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답양 2

강의2 에서 할 일:

algebraic fiber space $f: X \to Y$ 가 smooth 이고 general fiber F 가 general type 인 경우 Conjecture Q. 가 참임을 보인다.

($f: X \to Y$ is smooth if all fibers are smooth.)

정리 2-1. (Griffith)

f : X → Y 가 smooth algebraic fiber space

$f_* \omega_{X/Y}$ is a nef locally free sheaf .

($f_* \omega_{X/Y}$ local freeness 보이고 nefness 증명.)

nef locally free sheaf 란?

X = projective variety

$\mathcal{E}$ = locally free sheaf of finite rank

$\mathcal{E}|_U = \bigoplus_{i \in I} \mathcal{O}_X|_U$

I = finite

$\forall$ open $U \subset X$.

we say $\mathcal{E}$ is nef if $\xi = 0$ 이라고

$\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$ is nef on $\mathbb{P}_X(\mathcal{E})$

(즉, $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1) \cdot C \geq 0$ for any curve C on $\mathbb{P}_X(\mathcal{E})$)

(증명을 위한 준비)

Kodaira vanishing 복습.

$$H^i(X, \omega_X \otimes \mathcal{O}_X(H)) = 0 \quad \text{for any } i > 0$$

where H is an ample Cartier divisor on X.

정리 2-2. (Kollár's vanishing theorem)

If $f: X \to Y$ X : smooth projective, Y : projective

H = ample Cartier divisor on Y,

then $$H^i(Y, R^j f_* \omega_X \otimes \mathcal{O}_Y(H)) = 0$$

for any $i > 0$, $j \geq 0$.

- 증명은 생략
- Kodaira vanishing 의 일반화

  $X = Y$, $f = id: X \to X$ 로 놓으면

  Kodaira vanishing 복원
- Hodge theory 를 이용해서 증명.
- Kollár vanishing 을 더 일반화한 결과도 존재 (by Fujino)

정리 2-3.

X = projective

H = ample Cartier divisor on X such that
|H| is free

$\mathcal{F}$ = coherent sheaf on X such that

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(-iH)) = 0$$

for any $i > 0$.

Then $\mathcal{F}$ is generated by global sections.

증명은 연습시간에
(힌트: Castelnuovo Mumford regularity)

위의 두 정리(2-2,2-3)를 이용하면 다음 정리를 얻는다.

정리 2-4.

$f : X \to Y$ is an algebraic fiber space such that

X = smooth projective and

Y = projective, dim Y=m.

H = ample Cartier divisor on Y

Then $f_*\omega_X \otimes \mathcal{O}_Y((m+1)H)$ is generated by global sections.

증명:

By 정리 2-2 (Kollár vanishing),

$$H^i(Y, f_*\omega_X \otimes \theta_Y((m+1-i)H)) = 0$$

for any $i > 0$, $m = \dim Y$

$(m+1 > i > 0)$

By 정리 2-3,

$f_*\omega_X \otimes \theta_Y((m+1)H)$ is generated by global sections.

이제 이 강의2의 목표인 다음 정리의 증명을 시작

정리 2-1. (Griffith)

f : X ⟶ Y 가 smooth algebraic fiber space 이고 X, Y = smooth 일 경우

$f_*\omega_{X/Y}$ is a nef locally free sheaf .

증명:

(local freeness 보이고 nefness 증명.)

정리 2-5 (local freeness)

If $f: X \to Y$ is a smooth morphism with connected fibers and X, Y are smooth projective, then $f_* \omega_{X/Y}$ is locally free.

증명:

$\ell = \dim X - \dim Y$, $X_y = f^{-1}(y)$

$\dim_{\mathbb{C}} H^\ell(X_y, \mathbb{C})$ 는 $y \in Y$ 의 선택에 무관

Smooth 가정 →

Hodge decomposition 을 이용하면

$$H^\ell(X_y, \mathbb{C}) = \bigoplus_{p+q=\ell} H^q(X_y, \Omega^p_{X_y})$$

$\dim_{\mathbb{C}} H^q(X_y, \Omega^p_{X_y})$ 는 upper semicontinuous

$\Rightarrow$ $\dim_{\mathbb{C}} H^q(X_y, \Omega^p_{X_y})$ 는 $y \in Y$ 의 선택에 무관

$\Rightarrow$ $\dim_{\mathbb{C}} H^0(X_y, \Omega^\ell_{X_y}) = \dim_{\mathbb{C}} H^0(X_y, \omega_{X_y})$

는 성립

$\Rightarrow$ $f_* \omega_{X/Y}$ is locally free

("flat base change theorem" 등등 이용..)

