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담양4

강의 1,2,3에서...

algebraic fiber space $f: X \to Y$가

- smooth 일 경우(강의2), $f_* \omega_{X/Y}^{\otimes m}$ = nef locally finite sheaf
  $Var(f) = \dim Y$ 를 가정하면 $f_* \omega_{X/Y}^{\otimes m}$ 는 big

$\Rightarrow$ Conjecture Q

- smooth 가 아닌 일반적인 경우 (강의3), f의 weakly semistable reduction $f': X' \to Y'$에 대해서도

$f'_* \omega_{X'/Y'}^{\otimes m}$ = nef locally finite free sheaf

$Var(f) = \dim Y$ 를 가정하면 $f'_* \omega_{X'/Y'}^{\otimes m}$ 는 big

$\Downarrow$

$f_* \omega_{X/Y}^{\otimes m}$ 도 big.

$\Rightarrow$ Conjecture Q

"Conjecture Q $\Rightarrow$ Conjecture C"의 증명은 생략.

Settled cases of Iitaka Conjecture

An algebraic fiber space $f: X \to Y$ with general fiber F such that

1. F is of generel type,
(more generally F has a good minimal model)
2. Y is of general type, or
3. Y has maximal Albanese dimension.

There are many variations of Iitaka Conjecture.

Conjecture 1 (Popa)

If $f: X \to Y$ is a smooth algebraic fiber space, then

$$\dim F + \kappa(Y) \geq \kappa(X).$$

Equivalently, since dim F = dim X − dim Y ,

$$\dim X - \kappa(X) \geq \dim Y - \kappa(Y)$$
$$\| \qquad\qquad \|$$
$$\kappa_c(X) \qquad\qquad \kappa_c(Y).$$

- Not true if $f: X \to Y$ is not smooth.

There exists a surface S of general type with $q(S)=1$.
$\to$ $f: S \to E$ alg fiber space has a singular elliptic E. fiber.

정의

Let U be a smooth quasi-projective variety whcih can be considered as an open subset of some smooth projective variety X : $U \subseteq X$.
We can also assume that $X \setminus U = D$ is a simple normal crossing divisor on Y.
Now we define: $\kappa(U) := \kappa(Y, \omega_Y(D))$
$= \kappa_{Y+D}$

-independent of the compactification Y (연습문제 3 of Day 3).

Conjecture 2 (Popa)

Let $f: X \to Y$ be an algebraic fiber space which is smooth over an open subset $V \subseteq Y$. Then the following holds:

$$\kappa(V) + \kappa(F) \geq \kappa(X).$$

- Conj 2 implies Conj 1.

- Iitaka Conjecture + Popa's Conjecture 2
  $\Rightarrow$ If $f: X \to Y$ is smooth, then
  $\kappa(X) = \kappa(Y) + \kappa(F)$ holds.

- Conj 2 holds when Y is an abelian variety, Y=curve, Y=general type, dim Y-1= $\kappa(Y)$ 등등...

There is a numerical version of Iitaka conjecture.

numerical Kodaira dimension

$$\kappa_\sigma(X) = \max\left\{k \in \mathbb{Z}_{\geq 0} \;\middle|\; \limsup_{m\to\infty} \frac{h^0(X, mK_X + A)}{m^k} > 0\right\}$$

or $-\infty$ (A = suff general ample)

Conjecture

$f: X \to Y$ is an algebraic fiber space with gen fiber F.
Then

$$\kappa_\sigma(X) \geq \kappa_\sigma(F) + \kappa_\sigma(Y) \quad \text{holds}$$

- $\kappa(X) \leq \kappa_\sigma(X)$ holds in general.
- known if $\kappa_\sigma(F)$ is replaced by $\kappa(F)$ or $\kappa_\sigma(Y)$ by $\kappa(Y)$.
- Abundance Conjecture predicts that $\kappa(X) = \kappa_\sigma(X)$.

- "Okounkov body" was used in the proof.
  (An Okounkov body associated to a divisor D
  is a closed convex set in $\mathbb{R}^{\dim X}$ which
  encodes $\kappa(D)$ and volume $\mathrm{vol}(D)$, simultaneously.

Construction of Okounkov body

Given : a divisor $D = \sum a_i D_i$ on X

STEP 1. Fix an admissible flag $Y_\cdot$.

$$Y_\cdot : Y_1 \supset Y_2 \supset \cdots \supset Y_d = \{\text{pt}\}$$

Y_i = codim i irred subvar $\subseteq X$.

Y_i = smooth at Y_d.

