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Cosmology 2

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Basic Theory

- Dynamics : scale factor 의 시간에 따른 변화

아인슈타인 방정식

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Einstein 텐서 — geometry : RW metric

Energy-Momentum 텐서 — matter in the Universe, perfect fluid

Most simplest Universe: homogenous and isotropic

metric tensor

$$g_{\mu\nu} = (1, -R^2(t), -R^2(t), -R^2(t))$$
$$g^{\mu\nu} = (1, -R^{-2}(t), R^{-2}(t), -R^{-2}(t))$$

$$ds^2 = dt^2 - R(t)^2 d\mathbf{x}^2$$

energy momentum tensor

$$T^{\mu}_{\nu} = \mathrm{diag}(\rho, -p, -p, -p)$$

$$ds^2 = dt^2 - R(t)^2 d\mathbf{x}^2$$

dx: coordinate distance

R(t)dx: physical distance

Expansion of the Universe comes form R(t)

To know how R changes, we need to solve Einstein equation, with

Einstein tensor $$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu}}$$

With FRW metric,

$$g_{00} = 1$$

$$g_{0i} = 0$$

$$g_{ij} = -R^2(t)\left(\delta_{ij} + k\frac{x^i x^j}{1 - k\vec{x}^2}\right)$$

Ricci tensor $$R_{00} = -3\frac{\ddot{R}}{R}$$

$$R_{ij} = -\left[\frac{\ddot{R}}{R} + 2\frac{\dot{R}^2}{R^2} + \frac{2k}{R^2}\right] g_{ij}$$

Ricci scalar $$\mathcal{R} = -6\left[\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2}\right]$$

---

gives $$G_{00} = 3\left[\frac{\dot{R}^2}{R^2} + \frac{k}{R^2}\right] \qquad G_{ij} = \left[2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2}\right] g_{ij}$$

and Energy momentum tensor

$$T_{00} = g_{0\alpha}T^{\alpha}_{0} = \rho \qquad T_{ii} = g_{i\alpha}T^{\alpha}_{i} = g_{ii}T^{i}_{i} = g_{ii}(-p)$$

gives (0,0) component in the zero-th order : Friedmann-Lemaitre equation

$$\boxed{H^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho} \quad \text{(Eq.2)} \qquad H \equiv \frac{\dot{R}(t)}{R(t)}$$

gives (i,i) component in the zero-th order

$$\boxed{2\frac{\ddot{R}}{R} + H^2 + \frac{k}{R^2} = -8\pi G p} \quad \text{(Eq.3)}$$

The difference gives

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p)$$

for k=1

$$q_0 \equiv -\frac{1}{H_0^2}\frac{\ddot{R}(t_0)}{R(t_0)} = \Omega_0 \frac{1+3\omega}{2}$$

Eqs. 1, 2, 3 are connected by Bianchi identities and only two are independent. Usually Eqs. 1 and 2 are taken.

Energy-momentum conservation:

$$T^{\mu\nu}{}_{;\nu} = 0 \quad \xrightarrow[\mu = 0]{}$$

Continuity equation

$$\boxed{\dot{\rho} + 3H(\rho + p) = 0} \quad \textbf{(Eq.1)}$$

or $\quad d(\rho R^3) = -p d(R^3)$

with equation of state $\boxed{p = p(\rho)}$

Radiation	$(p = \frac{1}{3}\rho)$
(Cold) Matter	$(p = 0)$
Vacuum energy (cosmological constant)	$(p = -\rho)$

Many kinds of matter content of our interest can be described by a simple eq. of state

