_____/

답양3

강의3:

algebraic fiber space $f: X \longrightarrow Y$ 는 일반적으로 smooth morphism 이 아니다.
smooth 가 아닐 경우 " weakly semistable reduction " 을 통해서 극복.

정의 (weakly semistable morphism)

An algebraic fiber space $f: X \longrightarrow Y$ is called a weakly semistable morphism if

i) There are toroidal embeddings $(U_X \subset X)$, $(U_Y \subset Y)$ such that $U_X = f^{-1}(U_Y)$

ii) $f: X \longrightarrow Y$ is toroidal with respect to $(U_X \subset X)$, $(U_Y \subset Y)$

iii) f is equidimensional,

iv) all the fibers of f are reduced.

v) Y is smooth.

(X 까지 smooth 이면 f= semistable morphism)

* $(U_X \subset X)$ is a toroidal embedding if it is locally isomorphic to $(T \subset V)$

locally: 국소적으로.

T: dense torus

V: toric variety.

* $f: X \longrightarrow Y$ 가 toroidal 이라는 것은 local 하게 toric variety 사이의 map 과 isomorphic인 경우이다.

weakly semistable morphism 의 중요한 성질 I.

Lemma 3-2

If $f: X \to Y$ is weakly semistable , then X has only rational Gorenstein singularities.

Well-known :

rational Gorenstein iff canonical Gorenstein.

↓ MMP에서 많이 쓰이는 특이점

정의 (canonical/terminal)

For a given variety X, let $h: V \to X$ be a birational morphism. Then we can write

$$K_V = h^* K_X + \sum a_i E_i$$

i. $a_i = a(X, E_i)$ is called the discrepancy of X at E_i.

ii. X has terminal singularities if $\inf_E a(X,E) > 0$.

iii. X has canonical singularities if $\inf_E a(X,E) \geq 0$.

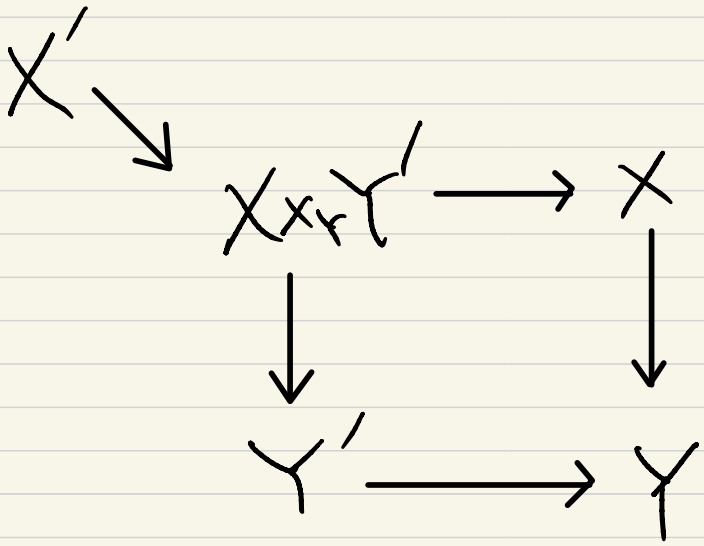
Fact:

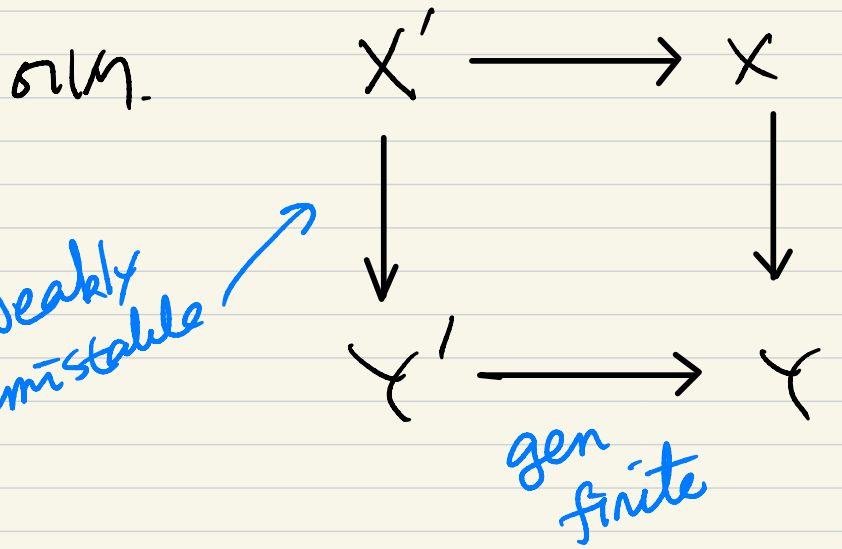
X : terminal (or canonical) → MMP → X' : terminal (or canonical)

정리 3-3 (weakly semistable reduction)

For any $f: X \to Y$ algebraic fiber space, there exists a generically finite surjective morphism $Y' \to Y$ with Y' smooth projective, projective birational morphism $X' \to X \times_Y Y'$, with normal projective X' such that $X' \to Y'$ is weakly semistable :

(due to Abramovich-Karu)





- X' 은 canonical singularity 만 갖는다.
- de Jong 의 alteration을 이용하여 증명.

weakly semistable morphism 의 중요한 성질 II.

