◎ Free boundary submanifolds in a domain

§. Rigidity

M^n, Riemannian mfd

$C \subset M$, strictly convex ($\mathrm{II} > 0$)

$\Sigma^{n-1} \subset C$, smooth cpt s.t. $\partial\Sigma \subset \partial C$

Def $\Sigma^{n-1} \subset C$, free boundary $\overset{\text{def}}{\Leftrightarrow}$ $\partial\Sigma \perp \partial C$ (& $\partial\Sigma \subset \partial C$)

(e.g.) $C = B^n \subset \mathbb{R}^n$

Thm (Nitsche '85)

$\Sigma^2 \subset B^3 \subset \mathbb{R}^3$, free boundary cmc disk

$\Rightarrow$ Σ : totally umbilic (spherical cap if $H \neq 0$; flat disk if $H = 0$)

(pf) (Ros-Souam '97)

$\Sigma^2 \subset B^3 \subset \overline{M}^3(c)$

conf. $\exists$ $D \ni z = x + iy$

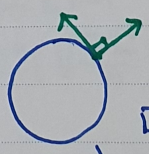
Let $h(z) := \frac{1}{4}\mathrm{II}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) - \mathrm{II}\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) - 2i\,\mathrm{II}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$

Known : h is holomorphic if $H \equiv$ const.

& $h = 0$ at $p \Leftrightarrow p$: umbilic.

(Joachimsthal) $\partial\Sigma$: a line of curvature in Σ

($\Rightarrow h_{ij} = 0$, $i \neq j$)



$\Rightarrow$ $0 = \mathrm{II}(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y})$ on $\partial\Sigma$

$\overset{(*)}{=} (x^2 - y^2)\,\mathrm{II}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) - xy\left(\mathrm{II}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) - \mathrm{II}\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right)\right)$

$\mathrm{Im}(z^2 h(z)) = \mathrm{Im}\left((x^2 - y^2 + 2xyi)\left(\mathrm{II}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) - \mathrm{II}\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) - 2i\,\mathrm{II}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\right)\right)$

harmonic $= -2(x^2 - y^2)\,\mathrm{II}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) + 2xy\left(\mathrm{II}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) - \mathrm{II}\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right)\right)$

$\Rightarrow \mathrm{Im}(z^2 h(z)) \equiv 0 = 0$ by (*)

$\Rightarrow \mathrm{Re}(z^2 h(z)) \equiv$ const (by Cauchy-Riemann)

$\Rightarrow z^2 h(z) \equiv 0 \Rightarrow h \equiv 0 \Rightarrow \Sigma$: totally umbilic ∎

Rmk

cpt minimal submanifolds in $\mathbb{S}^n$

$\updownarrow$

free boundary minimal submanifolds in B^n

(e.g.) ① Nitsche $\longleftrightarrow$ Almgren '66 : $M^2 \overset{\text{minimal}}{\subset} \mathbb{S}^3$, $M \approx \mathbb{S}^2 \Rightarrow M$: tot. geod.

② Calabi '67 : $\exists$ infinitely many $M \approx \mathbb{S}^2 \subset \mathbb{S}^n$ $(n \geq 4)$ which are not totally geod.

$\updownarrow$

Fraser-Schoen '15 : $\Sigma^2 \subset B^n \subset \mathbb{R}^n$, f.b.m.s $\Rightarrow$ $\Sigma = B^2$

③ Brendle '13 : $M^{55} \overset{T^2}{\subset} \mathbb{S}^3$, embedded minimal

$\Rightarrow$ $M = \mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^1(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^3$, Clifford

$\updownarrow$ (Lawson conj)

Conjecture (Nitsche '85, Fraser-Li '14)

$\Sigma^2 \subset B^3 \subset \mathbb{R}^3$, f.b. m. annulus $\Rightarrow$ Σ = a critical catenoid (open)

④ (Chern-do Carmo-Kobayashi 68, Lawson 69, Simons 68)

$\Sigma^n \subset \mathbb{S}^{n+1}$, cpt min. $|A|^2 \leq n$

$\Rightarrow$ either ① $|A| \equiv 0$ $(\Rightarrow \Sigma = \mathbb{S}^n)$

or ② $|A|^2 \equiv n$ $(\Rightarrow \Sigma = C_{m,n} = \mathbb{S}^m(\sqrt{\frac{m}{n}}) \times \mathbb{S}^n(\sqrt{\frac{n-m}{n}}))$ $1 \leq m \leq n-1$