STEP 2. Define a function

$$|D| = \{ D' \mid D \sim_{\mathbb{Q}} D' = \sum a_i D_i \geq 0 \}.$$

$$\longrightarrow \mathbb{R}^{\dim X}.$$

$$D' \longrightarrow (\nu_1, \nu_2, \ldots, \nu_d) = \nu_{Y_\cdot}(D')$$

$$\nu_1 = \text{mult}_{Y_1} D'$$

$$\nu_2 = \text{mult}_{Y_2}(D' - \nu_1 Y_1)|_{Y_1}$$

$$\vdots$$

$$\nu_d = \text{mult}_{Y_d}\left(\left((D' - \nu_1 Y_1)|_{Y_1} - \nu_2 Y_2 \right)|_{Y_2} \cdots \right)|_{Y_{d-1}}$$

STEP 3

$$\Delta_{Y_\cdot}(D) = \overline{\text{conv}\left(\{ \nu_{Y_\cdot}(D') \mid D' \in |D| \} \right)}^{\text{closure}}$$

Volumes also impose some restrictions on the fiber spaces.

$$\mathrm{Vol}(D) = \limsup_{m\to\infty} \frac{h^0(X, mD)}{m^d/d!} \qquad d = \dim X$$

Kawamata's volume formula

$f : X \to Y$ is an algebraic fiber space such that X, Y, F are of general type. Then

$$\frac{\mathrm{Vol}_X(K_X)}{\dim X!} \geq \frac{\mathrm{Vol}_Y(K_Y)}{\dim Y!} \cdot \frac{\mathrm{Vol}_F(K_F)}{\dim F!} \quad \text{holds.}$$

Also, "=" holds iff f is birationally isotrivial.

f is <u>birationally isotrivial</u> if X is a product of base × gen fiber.

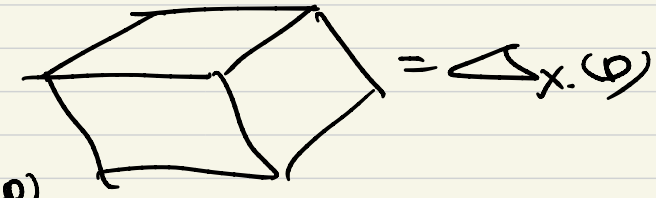
$$\begin{array}{ccc} X & \xleftarrow{} & X \times_Y Y' \\ \downarrow & & \downarrow \\ Y & \xleftarrow{\sigma} & Y' \end{array} \qquad \xrightarrow{h} \quad F \times Y'$$

σ = generically finite
h = birational.

정리 (Lazarsfeld-Mustaţă)

If D is big, then for an Okounkov body $\Delta_{X,\bullet}(D)$ for D with respect to any $X_\bullet$,

$$(\dim X)!\, \mathrm{vol}_{\mathbb{R}^{\dim X}}(\Delta_{X,\bullet}(D)) = \mathrm{vol}(D) \quad \text{holds.}$$

D on X $\longmapsto$  $= \Delta_{X,D}$

$\kappa(D) = \dim \Delta_{X,D}$

$\mathrm{Vol}(D) = \mathrm{Vol}_{\mathbb{R}^d}(\Delta_{X,D})\, d!$ $(d = \dim X)$

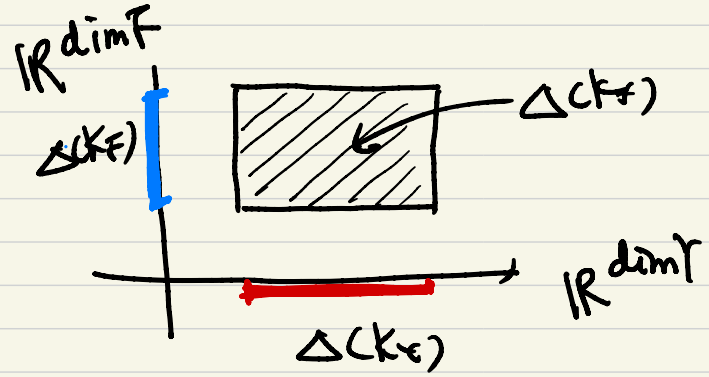
Okounkov body 를 이용한 Kawamata vol formula 증명스케치

$$\frac{\mathrm{Vol}(K_X)}{(\dim X)!} \geq \frac{\mathrm{Vol}(K_Y)}{(\dim Y)!} \cdot \frac{\mathrm{Vol}(K_F)}{(\dim F)!}$$

$$\Rightarrow \mathrm{Vol}_{\mathbb{R}^{\dim X}} \Delta_{X,D}(K_X) \geq \mathrm{Vol}_{\mathbb{R}^{\dim Y}} \Delta_Y(K_Y) \cdot \mathrm{Vol}_{\mathbb{R}^{\dim F}} \Delta_{F,D}(K_F).$$

$$\Rightarrow \Delta_{X,D}(K_X) \supseteq \Delta_{Y,D}(K_Y) \times \Delta_{F,D}(K_F)$$

- f = birationally isotrivial if and only if $\Delta_{X,D}(K_X)$ is a box shape.

$\Delta(K_X) =$ 

정리 [Chi-Kang Chang]

f:X→Y is an algebraic fiber space between normal projective Q-Gorenstein varieties with general fiber F. X has klt singularities and $-K_X$ is effective and the stable base locus $B(-K_f)$ does not dominate Y. Then

$$\kappa(-K) \leq \kappa(-K_X) + \kappa(-K_F).$$

이 외에도 algebraic fiber space 와 관련된
- Iitaka Conjecture over positive char field,
- Popa's Conjectures over positive char field,
- Kebekus-Kovac's Conjecture (settled by Taji)
- b-semiampleness conjecture of Prokhorov-Shokurov

등등 $\kappa(X)$, Var(f), isotriviality, volume 의 상관관계를 묻는 예상, 문제가 많다.

끝.