Fundamentals of

Quantum Phase Transitions

Contents

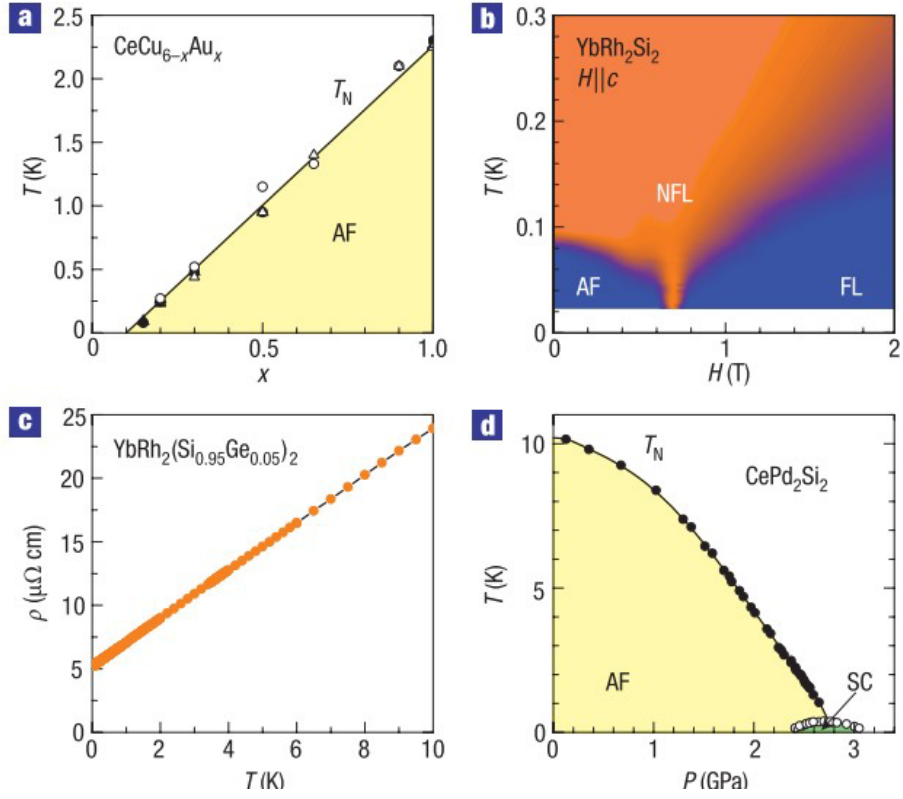
I. What is Quantum Phase Transition?

II. What do we know about Classical Phase Transitions?

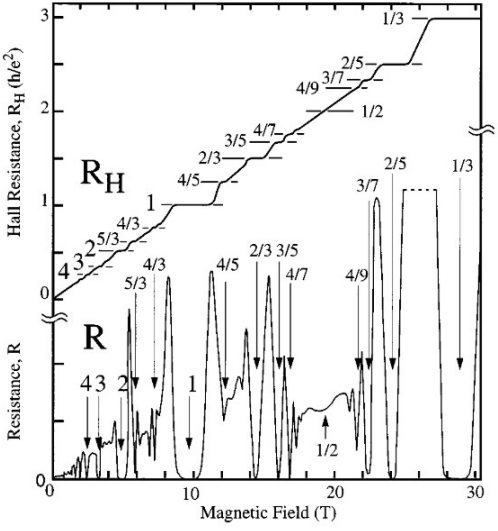
III. How can we understand Quantum Phase Transitions?

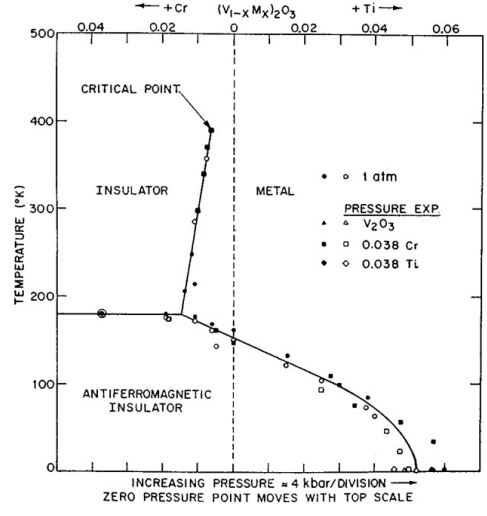
I. What is Quantum Phase Transition?

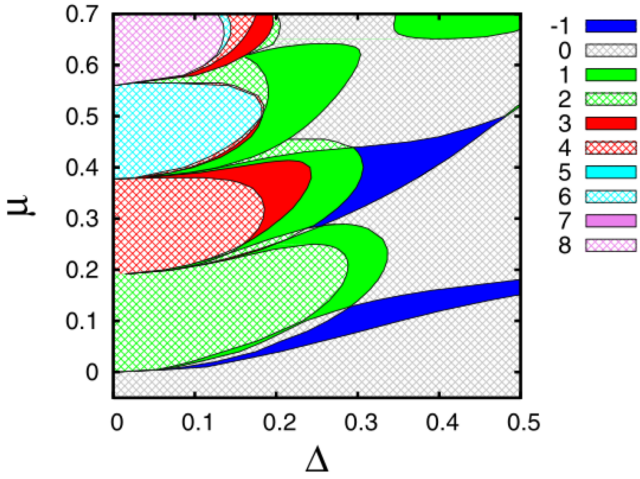
<Quantum phase transitions>

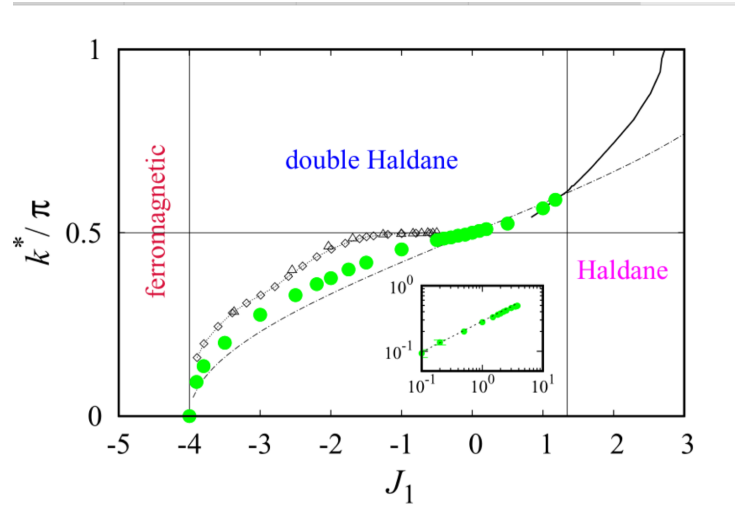


<Quantum phase transitions>






0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
μ
0
0.1
0.2
0.3
0.4
0.5
Δ
-1
0
1
2
3
4
5
6
7
8


1
0.5
0
k^*/π
ferromagnetic
double Haldane
Haldane
-5
-4
-3
-2
-1
0
1
2
3
J_1
10^0
10^{-1}
10^{-1}
10^0
10^1

< Quantum phase transitions >

- transitions at zero temperature
- competing ground states (quantum phases)
- driven by quantum fluctuations ($\Leftrightarrow$ thermal fluctuations)
- non-analyticity
  in the ground state energy of infinite-lattice system
- control parameters other than temperature - pressure, magnetic field, composition, ...

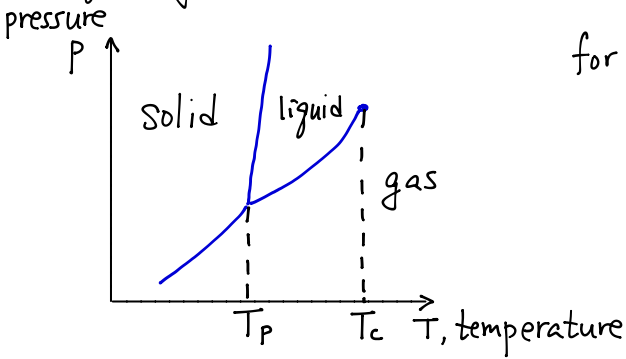
* quantum-classical mapping

d-dimensional quantum system
equivalent to (d+1)-dimensional classical system

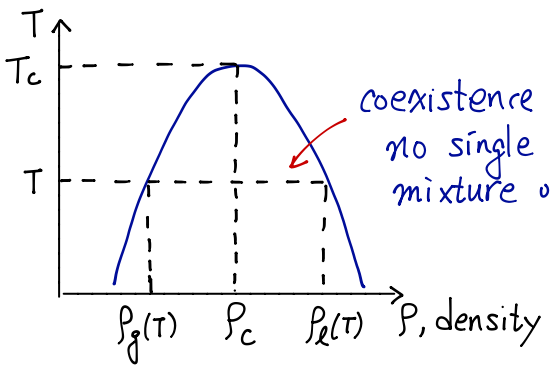
II. What do we know about Classical Phase Transitions?

phase transition : abrupt change of the system
by the variation of external parameters