(base change 에 대해서 invariant)

Lemma 3-4

If $f: X \to Y$ is a weakly semistable morphism, $\tau: Y' \to Y$ is a morphism with smooth projective Y' such that $\tau^{-1}(Y \setminus U_Y)$ is an snc divisor on Y'. ($U_Y \subset Y$ toroidal embedding)

Then $X' = X \times_Y Y' \to Y'$ is also a weakly semistable morphism:

$$\begin{array}{ccc} X \times_Y Y' = X' & \xrightarrow{\rho} & X \\ \downarrow f' & & \downarrow f \\ Y' & \xrightarrow{\tau} & Y \end{array}$$

오늘 증명할 정리

정리 3-5

$f: X \to Y$ is a weakly semistable algebraic fiber space with X = normal projective variety and Y = smooth projective variety.
Suppose that the general fiber F is of general type.

(i) Then $f_* \omega_{X/Y}^{\otimes m}$ is nef locally free sheaf on Y for any $m > 0$.

(ii) If, furthermore, $\mathrm{Var}(f) = \dim Y$, then $f_* \omega_{X/Y}^{\otimes m}$ is nef and big locally free sheaf on Y for some $k > 0$.

증명 (i)

STEP 1. $f_* \omega_{X/Y}^{\otimes m}$ 가 locally free 인 것 확인.
(see Remark)

STEP 2. (nefness)

$f: X \to Y$ 의 s-fold fiber product.

$$f^s : \underbrace{X \times_Y X \times_Y X \times_Y \cdots \times_Y X}_{= X^s} \longrightarrow Y$$

$S=2$ 인 경우 반례 증명

$$\begin{array}{ccc} X^2 = X \times_Y X & \xrightarrow{p} & X \\ \downarrow q & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

X = Gorenstein

f = flat

all fibers are reduced ...

"flat base change theorem" 을 쓰면

$$\omega_{X^2/X} \simeq p^* \omega_{X/Y}$$

$\Rightarrow X^2$ 는 Gorenstein $\Rightarrow$ CM

$X^2 = X \times_Y X$ 가 Smooth in codim 1

$\Rightarrow$ $X \times_Y X$ = normal

$f : X \to Y$ 는 toroidal 이고 X^2 = normal

이기 때문에 $X^2 = X \times_Y X$ 는 toroidal.

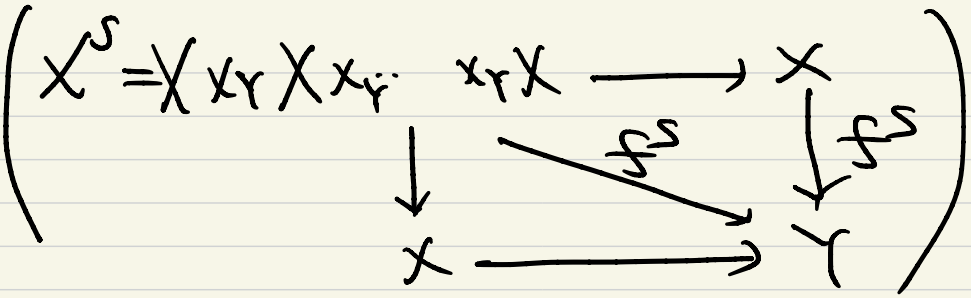
$\Rightarrow X^2$ 는 rational singularity 만 갖는다.

이 과정을 여러번 반복하면

$\Rightarrow X^S$ 는 normal, Gorenstein 이며

rational singularity 만 갖는다.

주장 : (증명은 생략)



$$f_*^S \omega_{X^S/Y}^{\otimes m} \simeq \overset{S}{\otimes} f_* \omega_{X/Y}^{\otimes m}$$

($f_* \omega_{X/Y}^{\otimes m}$ is locally free sheaf 인 사실과 projection formula 를 이용해서 증명)

* $f: X \to Y$ smooth 인 경우에 적용하는 relative dualizing sheaf 의 flat base change 를 이용해야 하는 훨씬 더 심오한 증명.

