Gradient flow.

$f: \mathbb{R}^n \longrightarrow \mathbb{R}$ smooth.

Calc. II.

$$\nabla_v f = \langle \text{grad} f, v \rangle$$

$$\text{grad} f = \langle \partial_1 f \dots \partial_n f \rangle.$$

$$(\|\nabla f\|)$$

$$\nabla \cdot f \in T_x^* \mathbb{R}^n \overset{\langle \cdot \rangle}{\rightsquigarrow} \text{grad} f \in T_x \mathbb{R}^n$$

$\frac{\text{grad} f}{|\text{grad} f|}$ $\rightsquigarrow$ direction f grows fastest.

Find (local) minimum of f?

or critical pts of f?

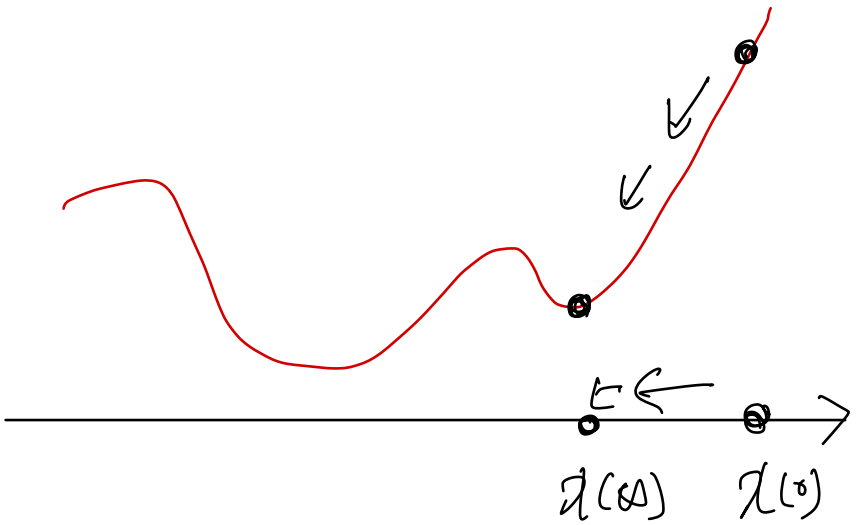
Grad. flow of f w.r.t. $\langle\,\rangle_{\text{gstd}}$.

$x(t) \in \mathbb{R}^n$, $x(0) = x_0$

$$\dot{x}(t) = -\nabla f(x(t)) \qquad t \geq 0.$$

- GF efficiently decreases f.

$$\frac{d}{dt} f(x(t)) = \nabla_{\dot{x}(t)} f = \langle \nabla f, -\nabla f \rangle = -|\nabla f|^2 (x(t))$$



- Many important geometric eqs follow GF structure. (∞-dim space)

MCF. MS. RF. YF, KMF, Harmonic Map "G_2 ..."

"$\exists$ energy f, metric g. ....

Q:
- $x(t)$ converges?
- to local min?
- convergence rate? $|x(\infty) - x(t)| \lesssim e^{-\delta t}$?

Examples

- $f(x) = \frac{c}{2}x^2 \qquad c > 0$

$$\dot{x} = -cx \Rightarrow x(t) = e^{-ct}x(0).$$

- $f(x_1, x_2) = \frac{c_1}{2}x_1^2 + \frac{c_2}{2}x_2^2 \qquad c_i > 0$

$$\dot{x}_i = -c_i x_i \Rightarrow \vec{x}(t) = \left(e^{-c_i t}x_i(0)\right]$$

$$\therefore \quad |x(t) - 0| \lesssim e^{-\min(c_i)t}.$$

If $\dot{x}(t) = -\nabla f(x(t))$ conv. to x_∞ as $t \to \infty$

& $D^2 f(x_\infty) \geq c I > 0$

$$|x(t) - x_\infty| \lesssim e^{-ct}$$

- $f(x) = \frac{1}{p}x^p \quad p > 2.$

$$\dot{x} = -x^{p-1} \Rightarrow \left(\frac{x^{2-p}}{2-p}\right)' = -1$$

$$\Rightarrow x^{2-p} = (p-2)(t+C) \Rightarrow x = C_1(t+C_2)^{-\frac{1}{p-2}}.$$

$|x| \sim t^{-\frac{1}{p-2}}$ algebraic decay.

