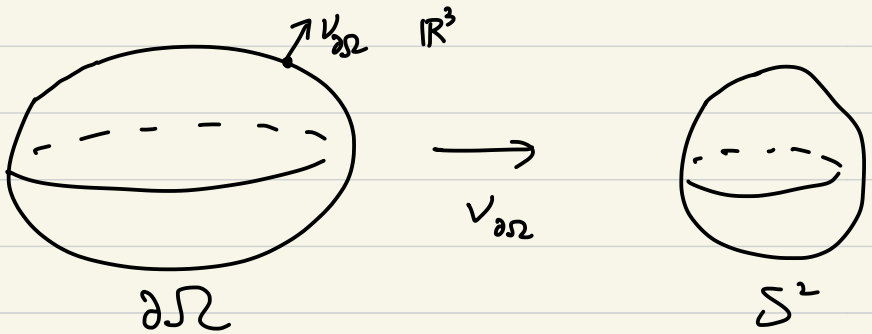
Regular solutions to L_p Minkowski problem.

Ω : convex body $\subset \mathbb{R}^3$.

(compact, convex set with non-empty interior.)



$$\Omega \mapsto \mu_\Omega$$

ex 1) surface area S_Ω

$$S_\Omega(\omega) = \left| \nu_\Omega^{-1}(\omega) \right|, \quad S_\Omega(S^2) = |\partial\Omega|$$

If Ω is smooth, $dS_\Omega = \frac{1}{K} d\sigma_{S^2}$.

ex 2) Cone volume C_Ω

$$C_\Omega(\omega) = \mathrm{Vol}\left(0 \# \nu^{-1}(\omega) \right)$$

$$C_\Omega(S^2) = \mathrm{vol}(\Omega)$$

$$dC_\Omega(\omega) = \frac{1}{3} h_\Omega \, dS_\Omega$$

Q. $\forall \Omega_0, \Omega_1, \exists \Omega$ s.t.

$$S_\Omega = S_{\Omega_1} + S_{\Omega_2} ?$$

Minkowski problem (1903).

$\mu \geq 0$ Borel measure on S^n.

$\exists \Omega$ s.t. $S_\Omega = \mu$?

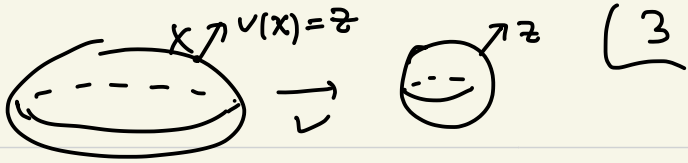
Rmk $d\mu = f d\sigma_{S^n}$, Ω : smooth. $(0 < c \leq f \leq C)$

$$\Rightarrow \frac{1}{K} d\sigma_{S^n} = dS_\Omega = d\mu = f\, d\sigma_{S^n}.$$

$$\Rightarrow K = 1/f.$$

Prescribed Gauss curvature problem.

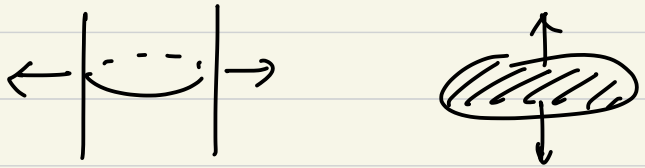
$\mu = S_\Omega$



① $\int_{S^n} z\, d\mu(z) = \int_{\partial\Omega} v(x)\, d\mathcal{H}^n(x) = 0.$

μ: zero centroid.