Simons : $\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + |A|^2(n - |A|^2) \Rightarrow \int |A|^2(n-|A|^2) \leq 0$

$\updownarrow$

(Ambrozio-Nunes '21)

$\Sigma^2 \subset B^3 \subset \mathbb{R}^3$, fbms $|A|^2 \langle x, \nu(x) \rangle^2 \leq 2$

$\Rightarrow$ either ① $|A|^2 \langle x, \nu(x) \rangle^2 = 0$ $(\Rightarrow \Sigma = B^2)$

or ② $|A|^2 \langle x, \nu(x) \rangle^2 = 2$ at p $(\Rightarrow \Sigma$ = critical catenoid)

Generalization

$C \subset \bar{M}^3(c)$, strictly convex

$\Sigma^2 \subset C$, f.b cmc H $(c+H^2 \geq 0)$

Define $f(r) = \begin{cases} \frac{1}{2}r^2, & c=0 \\ \cosh(kr), & c=-k^2 \\ \cos(kr), & c=k^2 \ \& \ r < \frac{\pi}{2\sqrt{c}} \end{cases}$ $(k>0)$

Let $\mathring{A} := A - H g_\Sigma$ (H: normalized), traceless 2nd f.f.

Note: $|\mathring{A}|^2 = (\lambda_1 - H)^2 + (\lambda_2 - H)^2 = |A|^2 - 2H^2$

Thm1 (Min-S, '22) $\Sigma^2 \subset C \subset \bar{M}^3(c)$, f.b cmc H, strictly convex $(c+H^2 \geq 0)$

(topology) $\frac{1}{2}|\mathring{A}|^2 (\nabla_\nu f)^2 \leq (f'' + H \nabla_\nu f)^2$

$\Rightarrow \Sigma \approx$ either a disk or an annulus

Thm2 (Min-S, '22)

$C = B^3_R(p)$ (a geodesic ball)

$\Rightarrow \Sigma =$ either a spherical cap or a Delaunay surface

Rmk In $\mathbb{R}^3$, $\Sigma^2 \subset B^3$, fbms

$\Rightarrow f(r) = \frac{1}{2}r^2$ & $\mathring{A} = A$

$\therefore \frac{1}{2}|\mathring{A}|^2 (\nabla_\nu f)^2 \leq (f'' + H \nabla_\nu f)^2$

$\| \qquad\qquad \| \qquad\qquad \|$

$\frac{1}{2}|A|^2 \langle r \nabla r, \nu \rangle \qquad 1$

$\|$

$|A|^2 \langle x, \nu \rangle$

pf of Thm 2

$Q = h_{11} - h_{22} + 2i h_{12}$, Hopf differential, holomorphic on Σ

Thm1 $\Rightarrow$ $\Sigma \approx$ a disk or an annulus

Case ①: $\Sigma \approx$ a disk

$\Rightarrow$ Σ = a spherical cap (by Nitsche)

Case ②: $\Sigma \approx$ an annulus

$\Rightarrow$ $\mathcal{A} := \{ x \in \Sigma : r(x) = \min_{\Sigma} r \}$

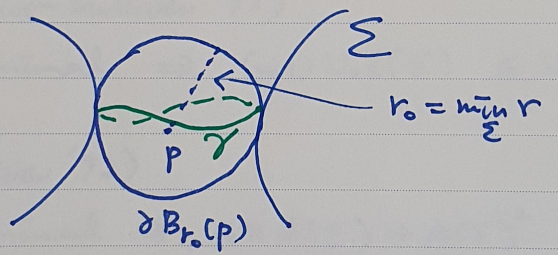
observe ① $\mathcal{A} \neq \emptyset$

② $\mathcal{A} \cap \partial C = \emptyset$

③ (nontrivial) $\mathcal{A}$ = the unique closed geodesic γ

By ③

$\Rightarrow$ Σ meets the geodesic sphere $\partial B_{r_0}(p)$ tangentially along γ



$\Rightarrow$ γ : also a geodesic in $\partial B_{r_0}(p)$ ($\because T_\gamma \Sigma = T_\gamma \partial B_{r_0}(p)$)

$\Rightarrow$ γ : a great circle $\subset$ totally geodesic plane P
(a geodesic circle)

$\Rightarrow$ $\Sigma \perp P$ along γ

$\Rightarrow$ Σ = a Delaunay surface (by the max. principle) ∎