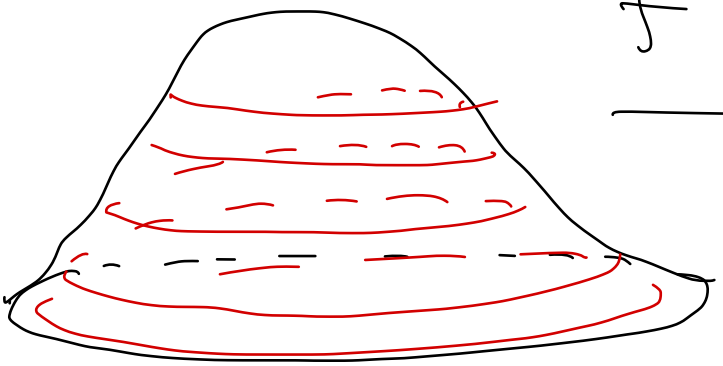
Uniqueness of limit.

On $\mathbb{R}^n$. Suppose $\exists\, t_i \to \infty$ s.t.

$$x(t_i) \to x_\infty \quad \text{as} \quad i \to \infty.$$

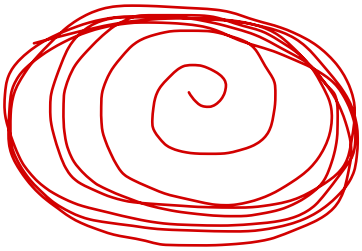
Q: $x(t) \to x_\infty$ as $t \to \infty$.

Counter example on $\mathbb{R}^2$



f smooth

$f \equiv 0$



$\int |\dot{x}| dt = \infty$

& $x(t)$ visits 

Theorem [L, IHES] If f is 'analytic' then $x(t) \to x_\infty$ & $\int |\dot{x}| dt < \infty$.

- Łojasiewicz gradient ineq.

If f is analytic, $\forall y, \exists U$ s.t. $x \in U$

$$|f(x) - f(y)|^\beta \leq C |\nabla f(x)|$$

for some. $\beta \in (0,1)$ & $C < \infty$.

pf of thm Suppose $\gamma(t_1) \in U$ some t_1.

$s(t)$ arc-length param of $\gamma(t)$

$$\Downarrow f$$

$$\frac{df}{ds} = \left\langle \nabla f, \frac{\dot\gamma}{|\dot\gamma|} \right\rangle = -|\nabla f| \leq -C\,|f(x)|^{\beta}$$

$$\frac{df}{ds} \leq -C f^{\beta}$$

$$\partial_s\left(f^{1-\beta}\right) \leq -C(1-\beta)$$

$$f^{1-\beta}(s(t_2)) - f^{1-\beta}(s(t_1)) \leq -C(1-\beta)(s_{t_2} - s_{t_1})$$

provided $\gamma(t) \in U \quad \forall t \in [t_1, t_2]$.

$$s_{t_2} - s_{t_1} \leq \frac{1}{C(1-\beta)} f^{(1-\beta}(s_{t_1})$$

If $x(t_1)$ is suffiently close to x_∞

& $f(s_{t_1})$ is suffi. small,

then $x(t)$ never leave U.

& $s_\infty - s_{t_1} < \infty \rightarrow x(t_1) \rightarrow x_\infty$

□

Indeed, what it shows.

($\exists U' \subset U$ s.t. if $x(t') \in U'$ some t'

then $x(t)$ never leave U &

$x(t_1) \rightarrow x_\infty$ some critical pt $x'_\infty \in U$.

& $\int |\dot{x}| < \infty$.)

□

- $|\nabla f(x)| \geq c|f(x) - f(y)|^B$

  is trivial if $\nabla f(y) \neq 0$.

- Lo, showed $B \in [1/2, 1)$

$f(x) = x^n \rightarrow \nabla f = n x^{n-1}$.

$$\frac{1}{n}|\nabla f| \geq (x^n)^{\frac{n-1}{n}} = |f(x) - f(0)|^{\frac{n-1}{n}}.$$

- $\frac{n-1}{n} \geq \frac{1}{2} \quad n \geq 2$.