$$\boxed{p = \omega\rho}$$

We can solve the continuity equation, $\boxed{\dot{\rho} + 3H(\rho + p) = 0}$

$$\frac{d\rho}{dt} + 3\frac{\dot{R}}{R}(1+\omega)\rho = 0$$

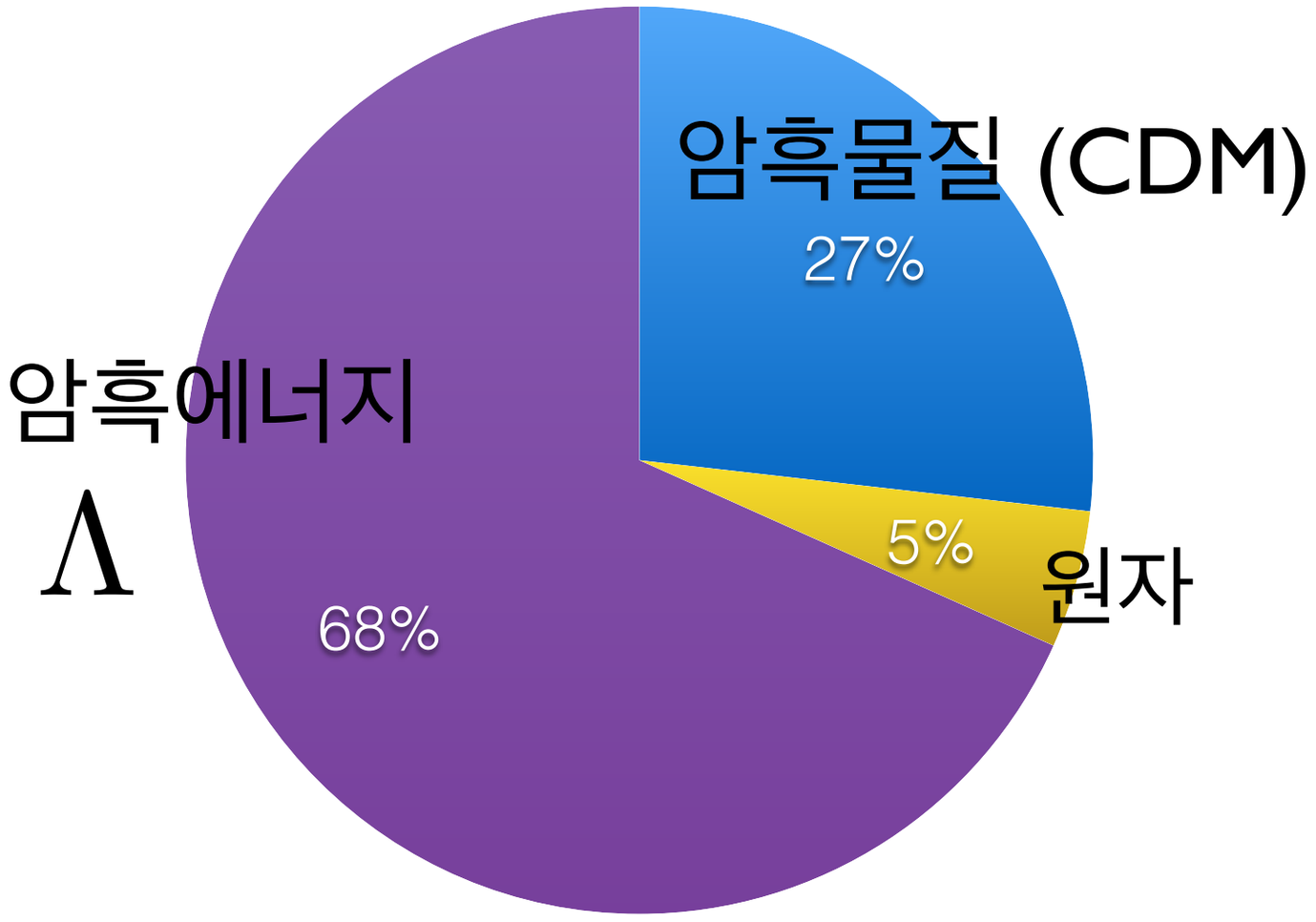
$$\frac{d\rho}{\rho} = -3(1+\omega)\frac{dR}{R}$$

$$\log\frac{\rho}{\rho_0} = -3(1+\omega)\log\frac{R}{R_0}$$

$$\rho(t) = \rho(t_0)\left(\frac{R(t)}{R(t_0)}\right)^{-3(1+\omega)}$$

Composition of present Universe

$$\rho_c = 3H_0^2 M_P^2 = 1.88 \times 10^{-29} \mathrm{g\,cm^{-3}}$$



Radiation : 0.005 %

Neutrinos : less than 5 %

: ordinary baryonic matters

Critical density

Friedmann equation $$\frac{k}{R^2H^2} = \frac{\rho}{3H^2/8\pi G} - 1 \equiv \Omega - 1$$

