Geometry of Equations

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Algebraic Geometry?

- Algebraic Geometry (AG) is a study on common zeroes of a system of multivariate polynomial equations ($\rightarrow$ Algebraic Variety).

$$\begin{aligned} f_1(x_1, \ldots, x_m) &= 0 \\ &\vdots \\ f_n(x_1, \ldots, x_m) &= 0, \end{aligned}$$

- In case $n = m = 1$, $\rightarrow$ an origin of modern algebra. Abel's theorem and Galois theory ...
- In case of all f_i having degree 1, $\rightarrow$ (undergraduate) Linear Algebra

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Examples

In general, algebraic variety (i.e. the solutions set) is quite complicate, but interesting. It's a ubiquitous object in science and engineering.

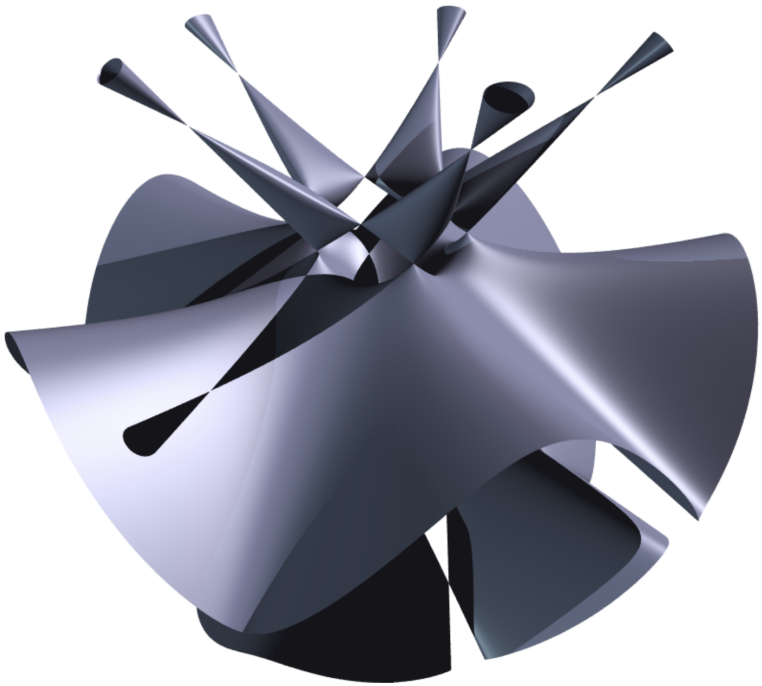


Figure: Togliatti's degree 5 surface

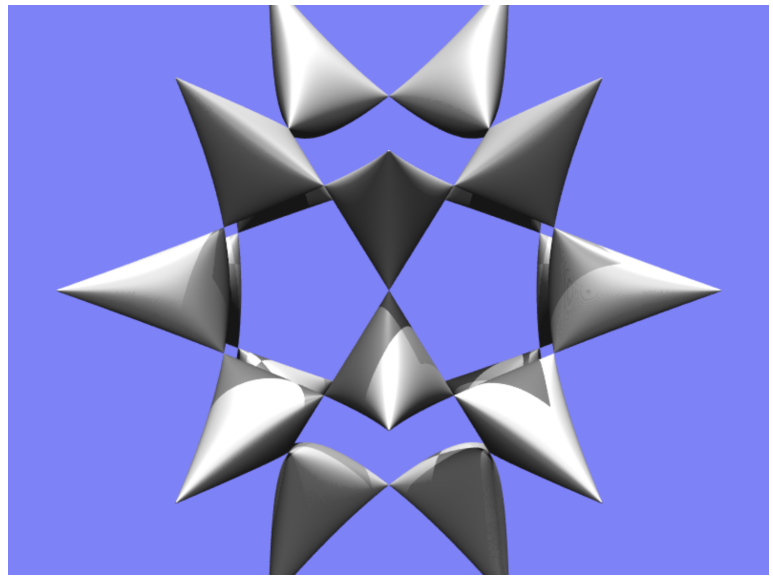


Figure: Barth's degree 6 surface

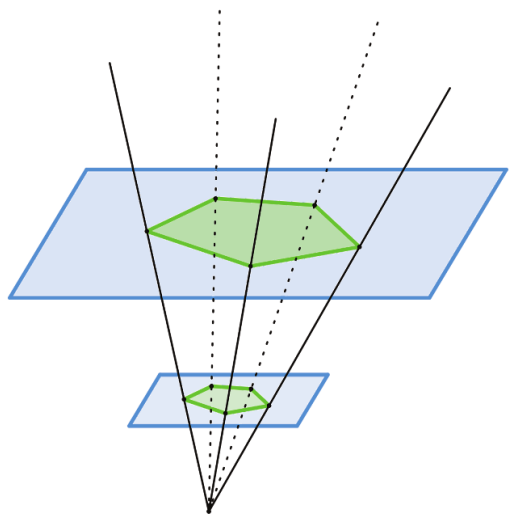
Projective n-space, Affine cone and Projective variety

Let V be an $(n+1)$-dimensional $\Bbbk$-vector space (mainly, $\Bbbk = \mathbb{R}$ or $\mathbb{C}$).

$\mathbb{P}V := V \setminus \{0\}/\text{non-zero scaling} = \{\text{1-dim'l subspaces of } V\}$
$\to \mathbb{P}V$ is called 'projective n-space'. Good ($\because$ cpt, Chow,...)

Let $W = S^d V^*$ be the d-th symmetric power of V^*. We can also think of W as the space of homogeneous polynomial functions (or forms) on V of degree d in $n+1$ variables.

For an $F \in W$ and any $\lambda \neq 0 \in \Bbbk$, $F(\lambda x) = \lambda^d F(x)$, so the zero set of F defines a '(affine) cone' in V and also a 'well-defined locus' in $\mathbb{P}V$.



