푸앙카레 계량의 일반화

§ Intro.

CIRM

Kähler Geometry = Complex Geometry ∩ Riemannian Geo.

Kähler manifold = $X + J + g$ satisfying $\nabla J \equiv 0$.

Q: What is the canonical or best Kähler metric on a given cpx mfd?

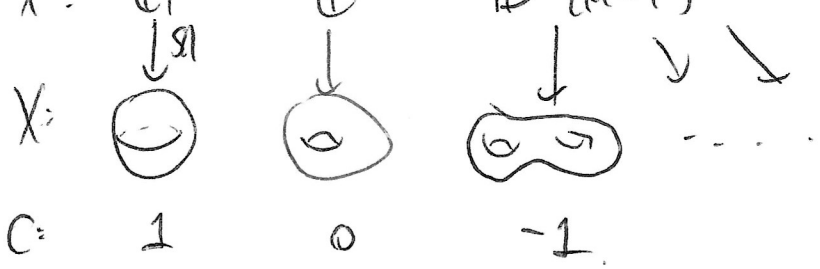
① : Riemannian Geometric aspects : KE metric (constant Ricci curv.)

② : Complex " " : Bergman metric (invariant under $\mathrm{Aut}(X) = \{\varphi : X \to X : \text{biholo}\}$)

§ Poincaré metric

$n=1$ X : cpt 1-dim cpx mfd (Riemann surface)

Uniformization thm ⇒ $\tilde{X}$: $\mathbb{CP}^1$ $\mathbb{C}$ $\mathbb{D}=\{|z|<1\}$



$$g_{\tilde{X}} := \frac{1}{(1+c(x^2+y^2))^2}(dx\otimes dx + dy\otimes dy), \quad \lambda_{\tilde{X}} := \frac{1}{(1+c(x^2+y^2))^2} = \frac{1}{(1+c|z|^2)^2}$$

$$K_{\tilde{X}} = -\frac{\Delta(\log\lambda)}{2\lambda} = 4c$$

Kähler potential

$$\lambda_{\tilde{X}} = -\frac{1}{2c}\frac{1}{4}\Delta(\log\lambda) = \frac{1}{4}\Delta\left(\frac{1}{c}\log\sqrt{\frac{1}{\lambda}}\right) = \frac{\partial^2}{\partial z\partial\bar{z}}\left(\frac{1}{c}\log(1+c|z|^2)\right)$$

$$\lambda_{\tilde{X}} = \frac{\partial^2}{\partial z\partial\bar{z}}\varphi, \quad \begin{cases} \varphi = \log(1+|z|^2) & c=1 \\ \varphi = |z|^2 & c=0 \\ \varphi = \log(1-|z|^2)^{-1} & c=-1 \end{cases}$$

Q) higher dim. ($n>1$) ? No uniformization thm ! ②

X:	$\mathbb{CP}^n$	$\mathbb{C}^n$	B^n
g_X:	$\frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log(1+\lvert z\rvert^2)$	$\frac{\partial^2}{\partial z_j \partial \bar{z}_k} \lvert z\rvert^2$	$\frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log(1-\lvert z\rvert^2)^{-1}$
$\mathrm{Hol}(g_X)$	2π	2π	2π
$\mathrm{Ric}(g_X)$	$(n+1)\pi$	0	$-(n+1)\pi$

$\lvert z\rvert^2 = \lvert z_1\rvert^2 + \cdots + \lvert z_n\rvert^2$

Poincare metric ($n=1$) $\quad \mathrm{Ric}(g_D) = -\frac{\Delta(\log\lambda)}{2\lambda} = -4$

$\boxed{\chi = \frac{1}{8}\Delta\varphi}$

$$\Leftrightarrow \frac{1}{4}\Delta(\log\lambda) = 2\lambda \Longleftrightarrow \frac{1}{4}\Delta\left(\log \Delta\left(\frac{\varphi}{8}\right)\right) = \frac{1}{4}\Delta\varphi$$

$$\Uparrow$$

(###) $\quad \log \Delta\varphi = \varphi$

$$\Updownarrow$$

$$\boxed{\Delta\varphi = e^{\varphi}}$$

§ KE metric

Def) Riemannian metric g is <u>hermitian</u> if $g(JX, JY) = g(X, Y)$

Rmk) g : hermitian $\Longleftrightarrow \omega(X, Y) := g(JX, Y)$

associate 2-form of type (0,1)

Def) hermitian metric g is <u>Kähler</u> if $d\omega = 0$.