Critical density $$\rho_c \equiv \frac{3H^2}{8\pi G}$$

Density parameters $\Omega_i \equiv \dfrac{\rho_i}{\rho_c}$ Total density $\Omega = \displaystyle\sum_i \Omega_i = \frac{\sum_i \rho_i}{\rho_c}$

Local geometry is related to the density in the Universe

Closed	$k > 0, \quad \Omega > 1$
Flat	$k = 0, \quad \Omega = 1$
Open	$k < 0, \quad \Omega < 1$

At present time, $\rho_c = 3H_0^2 M_P^2 = 1.88 \times 10^{-29} \mathrm{g\,cm^{-3}}$

Photons are in the Planck distribution and the energy density is

$$\rho_{\gamma,0} = a_B T_{\gamma,0}^4 = 4.64 \times 10^{-34}\,\mathrm{g\,cm^{-3}}$$

Radiation energy constant $a_B = \dfrac{8\pi^5 k_B^4}{15 h^3 c^3} = 7.5657 \times 10^{-15}\, erg\, cm^{-3}\, deg^{-4}$

Therefore $\boxed{\Omega_{\gamma,0} \simeq 5 \times 10^{-5}}$

Baryons are 5×10^{-9} numbers compared to the photons from BBN.

$$\rho_B = m_N N_B \simeq 938\,\mathrm{MeV} \times 5 \times 10^{-9} n_\gamma$$

$$n_\gamma = 421\ \mathrm{cm^{-3}}$$

Therefore $\boxed{\Omega_B \simeq 0.04}$

$\rho \propto R^{-4}$ Radiation

$\rho \propto R^{-3}$ (Cold) Matter

$\rho \propto \text{const}$ Vacuum energy

At present time,

$$\rho_R^0 = 5 \times 10^{-5} \rho_c^0$$

$$\rho_M^0 = \rho_{DM}^0 + \rho_B^0 = 0.32 \rho_c^0$$

$$\rho_{vac}^0 = 0.68 \rho_c^0$$

We can go backward in time with values at present.

Matter density = Vacuum density at

$$\rho_M = \rho_M^0 \left(\frac{R}{R_0} \right)^{-3} = \rho_v = \rho_v^0 \quad \Rightarrow \quad \frac{1}{1+z} = \frac{R}{R_0} = \left(\frac{0.68}{0.32} \right)^{-1/3} \simeq 0.78$$

or $z \simeq 0.29$

Why does the Vacuum energy dominates recently?

