Level-set forced mean curvature flow with the Neumann boundary condition

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This talk is based on joint work with Hung Tran, Jiwoong Jang (UW-Madison), and Hiroyoshi Mitake (University of Tokyo).

Overview

1. Mean curvature flow
2. Local Lipschitzness
3. Global Lipschitzness
4. Large-time behavior
5. Remarks on the coercivity assumption

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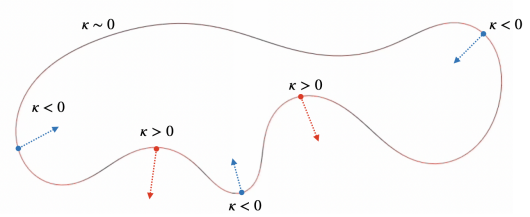
Mean curvature flow

- (Forced) mean curvature flow is a geometric flow of an evolving set, whose boundary moves with the normal velocity V given by

$$V = \kappa + c$$

where κ is the mean curvature and c is a forcing term.

- $c \equiv 0$:

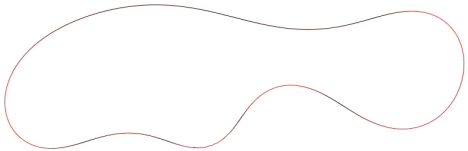


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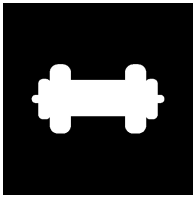
Our main equation: level-set forced mean curvature flow with the Neumann boundary condition

Set

$$\Gamma_t = \{x \in \mathbb{R}^n \mid u(x,t) = 0\}.$$

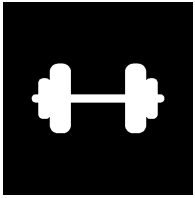
Since normal vector $\vec{n}$ is given by $\dfrac{Du}{|Du|}$, mean curvature is equal to

$$\kappa = \operatorname{div} \vec{n} = \operatorname{div}\left(\frac{Du}{|Du|}\right).$$



The level-set formulation of mean curvature flow:

$$\begin{aligned} \overbrace{\frac{u_t}{|Du|}}^{\text{normal velocity}} &= \overbrace{\operatorname{div}\left(\frac{Du}{|Du|}\right)}^{\text{mean curvature}} + c(x) \quad \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \vec{n}} &= 0 \quad \text{on } \partial\Omega \times [0,\infty), \\ u(x,0) &= u_0(x) \quad \text{on } \overline{\Omega}. \end{aligned}$$



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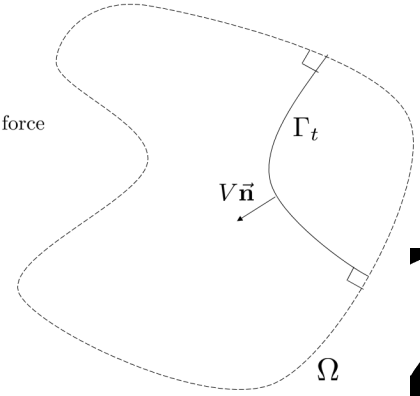
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The Neumann boundary condition

- $V = \kappa + c$

  κ : mean curvature, $c = c(x)$: a given force

- $\Gamma_t \perp \partial\Omega$





The questions of interests

Our main equation:

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1. Well-posedness
2. Regularity: a solution u is globally (or locally) Lipschitz
3. The large-time behavior

1. Well-posedness

For

$$F(p, X) = \operatorname{trace}\left(\left(I - \frac{p \otimes p}{|p|^2}\right) X\right),$$

we have

$$|Du| \operatorname{div}\left(\frac{Du}{|Du|}\right) = F(Du, D^2u), \text{ and } u_t = F(Du, D^2u) + |Du|c(x).$$

- In comparison to the usual Laplace operator $\operatorname{trace}(X)$, our operator is NOT uniformly elliptic, but degenerate elliptic:

$$0 \leq F(X, p) - F(Y, p) \text{ for } X \leq Y.$$

- The higher regularity (smoothness) is not expected.
- The well-posedness and the comparison principle are well established in the theory of viscosity solutions. See [Chen-Giga-Goto, 1991], [Evans-Spruck, 1991], [Giga-Sato, 1991], and [Giga-Sato, 1993], for instance.