Polynomial as a Point : Case of Quadrics

For instance, any nonzero real quadratic form $Q(\mathbf{x}) \in S^2V^*$ corresponds to a point in $\mathbb{P}S^2V^*$.

Further, $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ for a real symmetric $(n+1) \times (n+1)$-matrix.

$\rightarrow$ Most important information on A (i.e. rank, spectral theorem) can be understood geometrically!

$$A = U\Sigma U^T = \lambda_1 u_1 \cdot u_1^T + \cdots + \lambda_r u_r \cdot u_r^T \leftrightarrow Q = \lambda_1 u_1^2 + \cdots + \lambda_r u_r^2$$

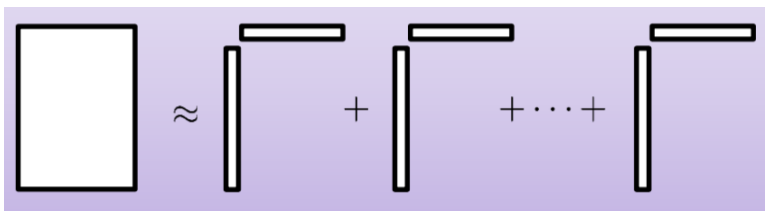


Figure: SVD of a matrix

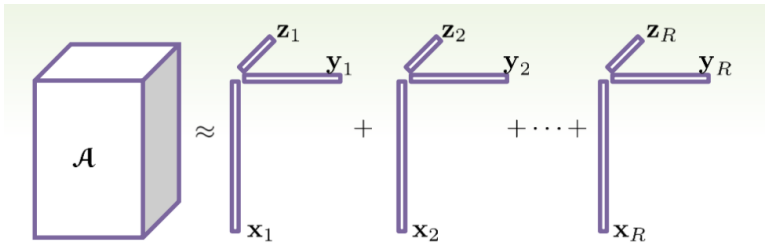


Figure: Tensor decomposition

Case of Higher degrees : Polynomial Waring Problem

Lagrange Four Square theorem : $n = a^2 + b^2 + c^2 + d^2$
$\rightarrow$ Waring Problem in number theory.

Similarly, for any nonzero form $F \in S^d V^*$ (corresponding to a point in $\mathbb{P}S^d V^*$), one can consider Polynomial Waring Problem as follows;

"Find an expression $F = \lambda_1 \ell_1^d + \cdots + \lambda_r \ell_r^d$ (ℓ_i: linear form) !!"

$\rightarrow$ Note that $\exists$ always such an expression for any F
$\rightarrow$ Finding an/all expression(s) : a Waring decomposition + Variety of Sums of Powers (VSP)
$\rightarrow$ Find approximation of rank k : useful **both** in theory and applications
$\rightarrow$ minimum r for F : Waring rank of F, $\mathrm{rank}_{\mathbb{k}}(F)$
$\rightarrow$ rank for a general (i.e. random) F : Generic rank of (d, n)
$\rightarrow$ maximum among all F's : Maximum rank of (d, n)
$\rightarrow$ etc etc...