다음 정리를 이용해서 nefness 증명.

정리 (Effective vanishing theorem, Popa-Schnell)

$f: X \to Y$ is a surjective morphism with
X=smooth projective, Y=projective of dimension n.
L=ample line bundle such that $|L|$ is free.

Then $H^i(Y, f_* \omega_X^{\otimes k} \otimes L^{\otimes \ell}) = 0$

for any $i > 0$ for any $\ell \geq nk + k - n$

Let $M = \omega_Y^{\otimes m} \otimes L^{\otimes m(\dim Y+1)}$ and

L = ample on Y such that $|L|$ is free

X^s의 위에 Popa-Schnell의 정리 + Castelnuovo–Mumford 를 적용하면

$f_*^s \omega_{X^s/Y}^{\otimes m} \otimes M$ 은 finitely generated by global sections $\forall s \geq 1$

$\Rightarrow \bigotimes^s \left(f_* \omega_{X^s/Y}^{\otimes m} \right) \otimes M$ is

finitely generated by global sections (by 정의)

$\Rightarrow f_* \omega_{X/Y}^{\otimes m}$ is nef by Lemma 2-5

Lemma 2-5

V = projective variety

$\mathcal{E}$ = locally free sheaf on V, $\mathcal{E} \neq 0$

If there is a line bundle M such that

$\mathcal{E}^{\otimes s} \otimes M$ is generated by global sections for any $s > 0$, then $\mathcal{E}$ is nef.

Remark : $f_* \omega_{X/Y}^{\otimes m}$ 의 local freeness.

$f : X \longrightarrow Y$ 가 Smooth 인 경우는 Siu 의 invariance of plurigenera를 써서 증명.

smooth 가 아닐 경우 다음 결과를 이용해서 증명

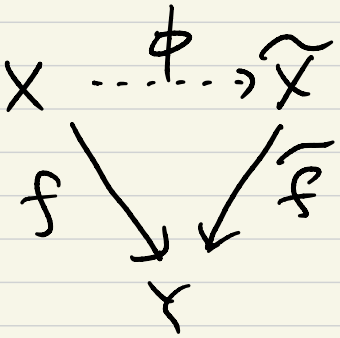
정리 3-10

f:X⟶ Y is an algebraic fiber space with general type F. Suppose X has canonical singularities. Then X has a good minimal model $\widetilde{X}$ over Y.

([Birkar-Cascini-Hacon-McKernan]의 결과를 이용해서 증명)

Sketch of proof:

STEP ① By [BCHM], or 지난 정리 3-10



φ : rel MMP on X/Y

X̃/Y good min model

X has rational Gorenstein sing

$\Leftrightarrow$ canonical Gorenstein

$\Rightarrow \tilde{X}$ has canonical singularities.

$\Rightarrow \tilde{f}_* \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}/X}) \simeq f_* \omega_{X/Y}^{\otimes m}$ for any $m \geq 0$

$\Rightarrow \tilde{f}_* \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}/X})$ 의 local freeness

$\Leftrightarrow$ 보이면 된다.

STEP 2

(2) (Adjunction & Inversion of adjunction)

fiber products

$$\begin{array}{ccccccccccc} X & \hookleftarrow & X_{H_1} & \hookleftarrow & X_{H_1 \cap H_2} & \hookleftarrow & \cdots & \hookleftarrow & X_{H_1 \cap \cdots \cap H_{n-2}} & \hookleftarrow & X_C \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ \forall p \in Y & \hookleftarrow & H_1 & \hookleftarrow & H_1 \cap H_2 & \hookleftarrow & \cdots & \hookleftarrow & H_1 \cap \cdots \cap H_{n-2} & \hookleftarrow & C \\ & & & & & & & & & & \| \\ & & & & & & & & & & H_1 \cap H_2 \cap \cdots \cap H_{n-1} \end{array}$$

H_i: general very ample Cartier divisor

$C = H_1 \cap \cdots \cap H_{n-1}$ = Smooth curve through p

$$\begin{array}{ccc} X & \hookleftarrow & X_C \\ \downarrow & & \downarrow \\ Y & \hookleftarrow & C \end{array} \quad \leftarrow \text{weakly semistable morphism}$$

$(\because X \to Y$ is $)$

$\Rightarrow X_C$ has rational Gorenstein sing.

(by inversion of adjunction)

$\Rightarrow (X_{H_1 \cap \cdots \cap H_{n-2}}, X_C)$ is plt near X_C

$\Rightarrow X_{H_1 \cap \cdots \cap H_{n-2}}$ is canonical Gorenstein.