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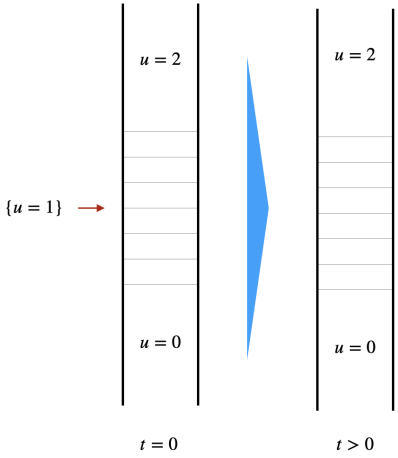
2. Lipschitzness

[Giga-Ohnuma-Sato, 1999]

- If the domain is convex, then $c \equiv 0$ gives global Lipschitzness.
- If the domain is NOT convex, then $c \equiv 0$ might not be good enough.

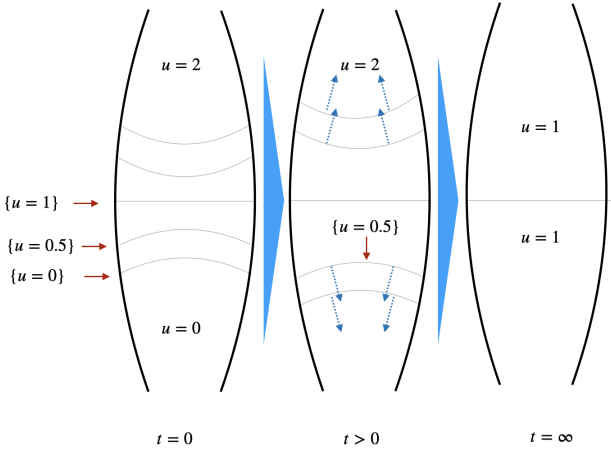
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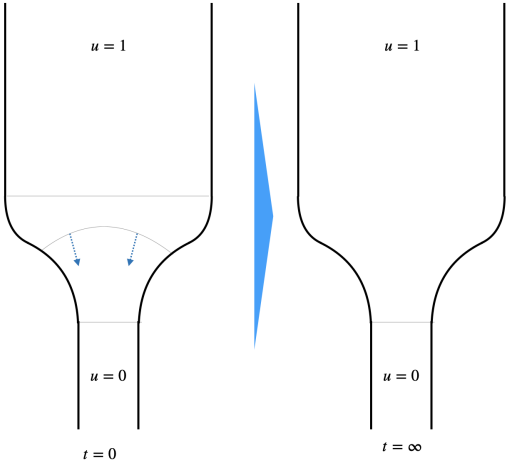
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2. Lipschitzness

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[Giga-Ohnuma-Sato, 1999]

Our main question

Question. What is the optimal condition of c to make sure that u is globally Lipschitz?

- Find the relation between c and the convexity of the domain.

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Local Lipschitzness

Theorem (Jang-K.-Tran-Mitake, 2022, JMPA)

Suppose that $c \in C^1(\overline{\Omega})$. Then there exists a unique viscosity solution u of (MCF). Moreover, there exists a constant $M > 0$ and for each $T > 0$, there exists a constant $C_T > 0$ depending on T such that

$$\begin{cases} |u(x,t) - u(x,s)| \le M|t-s|, \\ |u(x,t) - u(y,t)| \le C_T|x-y|, \end{cases} \qquad \text{for all } x, y \in \overline{\Omega},\ t, s \in [0,T].$$

An approximate problem

An approximate problem: for $\varepsilon > 0$, $T > 0$

$$\begin{cases} u^\varepsilon_t = \sqrt{\varepsilon^2 + |Du^\varepsilon|^2}\,\mathrm{div}\left(\frac{Du^\varepsilon}{\sqrt{\varepsilon^2+|Du^\varepsilon|^2}}\right) + c(x)\sqrt{\varepsilon^2 + |Du^\varepsilon|^2} & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u^\varepsilon}{\partial \vec{n}} = 0 & \text{on } \partial\Omega \times [0,T), \\ u^\varepsilon(x,0) = u_0(x) & \text{on } \overline{\Omega}. \end{cases} \tag{MCF_ϵ}$$

Set $b(p) = I_n - p \otimes p/(\varepsilon^2 + |p|^2)$. Then (MCF_ϵ) is expressed as

$$u^\varepsilon_t - b^{ij}(Du^\varepsilon)u^\varepsilon_{ij} - c(x)\sqrt{\varepsilon^2 + |Du^\varepsilon|^2} = 0 \quad \text{in } \Omega \times (0,\infty).$$

Here, we use the Einstein summation convention, and we write $f_i = \frac{\partial f}{\partial x_i}$ and $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ for $i, j = 1, \ldots, n$, where $f = f(x,t)$ is a given function.