Case of Higher degrees : Via Veronese map

There's a way to consider this Waring problem in a geometric manner: Use d-th Veronese embedding!

Let $W = S^d V$ be the d-th symmetric power of V (recall that we can also think of W as the space of homogeneous polynomials of degree d in $n+1$ variables).

$$v_d : \mathbb{P}V \to \mathbb{P}W, \quad v_d([\ell]) = [\ell^d] \ \ (\ell \in V).$$

The image $X = v_d(\mathbb{P}V)$ is called a Veronese variety. Then,

$$\begin{aligned} X &= \{d\text{-th power of linear forms}\} \\ &= \{\text{degree } d\text{-forms of Waring rank one}\} \\ &= \{\text{rank one symmetric } d\text{-way tensors}\} \end{aligned}$$

and we can interpret our Waring problem via k-secant plane to X.

X-rank

Similarly, for any variety $X \subset \mathbb{P}W$ (or equivalently, $\widehat{X} \subset W$, the affine cone of X), we can consider the addition map

$$+ : \widehat{X} \times \cdots \times \widehat{X} \longrightarrow W \ ,$$
$$+(v_1, \cdots, v_k) := v_1 + \cdots + v_k \ .$$

Then, for any nonzero $v \in W$, we define 'X-rank of v' $\mathrm{rank}_X(v)$ by the function $\mathrm{rank}_X : W \setminus \{0\} \to \mathbb{N}$

$$\mathrm{rank}_X(v) := \min\{r \mid v = +(v_1, \cdots, v_r),\ v_i \in \widehat{X}\} \ .$$

Note that $\mathrm{rank}_X(p)$ is well-defined for any $p \in X \subset \mathbb{P}W$, since $\mathrm{rank}_X(-)$ is invariant under scaling.

X-rank (cont'd)

- When $X = \{v_1 \otimes \cdots \otimes v_d\}$, that is $Seg(V_1 \times \cdots \times V_d)$ in $\mathbb{P}(V_1 \otimes \cdots \otimes V_d)$
- $\rightarrow$ (usual) tensor rank problem
- When $X = \{\ell^d\} = v_d(\mathbb{P}V) \rightarrow$ Waring rank (or Symmetric tensor rank) problem
- When $X = \{\ell_1 \ell_2 \cdots \ell_d\} \rightarrow$ Chow rank (or completely decomposable rank)
- Whatever X you like...

Theorem (Blekherman-Teitler, 2015) $\text{rank}_X^{max} \leq 2 \cdot \text{rank}_X^{gen}$ for any irreducible, nondegenerate $X \subset \mathbb{P}W$

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Waring rank of Monomials

Today, we consider $F = x_0^{a_0} x_1^{a_1} \dots x_n^{a_n}$, the monomial case.

Example (Binary case)

1. $xy = \frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2 \to \operatorname{rank}_{\mathbb{C}}(xy) = \operatorname{rank}_{\mathbb{R}}(xy) = 2$
2. $x^2y^2 = \frac{1}{18}(x+y)^4 + \frac{1}{18}\omega(x+\omega y)^4 + \frac{1}{18}\omega^2(x+\omega^2 y)^4$
   $\to \operatorname{rank}_{\mathbb{C}}(x^2y^2) = 3$
3. $x^2y^2 = \frac{1}{12}(x+y)^4 + \frac{1}{12}(x-y)^4 - \frac{1}{6}x^4 - \frac{1}{6}y^4 \to \operatorname{rank}_{\mathbb{R}}(x^2y^2) \leq 4$

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Waring rank of Monomials

Example (Ternary)

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$$+\omega^2(x+y+\omega^2 z)^6 + \omega^2(x+\omega^2 y+z)^6 + \omega^2(\omega^2 x+y+z)^6$$
$$\rightarrow \operatorname{rank}_{\mathbb{C}}(x^2y^2z^2) = 9$$
2. $$360x^2y^2z^2 = 4x^6 + 4y^6 + 4z^6 - 2(x+y)^6 - 2(x+z)^6 - 2(y+z)^6$$
$$-2(x-y)^6 - 2(x-z)^6 - 2(y-z)^6$$
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Waring rank of Monomials : Known results

Theorem (CCG and BBT, 2012)

For any monomial $X_0^{a_0} X_1^{a_1} \cdots X_n^{a_n}$ $(a_0 \geq a_1 \geq \cdots \geq a_0)$,

$$\operatorname{rank}_{\mathbb{C}}(X_0^{a_0} X_1^{a_1} \ldots X_n^{a_n}) = \frac{1}{a_n + 1} \prod_{i=0}^{n} (a_i + 1) \, .$$

Theorem (Boij-Carlini-Geramita, 2011)

For a binary monomial $x^a y^b$, $\operatorname{rank}_{\mathbb{R}}(x^a y^b) = a + b$.