(by I.A)

$\Rightarrow (X_{H_1 \cap \cdots \cap H_{n-3}}, X_{H_1 \cap \cdots H_{n-2}})$ is plt near $X_{H_1 \cap \cdots \cap H_{n-2}}$

$\Rightarrow X_{H_1 \cap \cdots \cap H_{n-3}}$ is canonical Gorenstein.

$\vdots$

$\Rightarrow (X, X_{H_1})$ is plt

Run relative MMP over Y

use inversion of adjunction

$$\begin{array}{ccccccccc} X & \hookleftarrow & X_{H_1} & \hookleftarrow & X_{H_1 \cap H_2} & \hookleftarrow & \cdots & \hookleftarrow & X_{H_1 \cap \cdots \cap H_{n-2}} \hookrightarrow X_C \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ \tilde{X} & \hookleftarrow & \tilde{X}_{H_1} & \hookleftarrow & \tilde{X}_{H_1 \cap H_2} & \hookleftarrow & \cdots & \hookleftarrow & \tilde{X}_{H_1 \cap \cdots \cap H_{n-2}} \hookleftarrow \tilde{X}_C \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ Y & \hookleftarrow & H_1 & \hookleftarrow & H_1 \cap H_2 & \hookleftarrow & \cdots & \hookleftarrow & H_1 \cap \cdots H_{n-2} \hookleftarrow C \end{array}$$

adjunction

(X, X_{H_1}) plt $\longrightarrow$ $(\tilde{X}, \tilde{X}_{H_1})$ plt

$\longrightarrow$ $\tilde{X}_{H_1}$ is normal & has canonical sing (by adjunction)

$(\tilde{X}_{H_1}, \tilde{X}_{H_1 \cap H_2})$ is plt

$\longrightarrow$ $\tilde{X}_{H_1 \cap H_2}$ is normal & has canonical sing

$\longrightarrow$ $\tilde{X}_C$ is normal & has canonical sing

$\tilde{X}_C \to C$ is dominant & flat

$\Rightarrow$ fibers are equidimensional

Since $p \in Y$ was arbitrary, $\tilde{f}: \tilde{X} \to Y$ has equidimensional fibers.

By construction and by applying adjunction repeatedly

$$\theta_{\tilde{X}}(mK_{\tilde{X}/X})\big|_{\tilde{X}_C} \simeq \theta_{\tilde{X}_C}(mK_{\tilde{X}_C/C})$$

By checking $\dim_{\mathbb{C}} H^0(\tilde{X}_y, \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}/Y}) \mid \tilde{X}_y)$ is independent of the choice of $y \in Y$,

we conclude $\tilde{f}_* \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}/Y})$ is locally free for $\forall m \geq 1$

so is $f_* \mathcal{O}_X(mK_{X/Y})$ by step ①. ∎

(ii)의 증명 스케치:

(ii) If, furthermore, Var(f)= dimY , then $f_* \omega_{X/Y}^{\otimes m}$ is nef and big locally free sheaf on Y for some k>0

즉, Conjecture Q 가 성립해서 $\Rightarrow C^+ \Rightarrow C$ (Iitaka Conjecture)

의견

- $\mathrm{Var}(f) := \min_K \mathrm{trans.deg}_K \in \{0, 1, 2, \dots \dim Y\}$.

$f: X \to Y$ alg fiber space

K = algebraically closed field, $\mathbb{C} \subseteq K \subseteq \overline{\mathbb{C}(Y)}$ for which there is a smooth projective variety V over K such that $V \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, \overline{\mathbb{C}(Y)} \overset{\text{bir}}{\dashrightarrow} X \times_Y \mathrm{Spec}\, \overline{\mathbb{C}(Y)}$

- Big sheaf

$\mathcal{F}$ = torsion free coherent sheaf.

$\mathcal{F}$ is big if there is an ample line bundle $\mathcal{H}$ & there is $n \in \mathbb{Z}_{>0}$ such that $\bigoplus_m \mathcal{H} \hookrightarrow \hat{S}^n(\mathcal{F})$ is generically isomorphic injection.

$f: X \to Y$ 가 weakly semistable 이고
$f_* \omega_{X/Y}^{\otimes m}$ 가 nef locally free sheaf 인 것을 가정.

$\Rightarrow f_* \omega_{X/Y}^{\otimes m}$ 가 nef 라서 $\det(f_* \omega_{X/Y}^{\otimes m}) =$ nef line bundle

$\Rightarrow \mathrm{Var}(f) = \dim Y$ 조건을 이용해서

$\left(\det(f_* \omega_{X/Y}^{\otimes m})\right)^{\dim Y} > 0$ 확인. 즉 $\det(f_* \omega_{X/Y}^{\otimes m})$ = big

$\Rightarrow f_* \omega_{X/Y}^{\otimes m}$ 도 big.