<u>Notation</u> $\quad \frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \sqrt{-1}\frac{\partial}{\partial y_j}\right) \qquad dz_j := dx_j + \sqrt{-1}\, dy_j$

Define $\quad g_{j\bar{k}} := g^{\mathbb{C}}\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}\right) \qquad g = \sum g_{j\bar{k}}\, dz^j \otimes \overline{dz^k}$

$$\omega = \sqrt{-1}\sum_{j,k=1}^{n} g_{j\bar{k}}\, dz^j \wedge \overline{dz^k}$$

CIRM

Note that: g: Kähler $\Leftrightarrow \frac{\partial}{\partial z_i} g_{j\bar{k}} = \frac{\partial}{\partial z_j} g_{i\bar{k}} \Leftrightarrow g_{j\bar{k}} = \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \varphi$ (potential)

Def) g: Kähler–Einstein

iff g: Kähler & $Ric_g(X,X) = \frac{Ric_g(X,X)}{g(X,X)} \equiv c$ (Einstein) (constant)

Note: $g \rightsquigarrow \omega = \sqrt{-1}\, g_{j\bar{k}}\, dz^j \wedge d\bar{z}^k$

$Ric_g \rightsquigarrow Ric_\omega = \sqrt{-1}\, R_{j\bar{k}}\, dz^j \wedge d\bar{z}^k = -\sqrt{-1}\, \partial\bar{\partial} \log \det g$.

$\therefore$ g: Kähler–Einstein $\Leftrightarrow$ $d\omega = 0$ & $Ric(\omega) = c\omega$.
(ω)

- $\omega \in [\omega_0] \overset{\partial\bar{\partial}\text{-lem.}}{\Longleftrightarrow} \omega = \omega_0 + i\partial\bar{\partial}\varphi =: \omega_\varphi = \omega_0 + \sqrt{-1} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \varphi\, dz_i \wedge d\bar{z}_j$

- $c_1(X) := c_1(TX) = -c_1(K_X) = \frac{1}{2\pi}[Ric\,\omega] \in H^{1,1}(X, \mathbb{R})$

$K_X := \bigwedge^n_1 \Omega^{1,0}_X = \mathrm{span}_{\mathbb{C}} \{ dz = dz_1 \wedge \cdots \wedge dz_n \}$

Canonical line bundle

- $\exists$ KE st. $Ric(\omega) = c\omega \Rightarrow c_1(X) = \frac{c}{2\pi}[\omega] = \begin{cases} < 0 & \text{general type} \\ = 0 & \text{CY} \\ > 0 & \text{Fano.} \end{cases}$

$Ric(\omega_\varphi) = c\omega_\varphi \Longleftrightarrow (\omega_0 + i\partial\bar{\partial}\varphi)^n = e^{f_0 + \varphi} \omega_0^n$

$\Longleftrightarrow \det\left(g_{j\bar{k}} + \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \varphi\right) = e^{f_0 + \varphi} \det(g_{j\bar{k}})$

Complex Monge–Ampère eqn

(nonlinear elliptic)

§ Bergman metric

Recall $\frac{2}{(1-|z|^2)^2} = \frac{\partial^2}{\partial z \partial \bar{z}} \log\left(\frac{1}{(1+|z|^2)^2}\right)$ $\quad \langle f, g\rangle = \int_D f\bar{g}\, d\eta$

$$\frac{1}{(1+|z|^2)^2} = \sum_{j=0}^{\infty} (j+1)|z|^{2j} = \sum_{j=0}^{\infty} \sqrt{j+1}\, z^j \cdot \sqrt{j+1}\, \bar{z}^j$$

$\left\{\frac{\sqrt{j+1}}{\sqrt{\pi}} z^j\right\}_{j=0}^{\infty}$: complete O.N.B. $\Rightarrow A^2(D) = L^2(D) \cap \mathcal{O}(D)$ Bergman space

$$B_D(z,\bar{z}) := \frac{1}{\pi(1+|z|^2)^2} = \sum_{j=0}^{\infty} \frac{j+1}{\pi}|z|^{2j}$$ Bergman kernel

$\frac{\partial}{\partial z \partial \bar{z}} \log B_D(z,\bar{z})\, dz \otimes d\bar{z}$ Bergman metric

Kobayashi X: cpx mfd.

$$\Omega^{n,0}_{(2)}(X) := \left\{ f \in H^0(X, K_X) : \|f\|^2_{L^2(X)} := \int_X f \wedge \bar{f} < +\infty \right\}$$

(f: hol. $(n,0)$-form)

$B_X := \sum_j e_j \wedge \bar{e}_j = B_X^*\, dz \wedge d\bar{z}$ where $dz = dz^1 \wedge \cdots \wedge dz^n$

$\{e_j\}_{j=0}^{N \le \infty}$ O.N.B of $\Omega^{n,0}_{(2)}(X)$ Bergman kernel form

$\omega_X := \sqrt{-1}\, \partial\bar{\partial} \log B_X^*$ Bergman metric (Kähler)

Rmks ① B_X, ω_X : invariant under $\mathrm{Aut}(X)$

② X: bdd homogeneous domain $\Rightarrow$ $\omega_X = KE$

③ X: cpt $\Rightarrow$ $\int_X B_X = \dim_{\mathbb{C}} \Omega^{n,0}_{(2)}(X) = \dim_{\mathbb{C}} \mathcal{H}^{n,0}_{(2)}(X)$