- $\nabla f(x) = 0 \Longrightarrow f(x) = f(y)$

  nearby critical pts have same potential

- $$f(x) = \begin{cases} e^{-\frac{1}{|x|}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Exercise Show that the convergence rate is no worse than algebraic decay

i.e. $\dot{x} = -\nabla f(x)$ f analytic

$x(t) \rightarrow 0$ as $t \rightarrow \infty$ then

$\exists\, C < \infty,\ \delta > 0$ s.t. $|x(t)| \leq C t^{-\frac{1}{\delta}}$

Thom's gradient conjecture.

$x(t) \longrightarrow x_\infty$ as $t \to \infty$.

$$\theta(t) = \frac{x(t) - x_\infty}{|x(t) - x_\infty|} \longrightarrow \theta_\infty \in S^{n-1} \; ?$$

Thm ['20. Ann. Math]

$x(t)$ has finite length & $\theta(t) \to \theta_\infty$

Pf by similar, but more sophisticated argument.

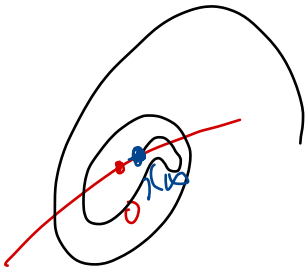
Łojasiewicz argument. (Review)

Suppose $\dot{x} = -\nabla f(x(t))$ & $\nabla f(0) = 0$

$f(0) = 0$

$\exists$ nbd U of 0, const $C > 0$, $\beta \in [\frac{1}{2}, 1)$

$$C\,|f(x)|^{\beta} \leq |\nabla f(x)|$$



Thm $\exists \varepsilon > 0$ s.t. if $\exists\, t_i \to \infty$

$|x(t_i) - 0| < \varepsilon$ then

$x(t) \to x_\infty \in U$ $\begin{cases} \nabla f(x_\infty) = 0 \\ f(x_\infty) = 0 \end{cases}$

(pf) $x(t) \in U$, then

$$\frac{df}{ds} \leq -C|f|^{\beta}$$

If $f(X(t)) \leq \delta < 0$, then $f(X(t)) \to -\infty$

$\therefore$ $f(X(t)) \equiv 0$ or $f(X(t)) > 0$.

$$\left(f^{1-\beta}\right)_s \leq -c(1-\beta)$$

$$f^{1-\beta}(S(t_2)) - f^{1-\beta}(S(t_1)) \leq -c(1-\beta)(S(t_2)-S(t_1))$$

$$S(t) - S(t_1) \leq \frac{1}{c(1-\beta)} f^{1-\beta}(X(t_1))$$

$t_* < \infty$ be maximal the s.t. $X(s) \in U$
$\forall s \in [t_1, t_*]$.

If $f^{1-\beta}(X(t_1))$ suff. small, then

$$S(t) - S(t_1) \leq \delta \to \underline{X(t) \in U}$$

$\therefore$ $\nexists$ $t_* = \infty$.

$S(t) < \infty$ $\lim_{t \to \infty} X(t) \to X_\infty \in U$

$\nabla f(X_\infty) = 0 \to \underline{f(X_\infty) = 0}$.

L. Simon '83 Ann. Math.

Σ closed Riem. mfd.

$u(x,t)$ $\quad x \in \Sigma,\ t \in [0,T)$

$$\dot{u} - \mathcal{M} u = Ru. \quad \text{or}$$

$$-\ddot{u} + \dot{u} - \mathcal{M} u = Ru.$$

$\mathcal{M} \sim -$ grad of some energy $\mathcal{E}$.

$R \sim$ <u>mild error</u>,

$$\mathcal{E}(u) = \int_\Sigma E(x, u, \nabla u)\, dx$$

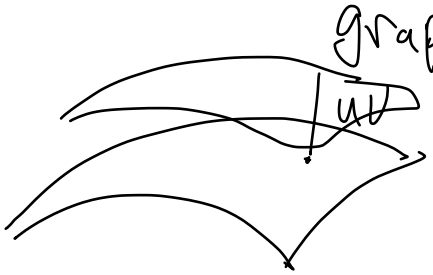
$E(x, z, p_x)$ analytic in z & p_x.