□

이제 $f_*\omega_{X/Y}$ 의 nefness 증명:

주어진 fiber space $f: X \to Y$의 s-fold fiber product를 생각해 본다:

$$f^s : \underbrace{X \times_Y X \times_Y X \times_Y \cdots \times_Y X}_{=: X^s} \longrightarrow Y$$

주장: $f^s_* \omega_{X^s/Y} \simeq \bigotimes_s f_* \omega_{X/Y}$ $\forall s \geq 1$

$s=1$: trivial

$s \geq 2$:

$$\begin{array}{ccc} X^s & \xrightarrow{p} & X^{s-1} \\ {\scriptstyle q}\downarrow & \searrow^{f^s} & \downarrow{\scriptstyle f^{s-1}} \\ X & \xrightarrow[f]{} & Y \end{array} \quad \text{commutes} \qquad \Rightarrow \begin{cases} f^s = f^{s-1} \cdot p \\ p_* q^* = (f^{s-1})^* f_* \end{cases}$$

(all the maps are smooth and X^s is smooth.)

$\omega_{X^s/X} \simeq p^* \omega_{X^{s-1}/Y}$ on X^s

(By "flat base change theorem")

$$\Rightarrow \quad \omega_{X^S/Y} \simeq \omega_{X^S/X} \otimes q^* \omega_{X/Y}$$

$$\simeq p^* \omega_{X^{S-1}/Y} \otimes q^* \omega_{X/Y}$$

$$f_*^S \omega_{X^S/Y} \simeq f_*^{S-1} p_* \left(p^* \omega_{X^{S-1}/Y} \otimes q^* \omega_{X/Y} \right)$$

$\left(f^S = f^{S-1} \cdot p \right)$

$$\simeq f_*^{S-1} \left(\omega_{X^{S-1}/Y} \otimes p_* q^* \omega_{X/Y} \right)$$

$\left(p_* q^* = (f^{S-1})^* f_* \right)$

$$\simeq f_*^{S-1} \left(\omega_{X^{S-1}/Y} \otimes (f^{S-1})^* f_* \omega_{X/Y} \right)$$

(projection formula)

$$\simeq f_*^{S-1} \omega_{X^{S-1}/Y} \otimes f_* \omega_{X/Y}$$

(induction)

$$= \overset{S}{\bigotimes} f_* \omega_{X/Y}$$

$$\Rightarrow \quad f_*^S \omega_{X^S/Y} = \overset{S}{\bigotimes} f_* \omega_{X/Y}$$

다음 정리를 이용해서 nefness 증명.

정리 2-4.

$f: X \to Y$ is an algebraic fiber space such that
X = smooth projective and
Y = projective, dim Y=m.

H = ample Cartier divisor on Y

Then $f_*\omega_X \otimes \mathcal{O}_Y((m+1)H)$ is genereted by global sections.

- $M = \omega_Y \otimes \mathcal{O}_Y((\dim Y+1)L)$ 라 놓으면
  $f_*^s \omega_{X^s/Y} \otimes M = f_*^s \omega_{X^s} \otimes \mathcal{O}_Y((\dim Y+1)L)$
- 위 정리를 $f^s: X^s \to Y$ 에 적용하면
  $f_*^s \omega_{X^s/Y} \otimes M$ is generated by global sections for any $s \geq 1$.

$\Rightarrow \left(\bigotimes^s f_*\omega_{X/Y}\right) \otimes M$ is generated by global sections for any $s \geq 1$.

by 곱개

$\Rightarrow f_*\omega_{X/Y}$ is nef by Lemma 2-5

Lemma 2-5

V = projective variety

$\mathcal{E}$ = locally free sheaf on V, $\mathcal{E} \neq 0$

If there is a line bundle M such that

$\mathcal{E}^{\otimes s} \otimes M$ is generated by global sections for any $s > 0$, then $\mathcal{E}$ is nef.

(증명은 생략)

주의 : Griffith의 원래 증명은 analytic.

사실은 더 일반적인 다음 결과가 성립한다.

정리 2-6

f : X $\longrightarrow$ Y 가 smooth algebraic fiber space.
X, Y = smooth 일 경우

$f_* \omega_{X/Y}^{\otimes m}$ is a nef locally free sheaf for any $m \geq 0$.

증명은 생략

- Siu 의 invariarce of plurigenera 를 이용해서
  $f_* \omega_{X/Y}^{\otimes m}$ 의 locally freeness 증명.

정리 2-7 (Invariance of plurigenera, Siu)

Let f:X $\longrightarrow$ Y be a SMOOTH projective morphism with X, Y smooth and with connected fibers. Then

$$\dim_{\mathbb{C}} H^0(X_y, \theta_{X_y}(mK_{X_0})) \qquad X_y = f^{-1}(y)$$

is independent of the choice of $y \in Y$.

- nefness는
  Kollár vanishing,
  Popa- Schnell 의 Effective vanishing Theorem 을
  이용해서 증명.
- 증명에 쓰이는 논법은 m=1 인 경우와 비슷.