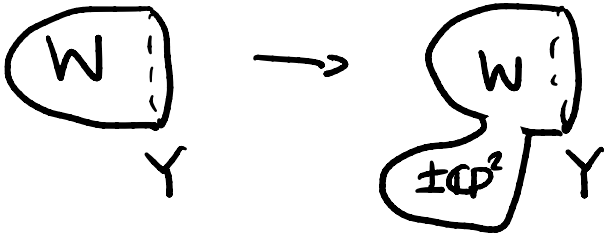
$\therefore$



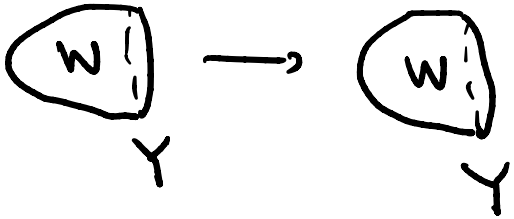
a generator of $H_2(W)$

Rmk

(K1)



(K2)



a basis change in $H_2(W)$

Algebraic aspects of Q

Q : a symmetric bilinear form

over $L \cong \mathbb{Z}^n$, $Q : L \times L \to \mathbb{Z}$

- rank $(Q) = rk(Q) = rk_{\mathbb{Z}}(L) = n$
- $L \otimes \mathbb{R} \rightsquigarrow \tilde{Q} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

  $b_+(Q) :=$ # of positive eigenvalues

  $b_-(Q) :=$ # of negative eigenvalues

  ($rk(Q) = b_+(Q) + b_-(Q)$

  if Q is nondegenerated)
- Signature of Q

  $$sig(Q) = b_+(Q) - b_-(Q)$$

  If $sig(Q) = b_+(Q)$ or $b_-(Q)$,

  Q is said to be definite

  otherwise, Q is indefinite
- Type of Q

Def Q is even if

$$Q(v,v) \equiv 0 \pmod 2$$

for $\forall v \in \mathbb{Z}^n$

Otherwise, Q is said to be odd.

Rmk Q is even $\Leftrightarrow Q(e_i, e_i) \equiv 0$ (2)

$\{e_1, \cdots, e_n\}$

a basis of L