Coincidence problem

$\rho \propto R^{-4}$ Radiation

$\rho \propto R^{-3}$ (Cold) Matter

$\rho \propto \text{const}$ Vacuum energy

At present time,

$$\rho_R^0 = 5 \times 10^{-5} \rho_c^0$$

$$\rho_M^0 = \rho_{DM}^0 + \rho_B^0 = 0.32 \rho_c^0$$

$$\rho_{vac}^0 = 0.68 \rho_c^0$$

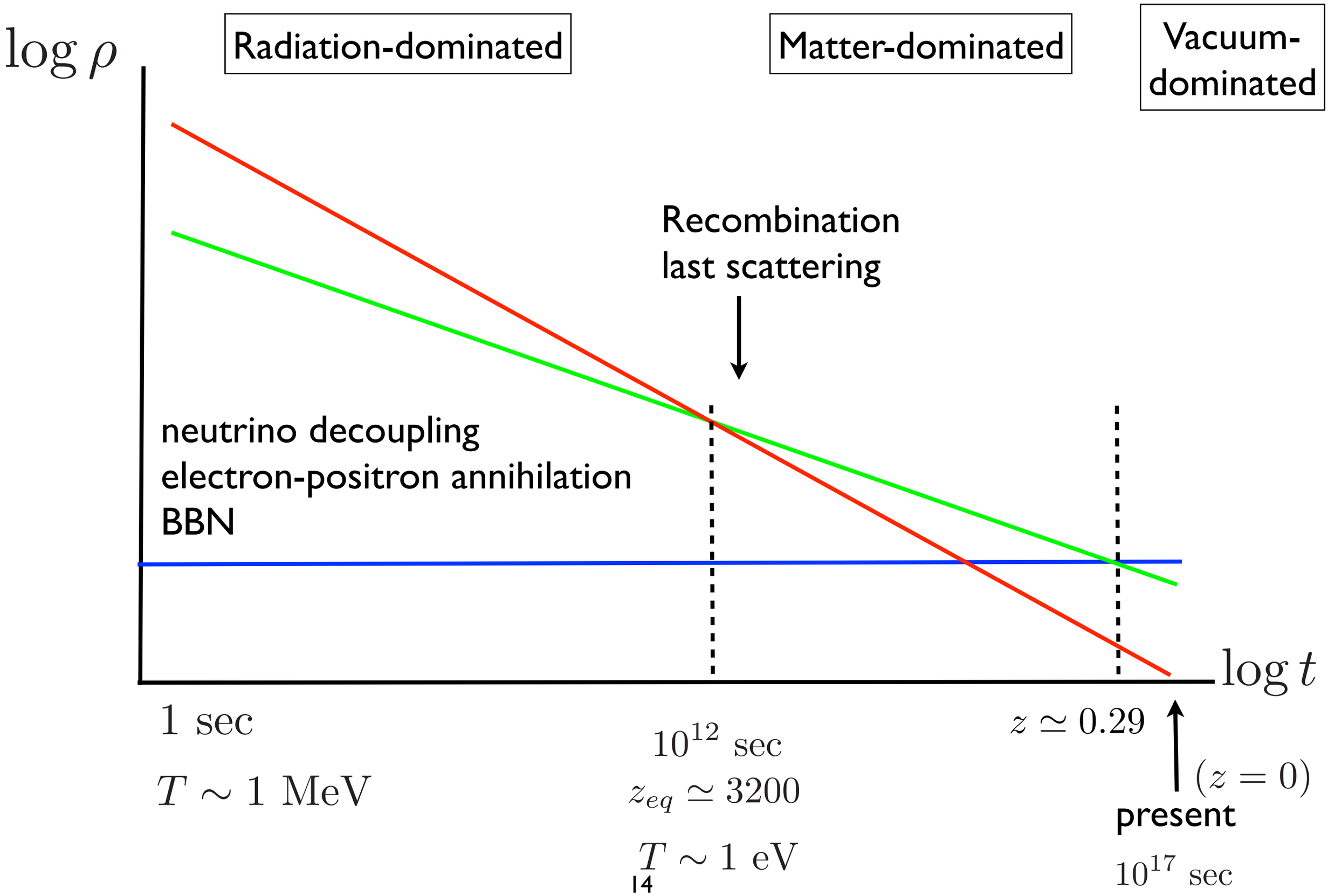
Radiation density = Matter density at

$$\left(\frac{R}{R_0} \right)^{-1} = \frac{\rho_M^0}{\rho_R^0} = \frac{0.32}{5 \times 10^{-5}} \simeq 6400$$

If we include neutrino in the relativistic particles,

$$z_{eq} \simeq 3200 \quad \text{(70,000 years)}$$

- Evolution of energy density



Scale factor as a function of time

We can solve Friedmann equation with a given matter type

For k=0 (flat), and $p = \omega\rho$ then from $H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho_i\left(\frac{R}{R_i}\right)^{-3(1+\omega)}$

we obtain $\boxed{R \propto t^{2/3(1+\omega)}}$

Radiation	$(p = \frac{1}{3}\rho)$	$R \propto t^{1/2}$
Matter	$(p = 0)$	$R \propto t^{2/3}$
Vacuum energy	$(p = -\rho)$	$R \propto \exp(Ht)$

The age of the Universe

$$\frac{R_0}{R} = 1 + z \equiv x^{-1}$$

$$\frac{R}{\dot{R}}\frac{R_0}{R}\frac{dR}{R_0} = \frac{dx}{Hx}$$

$$t = \int_0^t dt' = \int_0^{R(t)} \frac{dR'}{\dot{R}'}$$

$$= H_0^{-1} \int_0^{(1+z)^{-1}} \frac{dx}{[1 - \Omega_0 + \sum_i \Omega_{i0} x^{-1-3\omega_i}]^{1/2}}$$