Σ $\quad \mathbb{R}$ $\quad T_x\Sigma$

$$d\mathcal{E}_u v = -\int_\Sigma v \cdot \mathcal{M}u \; dVol_{g(x)}$$

$-\mathcal{M}u$ is $L^2(\Sigma)$ grad of $\mathcal{E}(u)$.

Example



- $\mathcal{E}(u)$ = Area of graph of u

$$= \int_\Sigma E(x, u, \nabla u)\, dx$$

- $\mathcal{M}(0)$ = mean curvature of Σ.
- $\mathcal{M}(u) \approx$ mean curvature of graph u

Assumption $\mathcal{M}(0) \equiv 0$

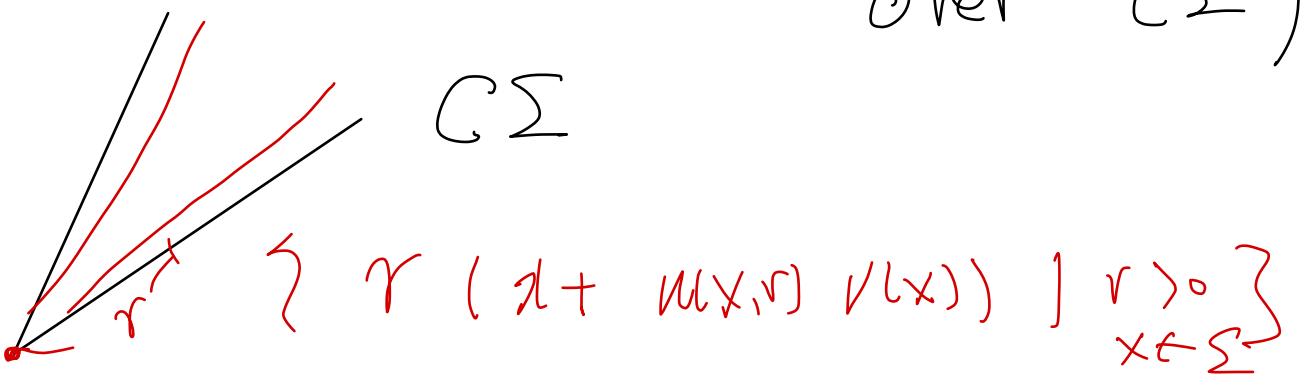
$\mathcal{M}u$ uniformly elliptic if $\|u\|_{C^1} < \varepsilon$.

- MCF (as a graph over $\Sigma \times [0,T]$)

$$u_t = \sqrt{1+|\nabla u|^2}\, Hu$$

$$= \underline{\mathcal{M}u + Ru}$$ "

- Minimal surface (as a graph over $C\Sigma$)



$$t = \ln r$$

$u(x,t)$ solves

$$\partial_{tt}^2 u + \mathcal{M}u + \partial_t u = R(u)$$

Rmk · Rescaled MCF, ok/A

· $\partial_t u$ is necessary

- If Σ_t MCF $\Sigma_{t_i} \to \Sigma$.

$\|u(\cdot, t_i)\|_{C^{2,\alpha}(\Sigma)} \to 0$ as $i \to \infty$.

- If $\mathcal{M}$ MS. $\lambda_i \mathcal{M} \to C\Sigma$.

$\exists\ t_i \to -\infty$ s.t.

$\|u(\cdot, t_i)\| \to 0$

Q: $\|u(\cdot, t)\| \to 0$ as $t \to \infty$ or $-\infty$?

Theorem [Łojasiewicz–Simon]

$\exists\ \sigma > 0$ s.t. if $\|u\|_{C^{2,\alpha}} < \sigma$,

then $\|\mathcal{M}(u)\|_{L^2} \geq |\mathcal{E}(u) - \mathcal{E}(0)|^{\beta}$

some $\beta \in [\frac{1}{2}, 1)$.

(cf. $|\nabla f(x)| \geq |f(x)-f(y)|^{\beta}$)

IDEA Lyapunov-Schmidt reduction.

$$\frac{\partial^2}{\partial s \partial t} \Sigma(sv_1 + tv_2)\Big|_{t=s=0} = \frac{\partial}{\partial s} \langle M(sv_1), v_2 \rangle$$

$$= - \langle L v_1, v_2 \rangle .$$

$L v_1 = \frac{\partial}{\partial s}\Big|_{s=0} M(sv_1)$ linearization of M at 0.