② $\mu(E) < \mu(S^n)$ $\forall$ E : great subsphere



- $\mu = S_\Omega$ for some Ω

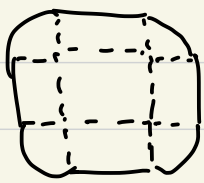
$\Leftrightarrow$ ① & ② (Minkowski, Alexandrov)

- $\exists\, \Omega$ s.t. $S_\Omega = S_{\Omega_0} + S_{\Omega_1}$

Rmk

$$\frac{d}{dt}\Big|_{t=0} V(\Omega_0 + t\Omega_1) = \int_{S^n} h_{\Omega_1}(z)\, dS_{\Omega_0}(z)$$

$$\Omega_0 + \Omega_1 = \{x+y : x \in \Omega_0, y \in \Omega_1\}$$

ex) $\square + \bigcirc =$ 

$$V^{\frac{1}{n}}(\Omega_0 + t\Omega_1) \geq V^{\frac{1}{n}}(\Omega_0) + t V^{\frac{1}{n}}(\Omega_1)$$

$\frac{d}{dt}|_{t=0}$ ↓

$$\frac{1}{n} V^{\frac{1}{n}-1}(\Omega_0) \cdot \int_{S^n} h_{\Omega_1}\, dS_{\Omega_0} \geq V^{\frac{1}{n}}(\Omega_1)$$

$$\Leftrightarrow \int_{S^n} h_{\Omega_1} \underbrace{dS_{\Omega_0}}_{= d\mu} \geq n\, V^{1-\frac{1}{n}}(\Omega_0)\, V^{\frac{1}{n}}(\Omega_1) \quad \forall\, \Omega_1$$

$V(\Omega_1) \geq 1.$

"=" if $\Omega_1 = \Omega_0$

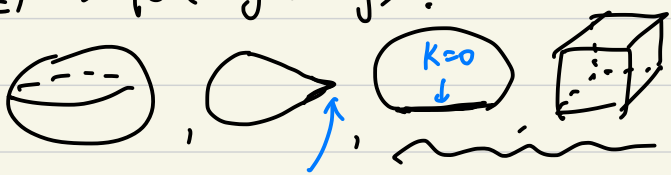
Consider

$$\inf \left\{ \int_{S^n} h_\Omega \, d\mu : V(\Omega) \geq 1 \right\}.$$

. Ω ?

① # of Ω ? unique (up to translation)

② shape (regularity) ?



$\mu = \mu_a + \mu_s$

$\mu_s \neq 0$.

$d\mu = f \, d\sigma_{S^n}$.

$f \in C^\infty \Rightarrow \Omega \in C^\infty$

(Pogorelov, Nirenberg, Cheng-Yau)

$f \in C^\alpha \Rightarrow \Omega \in C^{2,\alpha}$.

(Caffarelli)

- $h_{\Omega_0+\Omega_1} = h_{\Omega_0} + h_{\Omega_1}$.

- $h_{\Omega_0 +_p \Omega_1} = \left(h_{\Omega_0}^p + h_{\Omega_1}^p\right)^{1/p}$

$$\frac{d}{dt}\Big|_{t=0} V(\Omega_0 +_p t\Omega_1) = \frac{1}{p}\int_{S^n} h_{\Omega_1}^p \, dS_{\Omega_0,p}$$

L_p surface area measure.

- $p \geq 1$,
$$dS_{\Omega,p} = h_\Omega^{1-p}\, dS_\Omega .$$

$\forall p \in \mathbb{R}$. $p=1$ classical. origin

L_p Minkowski problem (Lutwak '93)

$\mu \geq 0$ Borel measure on S^n. $p \in \mathbb{R}$.

$\exists \Omega$ s.t. $S_{\Omega,p} = \mu$?

$p=1$ Classic. $p=0$ cone volume, $p=-n-1$ affine

$\mu = f\, d\sigma_{S^n}$.

PDE: $h^{1-p} \det(h_{ij} + h\sigma_{ij}) = \dfrac{h^{1-p}}{K} = f$ on S^n.

$h=0$ important, $\begin{cases} p>1 & \text{singular} \\ p<1 & \text{degenerate.} \end{cases}$ origin

- Self-Similar solution to (anisotropic) α-GCF
  f p K

Existence $\mu = f\, d\sigma_{S^n}$. $\exists\ \Omega$.