<liquid-gas transition>



for $T_p < T < T_c$
gas → liquid as p ↑
"liquid-gas transition"



for $T > T_c$, continuous change in ρ
for $T < T_c$, abrupt change in ρ
$\rho_g \to \rho_\ell$
"order parameter"
$\Delta\rho(T) \equiv \rho_\ell(T) - \rho_g(T)$

<magnetic transition>

magnet { no magnetization (paramagnetic) for $T > T_c$
finite magnetization (ferromagnetic) for $T < T_c$

alignment of spins ⇒ ferromagnetism

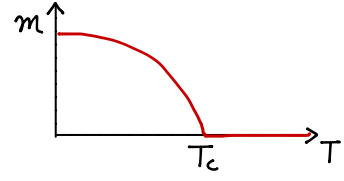
magnetization $\vec{m} \equiv \frac{1}{N} \sum_i \vec{S}_i$ "order parameter"

Two kinds of phase transitions

① continuous phase transition

order parameter is continuous at the transition

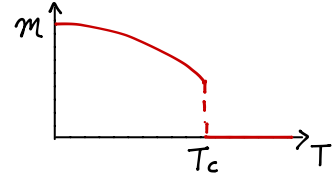
critical phenomena show up



② discontinuous transition

finite jump in order parameter

coexistence of two phases at the transition



In a conventional transition

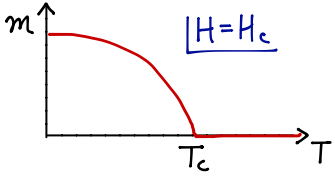
order parameter : first-order derivative of free energy

ex) $m = -\frac{\partial f}{\partial H}$ m : magnetization, f : free energy density, H : magnetic field

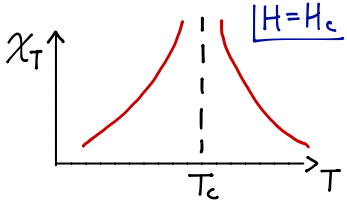
⇒ phase transition : singularity in free energy

some derivative of free energy is not continuous at transition

critical phenomena : at continuous transition, physical quantities show powe-law behaviors

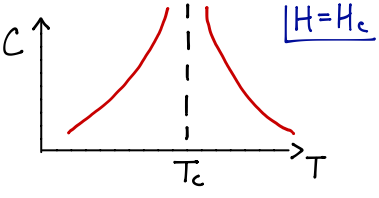


$m \sim (T_c - T)^{\beta}$

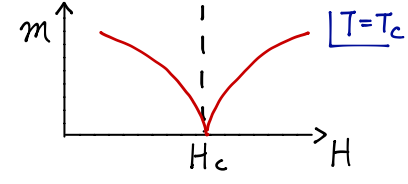


$\chi_T \sim |T - T_c|^{-\gamma}$

$C \sim |T - T_c|^{-\alpha}$



$m \sim |H - H_c|^{1/\delta}$



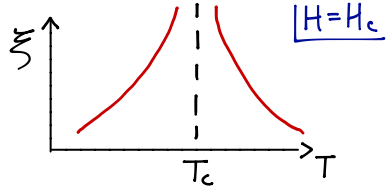
correlation length ξ

characteristic decay length for correlation function of order parameters

ex) corr. ftn. $G(\vec{r}-\vec{r}') = \langle \vec{S}(\vec{r})\vec{S}(\vec{r}') \rangle - \langle \vec{S}(\vec{r}) \rangle \cdot \langle \vec{S}(\vec{r}') \rangle$

generally $G(\vec{r}) \sim e^{-r/\xi}$: exponential decay

at critical point T_c $G(r) \sim \frac{1}{r^{d-2+\eta}}$: power-law decay $(\xi = \infty)$



$\xi \sim |T - T_c|^{-\nu}$

critical exponents : $\alpha, \beta, \gamma, \delta, \eta, \nu$

they depend only on { symmetries of the system
spatial dimension
short-rangeness of interactions

"universality class"

< mean-field theory >

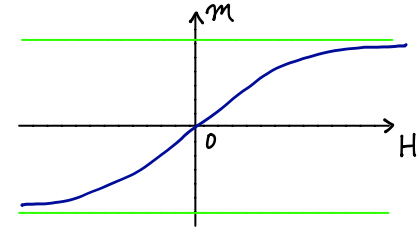
Weiss mean-field theory

nearest-neighbor Ising model

$$\mathcal{H}[S] = -H\sum_i S_i - J\sum_{\langle i,j\rangle} S_i S_j \qquad S_i = \pm 1,\ \langle i,j\rangle : \text{nearest-neighbor pairs}$$

i) $J = 0$: partition function $Z[H] = \sum_{\{S_i\}} \prod_i e^{-\beta H S_i} = \left[2\cosh(H/k_BT)\right]^N \qquad (\beta \equiv 1/k_BT)$

$$m = -\frac{1}{N}\frac{\partial F}{\partial H} = \frac{k_BT}{N}\frac{\partial}{\partial H}\ln Z[H] = \tanh\left(\frac{H}{k_BT}\right)$$



ii) $J \neq 0$: $\mathcal{H} \xrightarrow{\text{mean-field}} \mathcal{H}_{MF} = -\sum_i H_{MF}^{(i)} S_i$

$$\mathcal{H} = -\sum_i H_i S_i$$

$$H_i = H + J\sum_j{}' S_j = H + J\sum_j{}' \langle S_j\rangle + J\sum_j{}' (S_j - \langle S_j\rangle)$$

($\sum'$: n.n. of i)

neglect fluctuations from mean value

$$H_{MF}^{(i)} = H + zJm \qquad (z: \text{number of nearest neighbors},\ m = \langle S_i\rangle)$$

$$\therefore\ m = \tanh\left(\frac{H_{MF}^{(i)}}{k_BT}\right) = \tanh\left(\frac{H + zJm}{k_BT}\right)$$

Hamiltonian $\mathcal{H} = -H\sum_i S_i - J\sum_{\langle ij\rangle}(m+\tilde{S}_i)(m+\tilde{S}_j)$ $\quad(\tilde{S}_i \equiv S_i - m)$

$= -H\sum_i S_i - J\sum_{\langle ij\rangle}\left[m^2 + m(\tilde{S}_i+\tilde{S}_j) + \tilde{S}_i\tilde{S}_j\right]$ neglect

$\mathcal{H}_{MF} = -\sum_i S_i\,(H + \sum_j' Jm) + Jm^2\sum_{\langle ij\rangle}1 \;(=\tfrac{1}{2}zN)$

$H_{MF}^{(i)}$

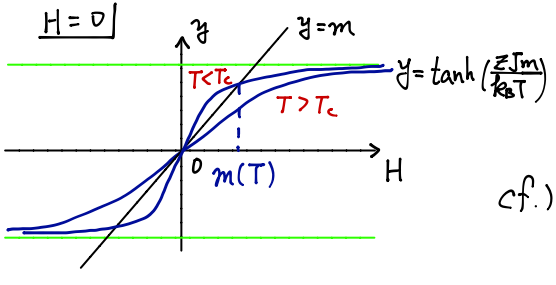
$= -(H+zJm)\sum_i S_i + \frac{1}{2}NzJm^2$

free energy $F = -k_BT \ln \sum_{\{S_i\}} e^{-\beta\mathcal{H}_{MF}} = \frac{1}{2}NzJm^2 - k_BTN\ln\left[2\cosh\left(\frac{H+zJm}{k_BT}\right)\right]$

minimize F : $\frac{\partial F}{\partial m} = 0 = NzJm - NzJ\tanh\left(\frac{H+zJm}{k_BT}\right)$

$\Rightarrow m = \tanh\left(\frac{H+zJm}{k_BT}\right)$

$H=0$



$y'(m=0)_{T=T_c} = 1 \Rightarrow \frac{zJ}{k_BT_c} = 1,\; T_c = \frac{zJ}{k_B}$

$\begin{cases} T > T_c : m = 0 \\ T < T_c : m \neq 0 \end{cases}$

cf.) $d=1,\ z=2$: $T_{c,MF}^{1D} = \frac{2J}{k_B}$, $T_{c,exact} = 0$

$d=2,\ z=4$ (square lattice) : $T_{c,MF}^{2D} = \frac{4J}{k_B}$ $\quad T_{c,exact} = \frac{2}{\ln(\sqrt{2}+1)}\frac{J}{k_B} \simeq 2.3\,J/k_B$

Critical phenomena in mean-field theory

$$m = \tanh\left(h + \frac{T_c}{T} m\right)$$

$$\Rightarrow h + \frac{T_c}{T} m = \tanh^{-1}(m) = m + \frac{1}{3} m^3 + \mathcal{O}(m^5)$$

$$h = t \cdot m + \frac{1}{3} m^3 \qquad \left(t \equiv 1 - \frac{T_c}{T} : \text{reduced temperature}, \; h \equiv \frac{H}{k_B T} \right)$$