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Moreover, $\operatorname{rank}_{\mathbb{C}} = \operatorname{rank}_{\mathbb{R}}$ *if and only if* $a_n = 1$.

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Waring rank of Monomials

Theorem (H. and Moon, 2021)

For a monomial $X_0^{a_0}X_1^{a_1}\dots X_n^{a_n}$ *with each* $a_i > 0$, *it holds that*

$$\operatorname{rank}_{\mathbb{R}}(X_0^{a_0}X_1^{a_1}\dots X_n^{a_n}) \leq \frac{1}{2}\left\{\prod_{i=0}^{n}(a_i+1) - \prod_{i=0}^{n}(a_i-1)\right\}. \quad (1)$$

Further, the same result is true for $\operatorname{rank}_{\mathbb{Q}}(X_0^{a_0}X_1^{a_1}\dots X_n^{a_n})$.

	$\mathbf{a} = (3,3,3)$	(4,3,3)	(4,4,3)	(4,4,4)	(5,5,5,5)	(7,7,7,7,7)	(10,9,8,7,6,5,4)	(7,7,7,7,7,7,7)
$\mathrm{UB}_{\mathrm{BT}}$	38	44	52	62	886	32902	8282766	8282766
$\mathrm{UB}_{\mathrm{CKOV}}$	36	42	49	64	1000	38416	2162160	7529536
$\mathrm{UB}_{\mathrm{HM}}$	28	34	41	49	520	12496	740880	908608

Table: Comparison of upper bounds for $\mathbb{R}$-rank of monomials

Finding explicit decompositions

Theorem (H. and Moon, 2022)

Let $\mathbf{a} = (a_0, a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^{n+1}$ *be a sequence of* $n+1$ *nonnegative integers with* $d = |\mathbf{a}|$. *Then, it holds that*

$$D_{\mathbf{a}} \cdot X_0^{a_0} X_1^{a_1} \cdots X_n^{a_n} = \sum_{Z_{\mathbf{a}} \subset A \subset E_{\mathbf{a}}} \sum_{(\mathbf{k},\mathbf{s}) \in K_A \times \overline{S_A}} C_{A,\mathbf{k},\mathbf{s}} \left(\ell_{A,\mathbf{k},\mathbf{s}}\right)^d \tag{2}$$

where

$$D_{\mathbf{a}} = (-1)^{|Z_{\mathbf{a}}|} \cdot 2^n \cdot \binom{d}{\mathbf{a}} \cdot \prod_{i \notin Z_{\mathbf{a}}} F_i(t^{a_i})$$

and

$$C_{A,\mathbf{k},\mathbf{s}} = (-1)^{|A|} \cdot 2^{|A|} \cdot (-1)^{\sum_{i \notin A} a_i \cdot s_i} \prod_{i \in A} F_i(1) \prod_{i \notin A} c(y^{k_i}, F_i)$$