Uniform bound of u_t

We start with a bound on u_t.

Lemma

Suppose that u^ε is the unique solution of (MCF_ϵ) for each $\varepsilon \in (0,1)$. Then, there exists $M > 0$ depending only on the forcing term c and the initial data u_0 such that

$$\|u_t^\varepsilon\|_{L^\infty(\overline{\Omega}\times[0,\infty))} \le \|u_t^\varepsilon(\cdot,0)\|_{L^\infty(\overline{\Omega})} \le M.$$

Differentiating (MCF_ϵ) with respect to t yields

$$(u_t^\varepsilon)_t - b^{ij}(u_t^\varepsilon)_{ij} - (b^{ij})_t u_{ij}^\varepsilon - c(x)\frac{(u_t^\varepsilon)_l u_l^\varepsilon}{\sqrt{\varepsilon^2+|Du^\varepsilon|^2}} = 0,$$

where

$$(b^{ij})_t = -\frac{(u_t^\varepsilon)_i u_j^\varepsilon}{\varepsilon^2+|Du^\varepsilon|^2} - \frac{u_i^\varepsilon (u_t^\varepsilon)_j}{\varepsilon^2+|Du^\varepsilon|^2} + \frac{2u_i^\varepsilon u_j^\varepsilon u_l^\varepsilon (u_t^\varepsilon)_l}{(\varepsilon^2+|Du^\varepsilon|^2)^2}.$$

Sketch of the proof: $\|u_t^\varepsilon\|_{L^\infty} \le \|u_t^\varepsilon(\cdot,0)\|_{L^\infty}$

Suppose that there exist $T > 0$, and $\delta > 0$ such that for some $(x_0, t_0) \in \overline{\Omega} \times (0, T]$

$$(x_0, t_0) \in \arg\max_{\overline{\Omega}\times(0,T]}(u_t^\varepsilon - \delta t).$$

At (x_0, t_0), we have

$$Du_t^\varepsilon = 0 \text{ and thus } (u_t^\varepsilon)_t - b^{ij}(u_t^\epsilon)_{ij} = 0.$$

On the other hand, $(u_t^\epsilon - \delta t)_t \ge 0$, $-b^{ij}(u_t^\varepsilon)_{ij} \ge 0$ at (x_0, t_0).

$$0 = (u_t^\varepsilon)_t - b^{ij}(u_t^\epsilon)_{ij} > \delta.$$

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Global Lipschitzness

Suppose that Ω is bounded, $c \in C^1(\overline{\Omega})$, and let

$$\begin{cases} C_0 = \max\{-\lambda : \lambda \text{ is an eigenvalue of } D\vec{n}|_{x_0}, x_0 \in \partial\Omega\}, \\ K_0 = \min\{d : d \text{ is the diameter of an open ball inscribed in } \Omega\}. \end{cases}$$

Suppose that there exists $\delta > 0$ such that

$$\frac{1}{n}c(x)^2 - |Dc(x)| - \delta > \max\left\{0,\ C_0|c(x)| + \frac{2nC_0}{K_0}\right\} \quad \text{for all } x \in \overline{\Omega}. \tag{C}$$

Theorem (Jang-K.-Tran-Mitake, 2022, JMPA)

Under the above assumptions, there exist constants $M, L > 0$, depending only the forcing term c and the initial data u_0, such that

$$\begin{cases} |u(x,t) - u(x,s)| \le M|t-s|, \\ |u(x,t) - u(y,t)| \le L|x-y|, \end{cases} \qquad \text{for all } x, y \in \overline{\Omega},\ t, s \in [0,\infty).$$

Remarks on the assumption

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Remark

- $\frac{1}{n}c(x)^2 - |Dc(x)| > 0$ appears in the homogenization of degenerate parabolic equation [Lions-Souganidis, 2005].
- If Ω is convex, then $C_0 \leq 0$.

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The Bernstein method

The Bernstein method considers a linearized equation:

1. Find an equation of $w = \sqrt{\varepsilon^2 + |Du^\varepsilon|^2}$
2. Apply the maximum principle at a maximizer of $|Du|$.

Fix a point $(x_0, t_0) \in \operatorname{argmax}_{\overline{\Omega} \times (0,T]} w^\varepsilon$.