① $H = 0$, $T \to T_c^-$: $m \sim (-t)^{\beta}$

$m\left(t + \frac{1}{3} m^2\right) = 0 \;\to\; m^2 = -3t$ $\qquad \beta = \frac{1}{2}$

② $T = T_c$, $H \to 0$: $m \sim |H|^{1/\delta}$

$h \simeq \frac{1}{3} m^3$ $\qquad \delta = 3$

③ $H = 0$, $T \to T_c$: $\chi_T \equiv \left(\frac{\partial m}{\partial H}\right)_T \sim |t|^{-\gamma}$

$\frac{\partial}{\partial H}$: $\frac{1}{k_B T} = \chi_T \left(t + m^2\right)$

i) $T \to T_c^+$: $m = 0$, $\chi_T \simeq \frac{1}{k_B (T - T_c)}$

ii) $T \to T_c^-$: $m \simeq \sqrt{-3t}$, $\chi_T \simeq \frac{1}{2 k_B (T_c - T)}$ $\qquad \gamma = 1$

④ $H = 0$, $T \to T_c$: $C_H \equiv \frac{1}{N} \frac{\partial E}{\partial T}\Big|_H \sim |t|^{-\alpha}$

$E = \langle \mathcal{H}_{MF} \rangle = -\frac{1}{2} N z J m^2$

i) $T > T_c$: $m = 0$, $E = 0$, $C_H = 0$

ii) $T \to T_c^-$: $m = \sqrt{-3t}$, $E \simeq \frac{3}{2} N k_B (T - T_c)$, $C_H \simeq \frac{3}{2} N k_B$ $\qquad \alpha = 0$

< phenomenological Landau theory>

Landau free energy $L[m]$ (m : order parameter)

① L should be consistent with symmetries of the system

② near the transition

L can be expanded in a power series of m

Landau free energy density

$$\mathcal{L} \equiv \frac{L}{V} = \sum_{n=0}^{\infty} a_n(K,T)\, m^n$$

③ find m_0 which minimizes $\mathcal{L}$

$m_0 = 0$: disordered phase ($T > T_c$)

$m_0 \neq 0$: ordered phase ($T < T_c$)

< Landau theory for Ising model >

near T_c $\quad \mathcal{L} = a_0 + a_1 m + a_2 m^2 + a_3 m^3 + a_4 m^4 + \cdots$ neglect

for $H=0$: $\mathcal{L}(m) = \mathcal{L}(-m) \Rightarrow a_1 = a_3 = 0$

$a_2(T) = a_2^{(0)} + t \cdot a_2^{(1)} + O(t^2)$ $\qquad t \equiv \frac{T - T_c}{T}$: reduced temperature

$a_4(T) = a_4^{(0)} + O(t)$

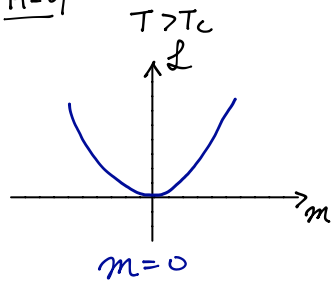
$\frac{\partial \mathcal{L}}{\partial m} = 0 = 2 a_2(t) m + 4 a_4 m^3 \Rightarrow m = 0$ or $\sqrt{\frac{-a_2(t)}{2a_4}}$

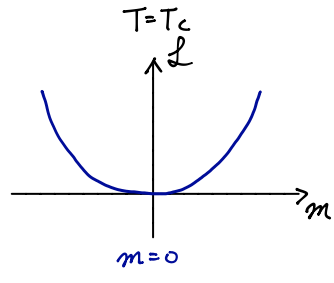
if $m = \begin{cases} 0 & t > 0 \\ \text{finite} & t < 0 \end{cases}$, $\Rightarrow a_2^{(0)} = 0 \quad a_4^{(0)} > 0, \quad a_2^{(1)} > 0$

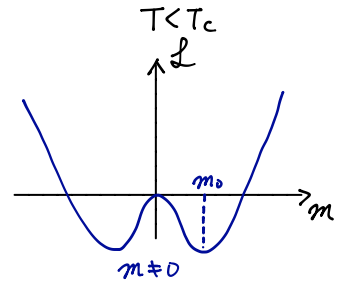
for $H \neq 0$, add $-Hm$ $\qquad$ cf.) $-H \sum_i S_i$

$\Rightarrow \mathcal{L} = -Hm + at m^2 + \frac{1}{2} b m^4 \quad (a \equiv a_2^{(1)}, \; b = 2 a_4^{(0)})$

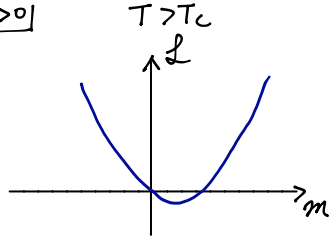
H=0

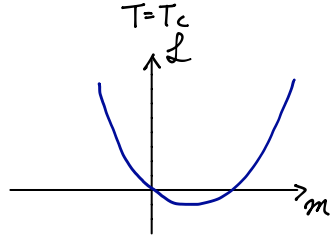


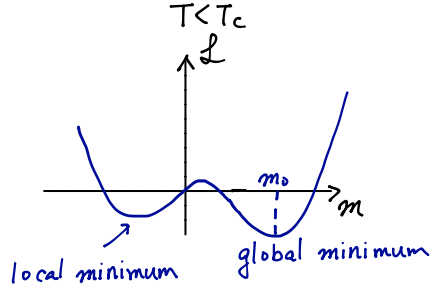




H>0

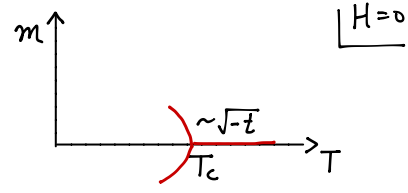


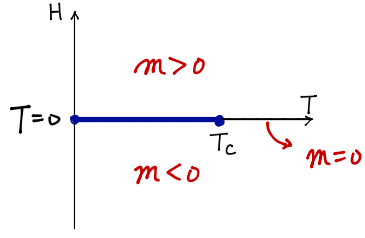




$$\frac{\partial \mathcal{L}}{\partial m} = 0 = -H + 2at\,m + 2bm^3$$

$H=0$: $m = \begin{cases} \cancel{0}\ \text{unstable}, \pm\sqrt{\frac{a}{b}(-t)} & t<0 \quad (T<T_c) \\ 0 & t>0 \quad (T>T_c) \end{cases}$





critical exponents

① $t<0$: $m \sim \sqrt{-t}$ $\quad \beta = \frac{1}{2}$

② $H=0$: $\mathcal{L} = \begin{cases} 0 & (t>0) \\ -\frac{a^2}{2b}t & (t>0) \end{cases} \Rightarrow C = -T\frac{\partial^2 \mathcal{L}}{\partial T^2} = \begin{cases} 0 & t>0 \\ \frac{a^2}{b}\cdot\frac{1}{T_c} & t<0 \end{cases}$

: finite jump $\quad \alpha = 0$

③ $t=0$: $H = 2bm^3 \Rightarrow \delta = 3$

④ $0 = \frac{\partial}{\partial H}\left(-H + 2atm + 2bm^3\right)\Big|_t = -1 + \left(\frac{\partial m}{\partial H}\right)_T (2at + 6bm^2)$

$$\chi_T = \left(\frac{\partial m}{\partial H}\right)_T = \frac{1}{2at^2 + 6bm^2}$$

$$= \begin{cases} \frac{1}{2at} & t>0 \\ -\frac{1}{4at} & t<0 \end{cases} \qquad \gamma = 1$$

if no symmetry on m

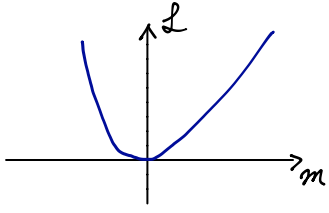
$$\mathcal{L} = -Hm + at\,m^2 + \frac{1}{2}b\,m^4 + a_3 m^3$$

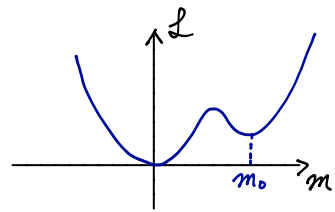
$H=0$: $\dfrac{\partial \mathcal{L}}{\partial m} = (2at + 3cm + 2bm^2)m = 0$

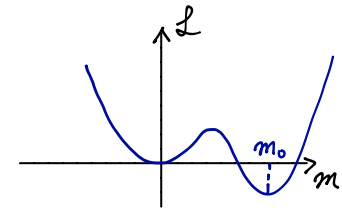
$$m = 0, \quad -c \pm \sqrt{c^2 - \frac{at}{b}} \qquad \left(c \equiv \frac{3a_3}{4b}\right)$$