For flat Universe, $(\Omega_0 = 1)$, the age of the present universe is

$$t_0 = \frac{2}{3(1+\omega)} H_0^{-1}$$

$$t_0 = \frac{1}{2} H_0^{-1} \quad \text{Radiation} \ (p = \frac{1}{3}\rho)$$

$$t_0 = \frac{2}{3} H_0^{-1} \quad \text{Matter} \ (p = 0)$$

$$H_0^{-1} = 9.78 h^{-1} \times 10^9 \,\text{yr}$$

$$= 3000 h^{-1} \,\text{Mpc}$$

$$\frac{2}{3} H_0^{-1} \simeq 9.3 \; Gyrs$$

Homogeneous and Isotropic Universe : FRW metric and Perfect fluid

$$g_{\mu\nu} = (1, -R^2(t), -R^2(t), -R^2(t))$$
$$g^{\mu\nu} = (1, -R^{-2}(t), R^{-2}(t), -R^{-2}(t))$$

$$\boxed{G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu - p g_{\mu\nu}$$

$$\boxed{H^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho}$$

$$\boxed{\dot{\rho} + 3H(\rho + p) = 0}$$

Radiation	$(p = \frac{1}{3}\rho)$	$R \propto t^{1/2}$
Matter	$(p = 0)$	$R \propto t^{2/3}$
Vacuum energy	$(p = -\rho)$	$R \propto \exp(Ht)$

$$\rho \propto R^{-3(1+\omega)} \begin{cases} \rho \propto R^{-4} & \text{Radiation} \quad (p = \frac{1}{3}\rho) \\ \rho \propto R^{-3} & \text{(Cold) Matter} \quad (p = 0) \\ \rho \propto \text{const} & \text{Vacuum energy (cosmological constant)} \quad (p = -\rho) \end{cases}$$

- 우주속 물질 상태의 진화

매우 차가운 거대 구조 형성



은하,은하단

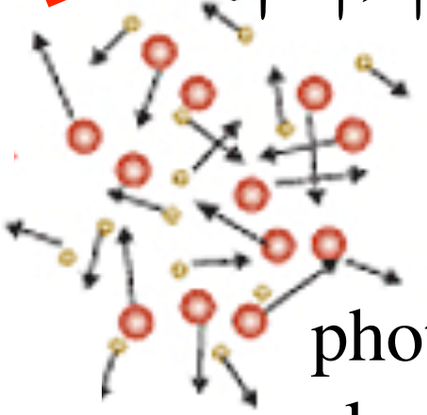


Recombination,
photon decoupling
: CMB

Gas

물질지배

복사지배



photons,
electrons,
light nuclei

매우 뜨거운

쿼크, 글루온,
전자의 플라즈마

열적평형 상태

양성자, 중성자,
전자의 플라즈마

Quark-hadron transition

BBN

The Standard Model of Particle Physics

Fundamental particles

BOSONS

Unified Electroweak spin = 1

Name	Mass GeV/c^2	Electric charge
γ photon	0	0
W^-	80.39	–1
W^+ W bosons	80.39	+1
Z^0 Z boson	91.188	0

Strong (color) spin =1

Name	Mass GeV/c^2	Electric charge
g gluon	0	0

Higgs 125 GeV

FERMIONS

matter constituents
spin = 1/2, 3/2, 5/2, …

Leptons spin =1/2

Flavor	Mass GeV/c^2	Electric charge
ν_L lightest neutrino*	$(0–0.13)\times10^{-9}$	0
e electron	0.000511	–1
ν_M middle neutrino*	$(0.009–0.13)\times10^{-9}$	0
μ muon	0.106	–1
ν_H heaviest neutrino*	$(0.04–0.14)\times10^{-9}$	0
τ tau	1.777	–1

Quarks spin =1/2

Flavor	Approx. Mass GeV/c^2	Electric charge
u up	0.002	2/3
d down	0.005	–1/3
c charm	1.3	2/3
s strange	0.1	–1/3
t top	173	2/3
b bottom	4.2	–1/3

• 열적 평형상태

In the non-expanding Universe,
with infinite of time, the system leads to the thermal equilibrium.