Two cases:

- Case 1: the interior case $x_0 \in \Omega$
- Case 2: the boundary case $x_0 \in \partial\Omega$

Case 1: the interior case $x_0 \in \Omega$

At (x_0, t_0), we have $w_t \geq 0$, $Dw = 0$, $D^2 w \leq 0$, and thus

$$|Dc(x_0)|\, w^2 \geq \frac{1}{n} \left(u_t - c(x_0) w\right)^2 .$$

Thanks to

$$\frac{1}{n} c(x)^2 - |Dc(x)| > \delta,$$

w is bounded in this case.

Case 2: the boundary case $x_0 \in \partial\Omega$

Note that the Neumann boundary condition $Du \cdot \vec{n} = 0$ gives $D^2u \cdot \vec{n} = -D\vec{n} \cdot Du$. Thus,

$$\frac{\partial w}{\partial \vec{n}} = \frac{D^2u \cdot Du}{w} \cdot \vec{n} = -\frac{Du \cdot D\vec{n} \cdot Du}{w} < C_0 w,$$

where $C_0 = \sup\{-\lambda : \lambda \text{ is an eigenvalue of } D\vec{n}|_x, x \in \partial\Omega\}$.

Remark

If Ω is convex, then

$$\frac{\partial w}{\partial \vec{n}} = \frac{D^2u \cdot Du}{w} \cdot \vec{n} = -\frac{Du \cdot D\vec{n} \cdot Du}{w} < 0,$$

This cannot happen since

$$(x_0, t_0) \in \operatorname{argmax}_{\overline{\Omega} \times (0,T]} w^\varepsilon.$$

Case 2: the boundary case $x_0 \in \partial\Omega$

- Denote by $\psi = \rho w$. Then, at (x_0, t_0),

$$\frac{\partial \psi}{\partial \vec{n}} = \frac{\partial(\rho w)}{\partial \vec{n}} = \rho \frac{\partial w}{\partial \vec{n}} + w \frac{\partial \rho}{\partial \vec{n}} < w \left(C_0 \rho + \frac{\partial \rho}{\partial \vec{n}} \right) = 0.$$

  We choose ρ such that $\rho(x_0, t_0) = 1$, $\rho \leq 1$ on $\partial\Omega$, $C_0\rho(x_0) + \frac{\partial \rho}{\partial \vec{n}}(x_0) = 0$.
- Then, the maximum of ψ is obtained inside, where we apply the maximum principle again.
- At a maximizer (x_1, t_1) of ψ inside Ω,

$$\frac{1}{n}(u_t - c(x_1)w)^2 \leq \left(\frac{2nC_0}{K} + |Dc| + C_0|c| \right) w^2,$$

  where K represents the radius of the ball inscribed inside and tangent to the boundary at x_0.
- Therefore, ψ (and thus w) is bounded.

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Theorem (Jang-K.-Tran-Mitake, 2022, JMPA)

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Large-time behavior

We next study the large time behavior of u under same condition.

Theorem (Jang-K.-Tran-Mitake, 2022, JMPA)

Assume $c \in C^1(\overline{\Omega})$ satisfies (C). *Let u be the unique viscosity solution to* (*MCF*). *Then,*

$$u(\cdot, t) \to v, \quad \text{as } t \to \infty,$$

uniformly on $\overline{\Omega}$ for some Lipschitz function v, which is a viscosity solution to

$$\begin{cases} |Dv| \operatorname{div}\left(\frac{Dv}{|Dv|}\right) + c(x)|Dv| = 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \vec{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

A Lyapunov function

A Lyapunov function of an ODE is a function V such that $\dot{V} < 0$ at nonzero points, V attains a minimum at 0. Investigate the time-derivative of the Lyapunov functional

$$I^\varepsilon(t) = \int_\Omega \sqrt{\varepsilon^2 + |Du^\varepsilon|^2}\, dx.$$

to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \sqrt{\varepsilon^2 + |Du^\varepsilon|^2}\, dx - \int_\Omega c(x)u^\varepsilon\, dx\right) \leq -\frac{1}{\sqrt{\varepsilon^2 + L^2}}\int_\Omega (u_t^\varepsilon)^2\, dx.$$

Finally,

$$\limsup_{\varepsilon\to 0}\int_0^T \int_\Omega (u_t^\varepsilon)^2\, dxdt \leq C,$$

Sketch of the proof

For every $\{t_k\} \to \infty$, by the Arzelà-Ascoli theorem, there exist a subsequence $\{t_{k_j}\}$ and a Lipschitz continuous function v such that

$$u_{k_j}(x,t) = u(x, t + t_{k_j}) \to v(x,t),$$

locally uniformly on $\overline{\Omega} \times [0, \infty)$.