$t < t^* \equiv \dfrac{bc^2}{a}$: two maxima and one minimum

$c<0$





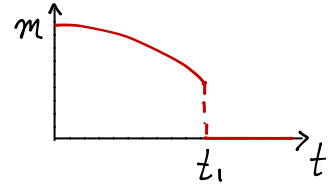


discontinuous jump at $t = t_1$: transition temperature

$m_0 = -c + \sqrt{c^2 - at/b}$

$\mathcal{L}(m_0) = m_0^2\left(at^2 + \dfrac{4bc}{3}m_0 + \dfrac{1}{2}bm_0^2\right) = 0$

$\Rightarrow t_1 = \dfrac{8bc^2}{9a}, \quad m_0(t_1) = -\dfrac{4}{3}c$



Allow some spatial variation of m

"local order parameter" $m(\vec{r})$

Landau free energy functional

$$L[m(\vec{r})] \equiv \int d^d\vec{r}\ \mathcal{L}(m(\vec{r}))$$

$$\mathcal{L}(m(\vec{r})) = at m^2 + \frac{1}{2} b m^4 - H(\vec{r}) m(\vec{r}) + \frac{1}{2}\gamma (\vec{\nabla} m(\vec{r}))^2$$

order parameter

$$\langle m(\vec{r})\rangle = -\frac{\delta F}{\delta H(\vec{r})} = k_B T \frac{1}{Z}\frac{\delta Z}{\delta H(\vec{r})} \qquad (Z: \text{partition function})$$

generalized isothermal susceptibility

$$\chi_T(\vec{r},\vec{r}') \equiv \frac{\delta \langle m(\vec{r})\rangle}{\delta H(\vec{r}')} = k_B T \frac{\delta}{\delta H(\vec{r}')}\left[\frac{1}{Z}\frac{\delta Z}{\delta H(\vec{r})}\right] = \frac{1}{k_B T}\left[\frac{1}{Z}\frac{\delta^2 Z}{\delta H(\vec{r})\delta H(\vec{r}')} - \frac{1}{Z}\frac{\delta Z}{\partial H(\vec{r})}\frac{1}{Z}\frac{\delta Z}{\partial H(\vec{r}')}\right]$$

$$= \frac{1}{k_B T}\left[\langle m(\vec{r}) m(\vec{r}')\rangle - \langle m(\vec{r})\rangle\langle m(\vec{r}')\rangle\right] \equiv \frac{1}{k_B T} G(\vec{r},\vec{r}') \quad \text{"correlation function"}$$

translationally invariant system $\quad G(\vec{r}-\vec{r}') = k_B T\, \chi_T(\vec{r}-\vec{r}')$

Fourier transf. $\quad \tilde{G}(\vec{k}) = k_B T\, \tilde{\chi}_T(\vec{k})$

static susceptibility $\quad \chi_T \equiv \lim_{\vec{k}\to 0} \tilde{\chi}_T(\vec{k}) = \frac{1}{k_B T}\tilde{G}(\vec{k}=0) = \frac{1}{k_B T}\int d^d\vec{r}\, G(\vec{r})$

correlation function

$$L = \int d^d\vec{r}\left[\frac{1}{2}\gamma(\vec{\nabla}m(\vec{r}))^2 + at\,m^2 + \frac{1}{2}bm^4 - H(\vec{r})m(\vec{r})\right]$$

$$\frac{\delta L}{\delta m(\vec{r})} = 0 \Rightarrow \quad -\gamma\vec{\nabla}^2 m(\vec{r}) + 2at\,m(\vec{r}) + 2b\,m(\vec{r})^3 - H(\vec{r}) = 0$$

$$\frac{\delta}{\delta H(\vec{r}')}\otimes: \quad \left[-\gamma\vec{\nabla}^2 + 2at + 6b\,m(\vec{r})^2\right]\chi_T(\vec{r}-\vec{r}') = \delta(\vec{r}-\vec{r}')$$

$$\frac{1}{k_BT}\left[-\gamma\vec{\nabla}^2 + 2at + 6b\,m(\vec{r})^2\right]\; G(\vec{r}-\vec{r}') = \delta(\vec{r}-\vec{r}')$$

for $t>0$, $m=0$: $\quad (-\vec{\nabla}^2 + \xi_>^{-2})\,G(\vec{r}-\vec{r}') = \frac{k_BT}{\gamma}\delta(\vec{r}-\vec{r}')$ $\qquad \xi_>(t) \equiv \left(\frac{\gamma}{2at}\right)^{1/2}$

for $t<0$, $m=\pm\sqrt{-\frac{at}{b}}$: $\quad (-\vec{\nabla}^2 + \xi_<^{-2})\,G(\vec{r}-\vec{r}') = \frac{k_BT}{\gamma}\delta(\vec{r}-\vec{r}')$ $\qquad \xi_<(t) \equiv \left(-\frac{\gamma}{4at}\right)^{1/2}$

$\xi(t)$: correlation length $\sim t^{-1/2}$ $\qquad \nu = \frac{1}{2}$

$$(-\vec{\nabla}^2 + \xi^{-2})\, G(\vec{r}-\vec{r}') = \frac{k_B T}{\gamma}\, \delta(\vec{r}-\vec{r}')$$

$$\xrightarrow{F.T.} (k^2 + \xi^{-2})\, \tilde{G}(\vec{k}) = \frac{k_B T}{\gamma}$$

$t=0\ (T=T_c)$: $\xi = \infty \longrightarrow \tilde{G}(\vec{k}, T_c) = \frac{k_B T}{\gamma}\frac{1}{k^2}$

$$G(\vec{r}, T_c) \propto \int d^d\vec{k}\; e^{i\vec{k}\cdot\vec{r}}\, \frac{1}{k^2} \sim \frac{1}{r^{d-2}} \int d^d\vec{\rho}\; e^{i\vec{\rho}\cdot\hat{r}}\, \frac{1}{\rho^2} \qquad (\vec{\rho} \equiv \vec{k} r)$$

at $T=T_c$, $G(r) \sim \frac{1}{r^{d-2+\eta}}$ $\qquad \eta = 0$

critical exponents for Ising model in Landau theory

$\alpha = 0,\ \beta = \frac{1}{2},\ \gamma = 1,\ \delta = 3,\ \nu = \frac{1}{2},\ \eta = 0$

< critical exponents for n.n. Ising model >

	mean-field	2D Ising	3D Ising
α	0	0	0.110(5)
β	$\frac{1}{2}$	$\frac{1}{8}$	0.325(2)
γ	1	$\frac{7}{4}$	1.2405(15)
δ	3	15	4.82(4)
ν	$\frac{1}{2}$	1	0.630(2)
η	0	$\frac{1}{4}$	0.032(3)

< Breakdown of Landau theory >

Ginzburg criterion

$$E_{LG} \equiv \frac{\left| \int_V d^d\vec{r}\, G(\vec{r}) \right|}{\int_V d^d\vec{r}\, m(\vec{r})^2} \ll 1 \qquad (V \equiv \xi(T)^d)$$

$$\int_V d^d\vec{r}\, m(\vec{r})^2 \simeq m_0^2 \cdot \xi(T)^d \simeq \frac{a}{b}|t| \cdot \xi_0^d |t|^{-\frac{d}{2}} \qquad (\xi(T) \equiv \xi_0 |t|^{-\frac{1}{2}})$$

$$\int_V d^d\vec{r}\, G(\vec{r}) \simeq k_B T_c \chi_T \simeq \frac{k_B T_c}{4a|t|}$$

$$\Rightarrow E_{LG} = \frac{k_B}{4\,\Delta C\, \xi_0^d\, |t|^{(4-d)/2}} \qquad (\Delta C \equiv \frac{a^2}{b}\frac{1}{T_c})$$

$$E_{LG} \ll 1 \iff |t|^{(4-d)/2} \gg \frac{k_B}{4\Delta C \xi_0^d}$$

i) $d > 4$: satisfied as $t \to 0$, correct critical phenomena

ii) $d < 4$: Landau theory is not self-consistent as $t \to 0$

$d = 4$: "upper critical dimension"

beyond which critical phenomena from Landau theory are correct

<upper critical dimension>

in general

$$\int_V d^d\vec{r}\, G(\vec{r}) \sim \chi_T \sim |t|^{-\gamma}$$

$$\int_V d^d\vec{r}\, m(\vec{r})^2 \sim \xi^d m_0^2 \sim |t|^{2\beta-\nu d}$$

$$E_{LG} \ll 1 \rightarrow |t|^{-\gamma} \ll |t|^{2\beta-\nu d} \quad \text{as} \quad t \to 0$$

$$\gamma < \nu d - 2\beta \Rightarrow d > \frac{\gamma+2\beta}{\nu} \equiv d_{UC}$$

$$\left(= \frac{2-\alpha}{\nu}\right) \quad \text{cf.) } \alpha+2\beta+\gamma = 2.$$