Decompositions of $X_0^4X_1^3X_2^2$

$$
\begin{aligned}
X_0^4X_1^3X_2^2 = \frac{1}{5040}\Bigg[& \frac{1}{(t^2-1)^2}(X_0+X_1+X_2)^9 + \frac{1}{(t^2-1)^2}(X_0+X_1-X_2)^9 - \frac{1}{(t^2-1)^2}(X_0-X_1+X_2)^9 \\
& - \frac{1}{(t^2-1)^2}(X_0-X_1-X_2)^9 - \frac{1}{t(t^2-1)^2}(X_0+tX_1+X_2)^9 - \frac{1}{t(t^2-1)^2}(X_0+tX_1-X_2)^9 \\
& + \frac{1}{t(t^2-1)^2}(X_0-tX_1+X_2)^9 + \frac{1}{t(t^2-1)^2}(X_0-tX_1-X_2)^9 - \frac{1}{t^2(t^2-1)^2}(X_0+X_1+tX_2)^9 \\
& - \frac{1}{t^2(t^2-1)^2}(X_0+X_1-tX_2)^9 + \frac{1}{t^2(t^2-1)^2}(X_0-X_1+tX_2)^9 + \frac{1}{t^2(t^2-1)^2}(X_0-X_1-tX_2)^9 \\
& + \frac{1}{t^3(t^2-1)^2}(X_0+tX_1+tX_2)^9 + \frac{1}{t^3(t^2-1)^2}(X_0+tX_1-tX_2)^9 - \frac{1}{t^3(t^2-1)^2}(X_0-tX_1+tX_2)^9 \\
& - \frac{1}{t^3(t^2-1)^2}(X_0-tX_1-tX_2)^9 - \frac{2}{t^2(t^2-1)}(X_1+X_2)^9 - \frac{2}{t^2(t^2-1)}(X_1-X_2)^9 \\
& + \frac{2}{t^3(t^2-1)}(tX_1+X_2)^9 + \frac{2}{t^3(t^2-1)}(tX_1-X_2)^9 - \frac{2(t^6+1)}{(t^2-1)^2}(X_0+X_1)^2 + \frac{2(t^6+1)}{(t^2-1)^2}(X_0-X_1)^9 \\
& + \frac{2}{t(t^2-1)^2}(X_0+tX_1)^9 - \frac{2}{t(t^2-1)^2}(X_0-tX_1)^9 + \frac{2}{t^2(t^2-1)^2}(tX_0+X_1)^9 - \frac{2}{t^2(t^2-1)^2}(tX_0-X_1)^9 \\
& - \frac{4(t^2+1)(t^4+1)}{t^2}X_1^9\Bigg],
\end{aligned}
$$

which gives us a 1-dimensional family of a Waring decomposition of $X_0^4X_1^3X_2^2$ (i.e. for all $t \in \Bbbk$ with $5040t^3(t^2-1)^2 \neq 0$) of $\frac{1}{2}(5\cdot 4\cdot 3 - 3\cdot 2\cdot 1) = 27$ summands.

Key algebraic tool: Apolar ideal

Let $\Bbbk = \mathbb{R}$ or $\mathbb{C}$. For any $F \in \Bbbk[X_0, \ldots, X_n]_d$,

$$F^\perp := \{\Psi \in \Bbbk[x_0, \ldots, x_n] : \Psi(\partial_0, \cdots, \partial_n) \circ F = 0\}$$

which is a homogeneous ideal in $\Bbbk[x_0, \ldots, x_n]$. We call this $F^\perp$ the apolar ideal of F.

Example

For a monomial $F = X^2Y^2$, $F^\perp = (x^3, y^3)$.
$J = (ax^3 + by^3) \subset F^\perp$ gives the $\text{rank}_{\mathbb{C}} = 3$ and $\text{rank}_{\mathbb{R}} > 3$ (recall that $x^2y^2 = \frac{1}{18}(x+y)^4 + \frac{1}{18}\omega(x+\omega y)^4 + \frac{1}{18}\omega^2(x+\omega^2 y)^4$).
$J = (x^3y - xy^3) \subset F^\perp$ gives the $\text{rank}_{\mathbb{R}} = 4$ (similarly, $x^2y^2 = \frac{1}{12}(x+y)^4 + \frac{1}{12}(x-y)^4 - \frac{1}{6}x^4 - \frac{1}{6}y^4$)

Key algebraic tool: Apolar ideal

Proposition (Apolarity Lemma)

Let $F \in S^d V$. There exist scalars $\lambda_1, \ldots, \lambda_r$ in $\Bbbk$ such that $F = \sum_{i=1}^{r} \lambda_i L_i^d$ with $L_i = c_{i0}X_0 + \cdots + c_{in}X_n$ ($c_{ij} \in \Bbbk$) if and only if $I_{\mathbb{X}} \subset F^{\perp}$, where $\mathbb{X} = \{[c_{10} : \cdots : c_{1n}], \cdots, [c_{r0} : \cdots : c_{rn}]\}$ is a set of reduced $\Bbbk$-points in $\mathbb{P}V$.

Difficulty : To find suitable $\mathbb{X}$ + To show that $I_{\mathbb{X}}$ is radical and saturated.

Thanks for your attention.