- Show that $v_t = 0$ weakly by using $\limsup_{\varepsilon \to 0} \int_0^T \int_\Omega (u_t^\varepsilon)^2 \, dxdt \leq C$,
- Show that v is independent of the choice of subsequence $\{t_k\}_k$ by using the comparison principle.

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The optimality

To obtain the global regularity, we assume

$$\frac{1}{n}c(x)^2 - |Dc(x)| > \max\left\{0,\ C_0|c(x)| + \frac{2nC_0}{K_0}\right\}$$

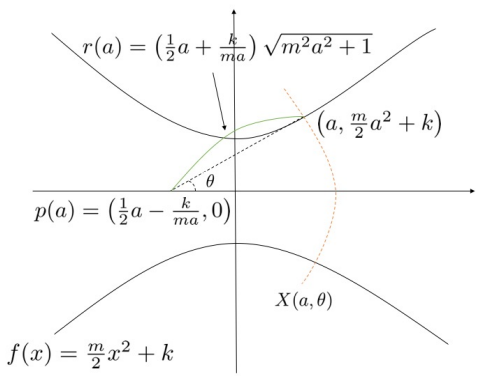
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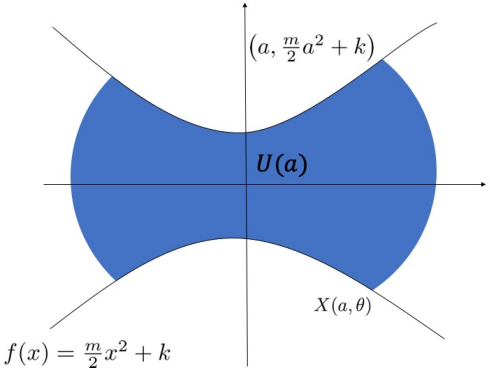
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Question. Is this condition optimal?

Example

The example below realizes an illustration in [Giga-Ohnuma-Sato].

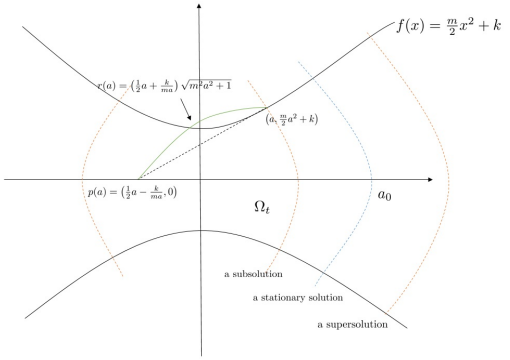




Example

Lemma

Let $a(t) := a_1 - l_1 e^{-\delta t}$ for $l_1 \in (0, a_1)$ and $\delta \in (0, \delta_0)$. Then, $U(a(t))$ is a set-theoretic subsolution with the Neumann boundary condition. Similarly, let $a(t) := a_1 + l_2 e^{-\delta t}$ for $l_2 \in (0, a_2 - a_1)$ and $\delta \in (0, \delta_0)$. Then, $U(a(t))$ is a set-theoretic supersolution with the Neumann boundary condition.



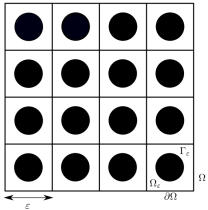
Discussion

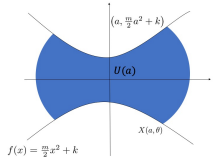
- Consider a periodic setting $\Omega = \Omega + \mathbb{Z}^n$, $\lambda(x) = \lambda(x+z)$ for all $z \in \mathbb{Z}^n$, and $\Omega_\epsilon = \epsilon\Omega$,

$$\frac{u_t}{|Du|} = \epsilon \operatorname{div}\left(\frac{Du}{|Du|}\right) + \lambda\left(\frac{x}{\epsilon}\right) \quad \text{in } \Omega_\epsilon \times (0,\infty),$$
$$Du \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega_\epsilon \times [0,\infty).$$

Prove this statement or provide a counterexample:

does the homogenization of the forced mean curvature motion occur in a periodic setting?





Thank you for your attention!