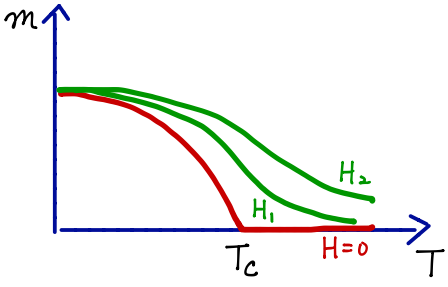
< Static Scaling Hypothesis >

(B. Widom '65)

$$m(t,h) = \begin{cases} t^{\beta} F_m^{+}(h/t^{\Delta}) & \text{for } t>0 \\ (-t)^{\beta} F_m^{-}(h/(-t)^{\Delta}) & \text{for } t<0 \end{cases}$$

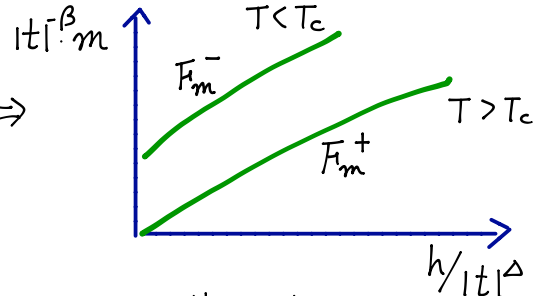
$$= |t|^{\beta} F_m^{\pm}(h/|t|^{\Delta})$$

: scaling hypothesis for order parameter



one curve for each H

$\Rightarrow$



all results collapse onto two curves

data collapse is perfect only for correct T_c, β, Δ

<Scaling Laws from Scaling Hypothesis>

$$m(t,h) = \begin{cases} t^{\beta} F_m^{+}(h/t^{\Delta}) & \text{for } t>0 \\ (-t)^{\beta} F_m^{-}(h/(-t)^{\Delta}) & \text{for } t<0 \end{cases}$$

① time-reversal symmetry $m(t,h) = -m(t,-h)$

$\Rightarrow F_m^{\pm}(x) = -F_m^{\pm}(-x)$: odd function

② limit of $h \to 0$: $m(t,0) = \begin{cases} 0 & \text{for } t>0 \\ (-t)^{\beta} \cdot \text{cnst} & \text{for } t<0 \end{cases}$

$\Rightarrow F_m^{+}(0) = 0$, $F_m^{-}(x) \to \text{cnst}$ as $x \to 0$

③ susceptibility $\chi_T(H=0) = \left(\frac{\partial m}{\partial H}\right)_{H=0} \sim |t|^{\beta-\Delta} F_m^{\pm\prime}(0)$

$\Rightarrow -\gamma = \beta - \Delta$ $\quad \Delta = \beta + \gamma$

④ limit of $h \to 0$ at T_c $(t=0)$: $m(0,h) \sim h^{1/\delta}$

if $F_m^{\pm}(x) \sim x^{\lambda}$ as $x \to \infty$

$t \to 0$: $m(0,h) \sim |t|^{\beta} \cdot \left(\frac{h}{|t|^{\Delta}}\right)^{\lambda} = h^{\lambda} \cdot |t|^{\beta - \lambda\Delta}$

$\Rightarrow \beta - \lambda\Delta = 0$, $\lambda = 1/\delta$ $\quad \Delta = \beta\cdot\delta$

$\beta\delta = \beta + \gamma$

Griffith

< Scaling Hypothesis for Free Energy >

$$f_s(t,h) = (-t)^{2-\alpha} F_f\left(h/(-t)^{\Delta}\right) \qquad (t<0)$$

↖ singular part of free energy

① heat capacity : $C_H \sim -T\left(\frac{\partial^2 f_s}{\partial T^2}\right)_{h=0} \sim (-t)^{-\alpha}$: $F_f(0) = \text{const}$

② magnetization $m = -\frac{\partial f_s}{\partial H} \sim (-t)^{2-\alpha-\Delta} F_f'\left(\frac{h}{(-t)^{\Delta}}\right)$

$h \to 0$: $2-\alpha-\Delta = \beta$

③ susceptibility $\chi_T = \frac{\partial m}{\partial H} \sim (-t)^{2-\alpha-2\Delta} F_f''\left(\frac{h}{(-t)^{\Delta}}\right)$

$2-\alpha-2\Delta = -\gamma$

$\Rightarrow$ $\alpha+2\beta+\gamma=2$ Rushbrooke

$\beta\delta = \beta+\gamma = 2-\alpha-\beta$ Griffith

< Scaling Hypothesis for Correlation Function >

$$G(\vec{r}, t, h) = \frac{1}{r^{d-2+\eta}} F_G\left(\vec{r}\,|t|^{\nu}, \frac{h}{|t|^{\Delta}}\right)$$

$\uparrow \vec{r}/\xi$

$$\chi_T \sim \int G(\vec{r})\, d^d\vec{r} \sim \xi^{-(d-2+\eta)} \xi^d \sim \xi^{2-\eta} \sim |t|^{-\nu(2-\eta)}$$

$\gamma = \nu(2-\eta)$: Fisher

dimensional analysis $[f_s] = L^{-d}$

$$\Rightarrow \frac{f_s}{k_B T} \sim \xi^{-d} \sim |t|^{\nu d} \quad \therefore \ \nu d = 2-\alpha \quad \text{Josephson}$$

$$\frac{\delta-1}{\delta+1} = \frac{\gamma/\beta}{(2-\alpha)/\beta} = \frac{\gamma}{2-\alpha} \Rightarrow d\,\frac{\delta-1}{\delta+1} = d\,\frac{\gamma}{2-\alpha} = \frac{d\nu}{2-\alpha}\cdot(2-\eta) = 2-\eta \quad : \text{B-G}$$

↑ Griffith ↑ Fisher ↑ Josephson

III. How can we understand Quantum Phase Transitions?

< Quantum Ising Model >

Hamiltonian $\hat{\mathcal{H}}_I = -J \sum_{\langle i,j \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z - gJ \sum_i \hat{\sigma}_i^x$ (spin $\frac{1}{2}$)

- $\langle i,j \rangle$: nearest neighbor (n.n.) pairs
- 1st term ($J>0$) : ferromagnetic n.n interactions along z-dir.
- 2nd term : magnetic field along x-dir.
- g : dimensionless coupling

at site i $\quad \hat{\sigma}_i^z |\uparrow\rangle_i = (+1)|\uparrow\rangle_i, \quad \hat{\sigma}_i^z |\downarrow\rangle_i = (-1)|\downarrow\rangle_i$

$$\hat{\sigma}_i^x |\uparrow\rangle_i = |\downarrow\rangle_i, \quad \hat{\sigma}_i^x |\downarrow\rangle_i = |\uparrow\rangle_i$$

eigenstates of $\hat{\sigma}_i^x$: $|\rightarrow\rangle_i = \frac{1}{\sqrt{2}} (|\uparrow\rangle_i + |\downarrow\rangle_i) \quad (\sigma_i^x = +1)$

$$|\leftarrow\rangle_i = \frac{1}{\sqrt{2}} (|\uparrow\rangle_i - |\downarrow\rangle_i) \quad (\sigma_i^x = -1)$$

$$[\hat{\sigma}_j^\alpha, \hat{\sigma}_k^\beta] = 2i \delta_{jk} \varepsilon_{\alpha\beta\gamma} \hat{\sigma}_j^\gamma \qquad (\alpha, \beta = x, y, z)$$

< Mapping : 1D classical Ising chain vs. quantum Ising spin >

1-dim nearest-neighbor Ising model (classical)

$$-\beta \mathcal{H}[S] = h \sum_i S_i + K \sum_i S_i S_{i+1} \qquad (S_i = \pm 1)$$

partition function (M spins)

$$Z(h,K) = \mathrm{Tr}\, \exp\left[h \sum_i S_i + K \sum_i S_i S_{i+1} \right]$$

$$\frac{1}{2}\sum_i (S_i + S_{i+1})$$

$$= \sum_{\{S_i\}} \prod_i \exp\left[\frac{1}{2} h (S_i + S_{i+1}) + K S_i S_{i+1} \right]$$

$$= \sum_{S_1} \sum_{S_2} \cdots \sum_{S_M} T_{S_1 S_2} T_{S_2 S_3} \cdots T_{S_M S_1}$$

$$T_{SS'} \equiv \exp\left[\frac{1}{2} h (S + S') + K S S' \right]$$

transfer matrix

$$\hat{T} = \begin{pmatrix} T_{11} & T_{1-1} \\ T_{-11} & T_{-1-1} \end{pmatrix} = \begin{pmatrix} e^{h+K} & e^{-K} \\ e^{-K} & e^{-h+K} \end{pmatrix}$$

$$\Rightarrow Z(h,K) = \sum_{S_1} \left(\hat{T}^M \right)_{S_1 S_1} = \mathrm{tr}\left(\hat{T}^M \right)$$

diagonalization

$$\hat{D} \equiv \hat{S}^{-1}\hat{T}\hat{S} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$Z(h,K) = tr\underbrace{\left(\hat{S}^{-1}\hat{T}\hat{S}\;\; \hat{S}^{-1}\hat{T}\hat{S}\cdots\hat{S}^{-1}\hat{T}\hat{S}\right)}_{M} = tr(\hat{S}^{-1}\hat{D}^{M}\hat{S})$$