Uniqueness $p \geq 1$ Unique.

$p < 1$ example of non-uniqueness.

If $f \equiv 1$, $p \in (-n-1, 1) \Rightarrow$ Unique.

(or f: even. i.e. $f(z) = f(-z)$)

Shape (regularity)





- $p \geq n+1$, $p \leq -n+1$: Ω smooth.
- $-n+2 < p < n+1$, $p \neq 1$.

  $\exists\, \Omega$ s.t. $k_i = \infty$ at a pt for some i.

  $y = |x|^{q}$, $q = \frac{2n}{n+p-1}$. $y = (|x|-1)_+^{\frac{n+p}{n+p-1}}$, $p < 1$.

Q. $-n+1 < p \leq -n+2$. ? ($p=0, n=2$)

important case!

Thm (Choi-Kim-L.)

$-n+1 < p \leq -n+2$, $f > 0$ semi convex.

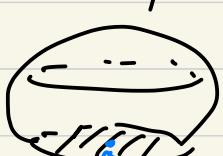
$\exists\, \Omega$ s.t. $S_{\Omega,p} = f\, d\sigma_{S^n}$.

with $\sup k_i \leq C$.

Rmk ① $\nexists$ sharp edge



Ω : regular.

② 

e ↓ origin

$dS_\Omega = \delta_e \cdot f_e + \cdots$

$h^{1+p} dS_\Omega = 0 + \cdots$

no pt mass.

pf) $X_t = -f(\nu)\cdot K^\alpha \nu$. or $h_t = -fK^\alpha$.

(rescaled) $h_\tau = h - fK^\alpha \Rightarrow h = fK^\alpha$

$\tau \downarrow 0$

$$\Rightarrow \frac{h^{1/\alpha}}{K} = f^{\frac{1}{\alpha}},$$

Goal $\sup \lambda_i(\Omega) \le C$.

① Ω : self-shrinker solution.

② $\frac{1}{C} \le K \le C$

$K = \lambda_1 \cdots \lambda_n$

③ $K^\alpha \cdot \frac{1}{\lambda_{min}} \le C$

$$(\lambda_1 \cdots \lambda_n)^\alpha \cdot \frac{1}{\lambda_{min}} \ge \lambda_1^{(n-1)\alpha} \cdot \lambda_n^\alpha \cdot \frac{1}{\lambda_1}$$

$$= \lambda_1^{(n-1)\alpha - 1} \lambda_n^\alpha .$$

$$\lambda_n^\alpha \le C\lambda_1^{1-(n-1)\alpha} \le C K^{\frac{1-(n-1)\alpha}{n}} \le C$$

if $1-(n-1)\alpha \ge 0$ $\left(\Leftrightarrow \alpha \le \frac{1}{n-1}\right.$

$\left.\Leftrightarrow p \le 2-n\right)$

- $F = fK^\alpha$.

$$F_t = \mathcal{L}F + \alpha F^2 H. \geq \mathcal{L}F + CF^{2+\frac{1}{n\alpha}}.$$

$\uparrow \mathcal{L} \alpha f K^\alpha b^{ij} \nabla_i \nabla_j F.$

$$\sim \lambda_2 \nabla_1 \nabla_1 F + \lambda_1 \nabla_2 \nabla_2 F.$$

$\alpha = 1$

- $\omega = \dfrac{F}{u - \rho_0}$:

$$\omega_t \leq \mathcal{L}\omega + C_0 + C_1 \omega - C_2 \omega^{1+\frac{1}{n\alpha}}$$

< 0 if $\omega \sim \infty$

$$\omega \leq C.$$

- $$\omega = \frac{F/\lambda_{min}}{R - |\nabla u|^2}$$

$$\omega_t \leq \mathcal{L}\omega - C\omega^2 + C\omega + C$$

< 0 if $\omega \sim \infty$.