$\dim(\ker L) = k \in \mathbb{N}$.

Π : proj on $\ker L$. & $\Pi^{\perp}$.

$N = \Pi + M$

$dN_0 = \Pi + L$ local iom.

$C^{2,\alpha} \to C^{0,\alpha}$.

$\exists\ \psi : N_1 \subset C^{0,\alpha} \longrightarrow N_2 \subset C^{2,\alpha}$.

$N \circ \psi = Id \qquad \psi \circ N = Id$

ψ is an analytic map

UPSHOT

$\ker L \overset{\sim}{\subset} \mathbb{R}^K$.

$f : \ker L \longrightarrow \mathbb{R}$

$f(\xi) := \mathcal{E}(\psi(\xi))$

f is an analytic map

$|\nabla f(\xi)| \geq |f(\xi) - f(0)|^{\beta}$

↑ classical Loj. inequality.

$f(\pi u) = \mathcal{E}(\psi(\pi u))$ kinds of

well approximate $\mathcal{E}(u)$.

Indeed,

- $|\nabla f(\pi u)| \lesssim \|Mu\|_{L^2}$

&

- $|\mathcal{E}(\psi(\pi u)) - \mathcal{E}(u)| \lesssim \|Mu\|_{L^2}^2$.

$$\therefore \|Mu\| \gtrsim |\mathcal{E}(\psi\pi u) - \mathcal{E}(u) + \mathcal{E}(u) - \mathcal{E}(0)|^\beta$$

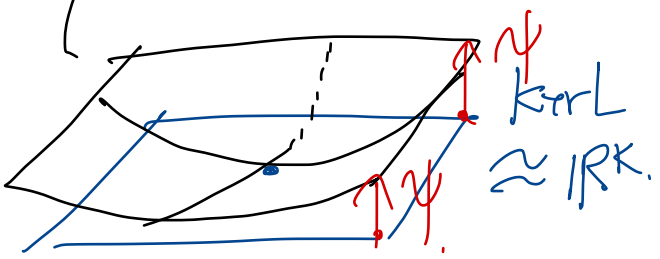
$$\gtrsim c\,|\mathcal{E}(u) - \mathcal{E}(0)|^\beta - C\,|\mathcal{E}(u) - \mathcal{E}(\pi u)|^\beta$$

$$\gtrsim |\mathcal{E}(u) - \mathcal{E}(0)|^\beta - C\,\|Mu\|^{2\beta}$$

$2\beta > 1$

$$\rightarrow \quad \underline{|\mathcal{E}(u) - \mathcal{E}(0)|^\beta \lesssim \|Mu\|}$$

graph ψ in $L^2(\Sigma)$.



$\checkmark$ solves eq on $(\ker L)^\perp$

$$\left[\left(\mathcal{M}[\psi(\xi)] \right)^\perp \equiv 0. \right.$$

$$\left. \mathcal{M}(\psi(\xi))^\top \approx -\nabla f(\xi) \right]$$

$$\textcircled{\Delta} \quad \{u \mid \mathcal{M}u \equiv 0\} = \{\psi(\xi) \mid \mathcal{M}(\psi(\xi)) \equiv 0\}$$

$$= \{\psi(\xi) \mid \nabla f(\xi) = 0\}$$

Critical pts are sub-variety of analytic k-dim submfd.

$$\|\nabla f(\pi u)\| \lesssim \|M(\psi\pi u)\|$$

$$\lesssim \|M(\psi\pi u) - M(u)\| + \|Mu\|$$

$$\lesssim \|\psi(\pi u) - \psi(\pi u + Mu)\|_{W^{1,2}} + \|Mu\|$$

$$u = \psi(\pi u + Mu) \lesssim \|Mu\| + \|Mu\|$$

$$|\mathcal{E}(\psi\pi u) - \mathcal{E}(u)|$$

$$= \left| \int \langle -M(u + t(\psi\pi u - u)),\ \psi\pi u - u \rangle \right|$$

$$\lesssim \|\psi\pi u - u\|_{L^\infty} \sup \|M(u + t(\ ))\|$$

$$\lesssim \|Mu\| \cdot \left(\|Mu\| + t\|\psi\pi u - u\|_{W^{1,2}} \right)$$

$$\lesssim \|Mu\|^2$$