$$= tr(\underbrace{\hat{S}\hat{S}^{-1}}_{=\mathbb{1}}\hat{D}^{M}) = tr(\hat{D}^{M}) = \lambda_1^M + \lambda_2^M \qquad (\lambda_1 > \lambda_2)$$

$$|\lambda\mathbb{1} - \hat{T}| = 0 = \begin{vmatrix} \lambda - e^{h+K} & -e^{-K} \\ -e^{-K} & \lambda - e^{-h+K} \end{vmatrix} = \lambda^2 - 2e^{K}\cosh h\,\lambda + e^{2K} - e^{-2K}$$

eigenvalues $\lambda_{1,2} = e^{K}\cosh h \pm \sqrt{e^{2K}\cosh^2 h - e^{2K} + e^{-2K}}$

$$= e^{K}\left[\cosh h \pm \sqrt{\sinh^2 h + e^{-4K}}\right]$$

$$Z = \lambda_1^M + \lambda_2^M = \lambda_1^M\left[1 + \left(\frac{\lambda_2}{\lambda_1}\right)^M\right] \xrightarrow[M\to\infty]{} \lambda_1^M$$

free energy per spin

$$f = -k_B T \ln \lambda_1 = -J - k_B T \ln\left[\cosh h + \sqrt{\sinh^2 h + e^{-4K}}\right]$$

spin correlation function

$$G(i,j) \equiv \langle (S_i - \langle S_i \rangle)(S_j - \langle S_j \rangle) \rangle \quad \text{correlation in fluctuations}$$

$$= \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle$$

$$\langle S_i \rangle = \frac{1}{Z} \mathrm{Tr}\left[S_i e^{-\beta \mathcal{H}} \right] = \frac{1}{Z} \mathrm{Tr}\left[T_{S_1 S_2} \cdots \underbrace{T_{S_{i-1} S_i} S_i T_{S_i S_{i+1}}} \cdots \right]$$

$$= \frac{1}{Z} tr(\hat{\sigma}_3 \hat{T}^M) \qquad \left[\hat{\sigma}_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]$$

$$= \frac{1}{Z} tr(\hat{S}^{-1} \hat{\sigma}_3 \hat{S} \hat{D}^M)$$

$$= T_{S_{i-1} S_i} S_i \delta_{S_i S_i'} T_{S_i' S_{i+1}}$$

$$= (\hat{T} \hat{\sigma}_3 \hat{T})_{S_i S_{i+1}}$$

two normalized eigenvectors of $\hat{T}$: $\vec{v}_1 = \begin{pmatrix} a \\ b \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} c \\ d \end{pmatrix}$

$$\hat{S} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \Rightarrow \hat{S}^{-1} \hat{\sigma}_3 \hat{S} = \begin{pmatrix} e & g \\ f & k \end{pmatrix}$$

$$\langle S_i \rangle = \frac{e\lambda_1^M + k\lambda_2^M}{\lambda_1^M + \lambda_2^M} \xrightarrow{M \to \infty} e \quad (k, h)$$

$$\langle S_i S_{i+j} \rangle = \frac{1}{Z} tr\left(\hat{\sigma}_3 \hat{T}^j \hat{\sigma}_3 \hat{T}^{M-j} \right)$$

$$= \frac{1}{Z} tr\left(\hat{S}^{-1} \hat{\sigma}_3 \hat{S} \hat{D}^j \hat{S}^{-1} \hat{\sigma}_3 \hat{S} \hat{D}^{M-j} \right)$$

$$= \frac{\lambda_1^M}{\lambda_1^M + \lambda_2^M} \left[e^2 + f \cdot g \left\{ \left(\frac{\lambda_2}{\lambda_1}\right)^j + \left(\frac{\lambda_2}{\lambda_1}\right)^{M-j} \right\} + k^2 \left(\frac{\lambda_2}{\lambda_1}\right)^M \right]$$

$$\xrightarrow[M\to\infty]{} e^2 + f \cdot g \left(\frac{\lambda_2}{\lambda_1}\right)^j$$

$$G(i, i+j) = \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle = f \cdot g \left(\frac{\lambda_2}{\lambda_1}\right)^j \sim e^{-j/\xi}$$

correlation length $\xi = \dfrac{1}{\ln(\lambda_1/\lambda_2)}$

for $h = 0$, $\lambda_1 = 2\cosh K$, $\lambda_2 = 2\sinh K$

$$\therefore \xi = \frac{1}{\ln \coth K}$$

scaling limit (critical region)

system size $L = Ma$ (a: lattice spacing)

$\xi/a \gg 1$: $a \to 0$, $\tilde{h} \equiv h/a$, L : fixed

$$\frac{a}{\xi} = \ln \frac{e^{K}+e^{-K}}{e^{K}-e^{-K}} \simeq 2e^{-2K} \quad (K \gg 1)$$

$$\lambda_{1,2} = e^{K}\left[\cosh h \pm \sqrt{\sinh^2 h + e^{-4K}}\right] \simeq \left(\frac{2\xi}{a}\right)^{1/2}\left[1 \pm \left\{(a\tilde{h})^2 + \left(\frac{a}{2\xi}\right)^2\right\}^{1/2}\right]$$

$$\simeq \left(\frac{2\xi}{a}\right)^{1/2}\left[1 \pm \left(\frac{a}{2\xi}\right)(1+4\tilde{h}^2\xi^2)^{1/2}\right]$$

$$f \equiv -\frac{k_B T}{L}\ln Z = -\frac{k_B T}{L}\ln\left[\varepsilon_1^{M} + \varepsilon_2^{M}\right]$$

$$\simeq -\frac{J}{a} - \frac{k_B T}{L}\ln\left[2\cosh\left\{\left(\frac{L^2}{4\xi^2} + L^2\tilde{h}^2\right)^{1/2}\right\}\right]$$

$$= \varepsilon_0 + \frac{k_B T}{L}\underbrace{\Phi_f\left(\frac{L}{\xi}, \tilde{h}L\right)}$$

universal scaling function

dep. on two dimensionless ratios L/ξ, $L/(1/\tilde{h})$

< Mapping : 1D classical Ising chain vs. quantum Ising spin >

$$\hat{T} = \begin{pmatrix} e^{h+K} & e^{-K} \\ e^{-K} & e^{-h+K} \end{pmatrix} = e^{K}\left(e^{h\hat{\sigma}^z} + e^{-2K}\hat{\sigma}^x \right)$$

$$= e^{K}\left(e^{a\tilde{h}\hat{\sigma}^z} + \frac{a}{2\xi}\hat{\sigma}^x \right) \simeq e^{K}\left[1 + a\left(\tilde{h}\hat{\sigma}^z + \frac{1}{2\xi}\hat{\sigma}^x \right) \right] + \mathcal{O}(a^2)$$

$$= e^{-a\hat{\mathcal{H}}_Q} \quad \left(\hat{\mathcal{H}}_Q \equiv -\frac{1}{2\xi}\hat{\sigma}^x - \tilde{h}\hat{\sigma}^z - \frac{K}{a} \right)$$

$$Z = \mathrm{Tr}\,\hat{T}^M \simeq \mathrm{Tr}\, e^{-Ma\hat{\mathcal{H}}_Q} \equiv \mathrm{Tr}\, e^{-\hat{\mathcal{H}}_Q/T}$$

$$\hat{\mathcal{H}}_Q \equiv -\frac{\Delta}{2}\hat{\sigma}^x - \tilde{h}\hat{\sigma}^z + E_0$$

$$\left(\frac{1}{T} \equiv Ma = L_\tau \,, \quad \Delta \equiv \frac{1}{\xi} \right)$$

1D classical Ising chain

$$-\beta \mathcal{H} = K \sum_i \sigma_i^z \sigma_{i+1}^z + h \sum_i \sigma_i^z$$

length of chain L_z

inverse correlation length ξ_z^{-1}

(imaginary time direction)

quantum Ising spin

$$\hat{\mathcal{H}}_Q \equiv -\frac{\Delta}{2} \hat{\sigma}^x - \tilde{h} \hat{\sigma}^z$$

inverse temperature $\frac{1}{T}$

energy gap $\Delta = E_1 - E_0$

< quantum-classical mapping >

partition function of quantum Ising chain ($d=1$)

$$Z = \mathrm{Tr}\, e^{-\hat{\mathcal{H}}_I/T} = \mathrm{Tr}\, \exp[-a\hat{\mathcal{H}}_I]^{M_\tau} \qquad (L_\tau = M_\tau a = 1/T)$$

$$\exp[-a\hat{\mathcal{H}}_I] \simeq \hat{T}_1 \hat{T}_2 + \mathcal{O}(a^2)$$

$$\hat{T}_1 \equiv \exp\Big[Jga \sum_i \hat{\sigma}_i^x\Big], \quad \hat{T}_2 \equiv \exp\Big[Ja \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z\Big]$$

