Project 2: Ising spins on a ladder and universal scaling function for free energy

In this project we consider two-dimensional Ising models on $2 \times M$ square lattice. We will call it a *ladder*. The Hamiltonian is given by

$$-\beta\mathcal{H} = K \sum_{n=1}^{2} \sum_{m=1}^{M} (S_{m,n}S_{m+1,n} + S_{m,n}S_{m,n+1})$$

and we adopt the periodic boundary conditions in both directions, which mean $S_{M+1,n} \equiv S_{1,n}$ and $S_{m,3} \equiv S_{m,1}$.

1. Consider $\boldsymbol{v}_m \equiv (S_{m,1}, S_{m,2})$ and show that

   $$-\beta\mathcal{H} = \sum_{m=1}^{M} [K_1(\boldsymbol{v}_m, \boldsymbol{v}_{m+1}) + K_2(\boldsymbol{v}_m)]$$

   and that the partition function is represented by the product of transfer matrices

   $$Z = \sum_{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_M} T_{\boldsymbol{v}_1\boldsymbol{v}_2} T_{\boldsymbol{v}_2\boldsymbol{v}_3} \cdots T_{\boldsymbol{v}_M\boldsymbol{v}_1}.$$

   Find the 4×4 transfer matrix $\boldsymbol{T}$ and its eigenvalues.

2. Calculate the correlation length ξ by the use of the two largest eigenvalues.

3. In the scaling limit find the universal scaling function Φ_f for free energy.

4. Compare the scaling function with that of one-dimensional nearest-neighbor Ising model, and discuss its implication.