2^M basis states $|\{m_i\}\rangle$ $(m_i = \pm 1)$

$$\hat{\sigma}_j^z |\{m_i\}\rangle = m_j |\{m_i\}\rangle$$

$$\Rightarrow \hat{T}_2 |\{m_i\}\rangle = \exp\Big[Ja \sum_j m_j m_{j+1}\Big] |\{m_i\}\rangle$$

$$Z \simeq \sum_{\{m_i(l)\}} \prod_{l=1}^{M_\tau} \langle \{m_i(l)\}| \hat{T}_1 \hat{T}_2 |\{m_i(l+1)\}\rangle$$

$$\simeq \sum_{\{m_i(l)\}} \prod_{l=1}^{M_\tau} \langle \{m_i(l)\}| \hat{T}_1 |\{m_i(l+1)\}\rangle \exp\Big[Ja \sum_j m_j(l+1)\, m_{j+1}(l+1)\Big]$$

$$\langle \{m_i(l)\} | \hat{T}_l | \{m_i(l+1)\} \rangle = \prod_{i=1}^{M} \langle m_i(l) | \exp[Jga \hat{\sigma}_i^x] | m_i(l+1) \rangle$$

for $m=\pm 1, m'=\pm 1$ $\quad \langle m | \exp[Jga\hat{\sigma}_x] | m' \rangle = A \exp[B m m']$

$$\exp[Jga\hat{\sigma}_x] = \cosh(Jga)\,\mathbb{1} + \sinh(Jga)\,\hat{\sigma}_x$$

$m = m' = \pm 1$: $\cosh(Jga) = A e^{B}$ $\quad \cdots$ ①

$m = -m' = \pm 1$: $\sinh(Jga) = A e^{-B}$ $\quad \cdots$ ②

② ÷ ① : $e^{-2B} = \tanh(Jga)$

$$Z \sim \sum_{\{m_i(l)\}} \exp\left[\sum_{i,l} \{ Ja\, m_i(l) m_{i+1}(l) + B\, m_i(l) m_i(l+1) \} \right]$$

"Quantum - Classical mapping"

1dim 2-dim

[d-dim.] [(d+1)-dim.]

< 1D Quantum Ising Model >

strong coupling limit ($g \gg 1$)

ground state $|0\rangle = \prod_i |\rightarrow\rangle_i$ to leading order in $1/g$

$\langle 0 | \hat{\sigma}_i^z \hat{\sigma}_j^z | 0 \rangle \sim e^{-|x_i - x_j|/\xi}$ decay exponentially

(ξ : correlation length, $\xi \uparrow$ as $g \downarrow$).

weak coupling limit ($g \ll 1$)

two degenerate ground states $|\uparrow\rangle = \prod_i |\uparrow\rangle_i$, $|\downarrow\rangle = \prod_i |\downarrow\rangle_i$

"spontaneous breaking" of Z_2 symmetry (to leading order in g)

$\langle 0 | \hat{\sigma}_i^z \hat{\sigma}_j^z | 0 \rangle \xrightarrow[|x_i - x_j| \to \infty]{} N_0^2 \ (\neq 0)$ ($|0\rangle = |\uparrow\rangle$ or $|\downarrow\rangle$)

$\langle 0 | \hat{\sigma}_i^z | 0 \rangle = N_0$ "magnetization", order parameter

expectation : at least one "quantum phase transition" at $g = g_c$

< quantum paramagnetic phase ($g \gg 1$) >

$$\hat{\mathcal{H}}_I = -gJ \sum_i \hat{\sigma}_i^x - J \sum_{\langle i,j \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z$$

ground state $|0\rangle = \prod_i |\rightarrow\rangle_i$: energy $E_0 = -NJg + \mathcal{O}(1/g)$

"one-particle" excited state $|i\rangle \equiv |\leftarrow\rangle_i \prod_{j \neq i} |\rightarrow\rangle_j$

$$\langle i | \hat{\mathcal{H}}_I | i \rangle = E_0 + 2gJ$$

$$\langle i | \hat{\mathcal{H}}_I | i+1 \rangle = \langle i+1 | \hat{\mathcal{H}}_I | i \rangle = -J$$

Fourier-transformed state $|k\rangle \equiv \frac{1}{\sqrt{N}} \sum_j e^{ikx_j} |j\rangle$

$$\mathcal{H}_I |k\rangle = \frac{1}{\sqrt{N}} \sum_j e^{ikx_j} \underbrace{\hat{\mathcal{H}}_I |j\rangle}_{(E_0+2gJ)|j\rangle - J(|j+1\rangle + |j-1\rangle)} = \frac{1}{\sqrt{N}} \sum_j (E_0 + 2gJ - Je^{-ika} - Je^{ika}) e^{ikx_j} |j\rangle$$

$$= (E_0 + \varepsilon_k) |k\rangle \quad \text{with} \quad \varepsilon_k = 2J(g - \cos ka)$$

energy gap $\Delta = \varepsilon_{k=0} = 2J(g-1)$: $\Delta \downarrow$ as $g \downarrow$

< ferromagnetic phase ($g \ll 1$) > ($d=1$)

$$\hat{\mathcal{H}}_I = -J \sum_{\langle i,j \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z - gJ \sum_i \hat{\sigma}_i^x$$

two degenerate ground states $|\uparrow\rangle = \prod_i |\uparrow\rangle_i$, $|\downarrow\rangle = \prod_i |\downarrow\rangle_i$

single domain wall $|i\rangle \equiv \prod_{j \le i} |\uparrow\rangle_j \prod_{j > i} |\downarrow\rangle_j$

$$\langle i | \hat{\mathcal{H}}_I | i \rangle = -MJ + 2J \equiv E_0 + 2J + \mathcal{O}(g)$$

$$\langle i | \hat{\mathcal{H}}_I | i+1 \rangle = \langle i+1 | \hat{\mathcal{H}}_I | i \rangle = -gJ$$

$$|k\rangle \equiv \frac{1}{\sqrt{M}} \sum_j e^{ikx_j} |j\rangle$$

$\Rightarrow$ energy $E_0 + \varepsilon_k$ with $\varepsilon_k = 2J(1 - g\cos ka)$

energy gap $\Delta = \varepsilon_{k=0} = 2J(1-g)$: $\Delta \downarrow$ as $g \uparrow$

< ferromagnetic phase ($g \ll 1$) > ($d=2$)

$$\hat{\mathcal{H}}_I = -J \sum_{\langle i,j \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z - gJ \sum_i \hat{\sigma}_i^x$$

two degenerate ground states $|\uparrow\rangle = \prod_i |\uparrow\rangle_i$, $\left(|\downarrow\rangle = \prod_i |\downarrow\rangle_i \right)$

single-particle states $|i\rangle \equiv |\downarrow\rangle_i \prod_{j \neq i} |\uparrow\rangle_j$

$$\langle i | \hat{\mathcal{H}}_I | i \rangle = -2MJ + gJ \equiv E_0 + gJ + \mathcal{O}(g)$$

$$\langle i | \hat{\mathcal{H}}_I | j \rangle = \langle j | \hat{\mathcal{H}}_I | i \rangle = 0 \qquad \text{for } i \neq j$$

second-order degenerate perturbation

$\Rightarrow$ energy $E_0 + \mathcal{E}_k$ with $\mathcal{E}_k = J\left[g - \frac{g^2}{4} (1 + \cos k_x a + \cos k_y a) \right]$

energy gap $\Delta = \mathcal{E}_{\vec{k}=0} = gJ\left(1 - \frac{3}{32} g^2\right)$: $\Delta \downarrow$ as $g \uparrow$

Exact spectrum of quantum Ising model (d=1)

① Jordan-Wigner transformation

$$\begin{cases} \hat{\sigma}_i^x = 1 - 2c_i^+ c_i \\ \hat{\sigma}_i^z = -\left[\prod_{j<i} (1 - 2c_j^+ c_j)\right](c_i + c_i^+) \end{cases}$$

$$\left(\begin{matrix} \hat{\sigma}_i^x = +1 \\ -1 \end{matrix} \iff \begin{matrix} c_i^+ c_i = 0 \\ = 1 \end{matrix} \right)$$

c_i^+ / c_i : fermion creation/annihilation operator

$$\{c_i^+, c_j\} = \delta_{ij}, \quad \{c_i, c_j\} = \{c_i^+, c_j^+\} = 0$$

$$\iff [\hat{\sigma}_j^\alpha, \hat{\sigma}_l^\beta] = 2i\,\delta_{jl}\,\varepsilon_{\alpha\beta\gamma}\,\hat{\sigma}_j^\gamma$$

$$\Rightarrow \hat{\mathcal{H}}_I = -J \sum_i \left[c_i^+ c_{i+1} + c_{i+1}^+ c_i + c_i^+ c_{i+1}^+ + c_{i+1} c_i + g(1 - 2c_i^+ c_i)\right]$$

no conservation of fermion numbers

② Fourier transformation

$$\begin{cases} c_k = \frac{1}{\sqrt{M}} \sum_j c_j e^{+ikx_j} \\ c_k^\dagger = \frac{1}{\sqrt{M}} \sum_j c_j^\dagger e^{-ikx_j} \end{cases} \qquad (M = \text{number of sites})$$

$$\Rightarrow \mathcal{H}_I = J \sum_k \left[2\{g - \cos(ka)\} c_k^\dagger c_k + i \sin(ka) \{ c_{-k}^\dagger c_k^\dagger + c_{-k} c_k \} - g \right]$$

③ Bogoliubov transformation

$$\gamma_k \equiv u_k c_k - i v_k c_{-k}^\dagger, \quad \gamma_k^\dagger = u_k c_k^\dagger + i v_k c_{-k} \qquad \begin{pmatrix} u_k = u_{-k} \\ v_k = -v_{-k} \end{pmatrix}$$

$$\{\gamma_k, \gamma_k^\dagger\} = u_k^2 + v_k^2 = 1, \quad \{\gamma_k^\dagger, \gamma_{-k}^\dagger\} = \{\gamma_k, \gamma_{-k}\} = 0$$

$$c_k = u_k \gamma_k + i v_k \gamma_{-k}^\dagger$$

$$\tan\theta_k = \frac{\sin ka}{g - \cos ka} \qquad \left(u_k = \cos\frac{\theta_k}{2}, \; v_k = \sin\frac{\theta_k}{2} \right)$$

$$\mathcal{H}_I = \sum_k \{ \varepsilon_k (\gamma_k^\dagger \gamma_k - \tfrac{1}{2}) - gJ \} \qquad \varepsilon_k = 2J (1 + g^2 - 2g \cos ka)^{1/2}$$

$$\mathcal{H}_I = \sum_k \{\varepsilon_k(\gamma_k^+\gamma_k - \tfrac{1}{2}) - gJ\} \qquad \varepsilon_k = 2J(1+g^2-2g\cos ka)^{1/2}$$

ground state $|0\rangle$

$$\gamma_k|0\rangle = 0 \quad \text{for all } k$$

excited states

$$\gamma_{k_1}^+\gamma_{k_2}^+\cdots\gamma_{k_n}^+|0\rangle \quad \text{with energy} \quad E = E_0 + \sum_i \varepsilon_{k_i} \qquad (\varepsilon_k \geq 0)$$

energy gap $\Delta \equiv E_1 - E_0$

$$\Delta = \varepsilon_{k=0} = 2J|1-g| : \qquad \Delta = 0 \Rightarrow g = g_c \equiv 1$$

limiting cases

① $g \ll 1$: $\varepsilon_k \simeq 2J(1-g\cos ka) + O(g^2)$

② $g \gg 1$: $\varepsilon_k = 2Jg\left(\frac{1}{g^2}+1-\frac{2}{g}\cos ka\right)^{1/2} \simeq 2Jg\left(1-\frac{1}{g}\cos ka\right) + O\left(\frac{1}{g}\right)$

< Finite-temperature crossover >

correlation function

$$C(x_i,t) \equiv \langle \hat{\sigma}^z(x_i,t)\,\hat{\sigma}^z(0,0)\rangle = \mathrm{Tr}\left(e^{-\hat{\mathcal{H}}_I/T} e^{+i\hat{\mathcal{H}}_I t} \hat{\sigma}_i^z e^{-i\hat{\mathcal{H}}_I t} \hat{\sigma}_0^z \right)/Z$$

partition function $Z \equiv \mathrm{Tr}\left(e^{-\hat{\mathcal{H}}_I/T}\right)$

equal-time ($t=0$), long distance ($|x| \gg a$), finite temperature ($T>0$)

$$C(x,0) = Z_0 T^{1/4} G_I(\bar{\Delta}/T) \exp\left(-\frac{T}{c}|x|\, F_I(\bar{\Delta}/T)\right) \equiv N_0^2 e^{-|x|/\xi}$$

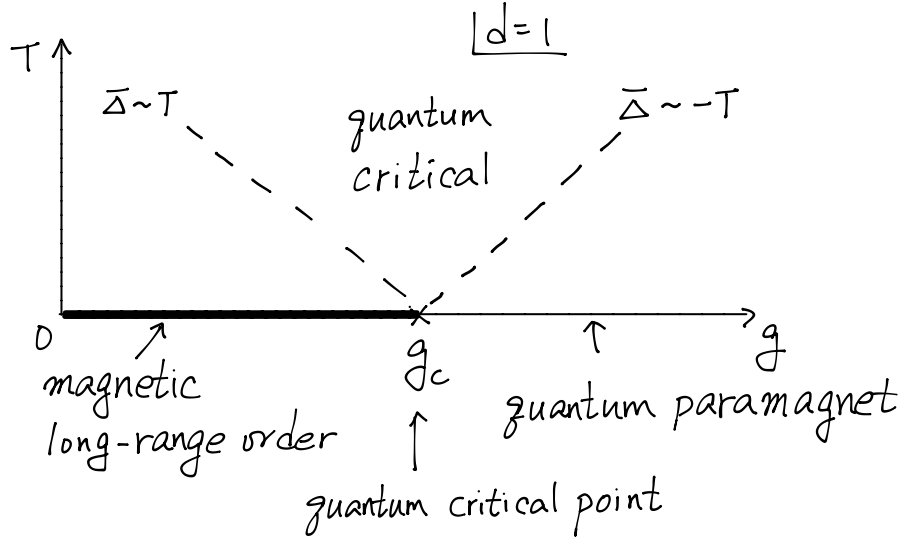
$$\left[\begin{array}{l} \bar{\Delta} \equiv 2J(1-g) \\ c = 2Ja \end{array}\right]$$

correlation length ξ

$$\xi^{-1} = \frac{T}{c} F_I\left(\frac{\bar{\Delta}}{T}\right)$$

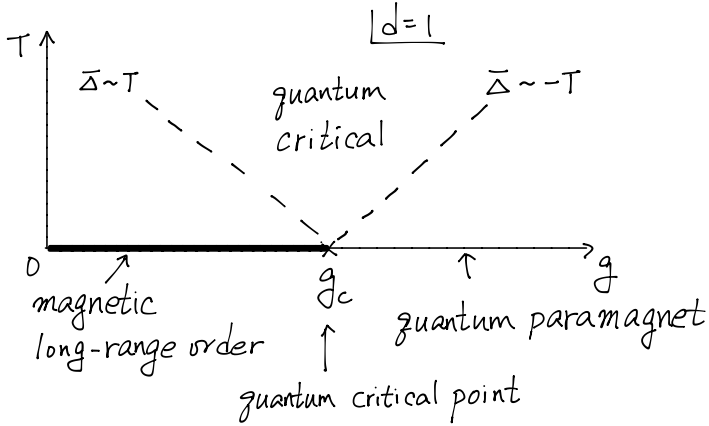
$$\begin{cases} F_I(s) = |s|\,\theta(-s) + \dfrac{1}{\pi}\displaystyle\int_0^\infty dy \ln \coth \frac{\sqrt{y^2+s^2}}{2} \\ \ln G_I(s) = \displaystyle\int_s^1 \frac{dy}{y}\left[\left(\frac{dF_I}{dy}\right)^2 - \frac{1}{4}\right] + \int_1^\infty \frac{dy}{y}\left(\frac{dF_I}{dy}\right)^2 \end{cases}$$

$$\xi = \begin{cases} c\sqrt{\dfrac{\pi}{2\bar{\Delta}T}}\, e^{\bar{\Delta}/T} & (\bar{\Delta} \gg T) \\ \dfrac{4c}{\pi T} & (|\bar{\Delta}| \ll T) \\ \dfrac{c}{|\bar{\Delta}|} & (\bar{\Delta} \ll -T) \end{cases} \qquad [\ \bar{\Delta} \equiv 2J(1-g) \quad c = 2Ja\ ]$$



$$\xi = \begin{cases} c\sqrt{\frac{\pi}{2\bar{\Delta}T}}\, e^{\bar{\Delta}/T} & (\bar{\Delta} \gg T) \\ \frac{4c}{\pi T} & (|\bar{\Delta}| \ll T) \\ \frac{c}{|\bar{\Delta}|} & (\bar{\Delta} \ll -T) \end{cases}$$

$[\ \bar{\Delta} \equiv 2J(1-g) \quad c = 2Ja\ ]$



1) Zero temperature

for $\bar{\Delta} > 0$ $(g < g_c \equiv 1)$

$N_0 = \sqrt{Z_0 \bar{\Delta}^{1/4}} \sim (g_c - g)^{1/8}$

magnetic long-range order

for $\bar{\Delta} < 0$ $(g > g_c)$

ξ ; finite : no long-range order

$\xi \sim |\bar{\Delta}|^{-1} \sim (g - g_c)^{-1}$: $\nu = 1$

(* 2D classical Ising

$\beta = 1/8$, $\nu = 1$)

2) quantum critical region

$(T \gg |\bar{\Delta}|)$

no well-defined quasi-particles