Thermal equilibrium: Kinetic eq. + Chemical eq.

$A + B \leftrightarrow A + B$

by elastic scattering
: changes energy and reshape
the distribution function
to Fermi-Dirac for fermions and
Bose-Einstein for bosons

$A + B \leftrightarrow C + D$

by inelastic scattering
: changes numbers of the
species and satisfy
between chemical potentials

$$\mu_A + \mu_B = \mu_C + \mu_D$$

In thermal equilibrium the entropy is maximized and constant.

Thermal equilibrium:

phase distribution function $$f(\tilde{\mathbf{p}}) = \frac{\mathbf{1}}{\exp((\mathbf{E}-\mu)/\mathbf{T}) \pm \mathbf{1}}$$

$$E^2 = |\vec{\mathbf{p}}|^2 + m^2$$

μ : chemical potential

Fermi-Dirac for fermions: +1

Bose-Einstein for bosons: -1

number density $$n = \frac{g}{(2\pi)^3} \int f(\tilde{\mathbf{p}}) \mathbf{d^3 p}$$

energy density $$\rho = \frac{g}{(2\pi)^3} \int E(\tilde{\mathbf{p}}) \mathbf{f}(\tilde{\mathbf{p}}) \mathbf{d^3 p}$$

pressure $$p = \frac{g}{(2\pi)^3} \int \frac{|\vec{\mathbf{p}}|^2}{3E}(\tilde{\mathbf{p}}) \mathbf{f}(\tilde{\mathbf{p}}) \mathbf{d^3 p}$$

g : internal degrees of freedom

In the thermal (kinetic) equilibrium, $f(\tilde{\mathbf{p}}) = \dfrac{1}{\exp((\mathbf{E}-\mu)/\mathbf{T}) \pm 1}$

In the relativistic limit (at high temperature), Radiation-like $\boxed{p = \dfrac{\rho}{3}}$

$$(m \ll T), \quad \text{for} \quad \mu \ll T$$

Bosson $\quad \rho = \dfrac{\pi^2}{30} g T^4 \qquad n = \dfrac{\zeta(3)}{\pi^2} g T^3 \qquad \langle E \rangle \equiv \dfrac{\rho}{n} = \dfrac{\pi^4}{30\zeta(3)} T \simeq 2.701 T$

Fermion $\quad \rho = \dfrac{7}{8}\left(\dfrac{\pi^2}{30}\right) g T^4 \qquad n = \dfrac{3}{4}\dfrac{\zeta(3)}{\pi^2} g T^3 \qquad \langle E \rangle = \dfrac{7\pi^4}{180\zeta(3)} \simeq 3.151 T$

In the non-relativistic limit (at low temperature), Matter-like

$$m \gg T \text{ , Boltzmann suppressed,}$$

For both Boson and Fermion,

$$\rho = mn \qquad n = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-(m-\mu)/T} \qquad \langle E \rangle = m + \frac{3}{2}T$$

$$\boxed{p = nT \ll \rho}$$

Total Energy density: $\rho_R = \frac{\pi^2}{30} g_* T^4$

mostly from relativistic particles

$$g_* = \sum_{bosons} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{fermions} g_i \left(\frac{T_i}{T}\right)^4$$

Entropy density $s = \frac{2\pi^2}{45} g_{*s} T^3$

$$g_{*S} = \sum_{bosons} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{fermions} g_i \left(\frac{T_i}{T}\right)^3$$

Entropy is conserved in the comoving volume, $s(T) R^3 = \text{constant}$

Time-temperature relation

$$t = 0.301 g_*^{-1/2} \frac{M_P}{T^2} \sim \left(\frac{T}{\text{MeV}}\right)^{-2} \text{sec}$$

$$T \gg M_h$$

At high temperature much larger than, electro-weak scale, all SM particles are massless and in the thermal equilibrium and relativistic.

$$\rho_R = \frac{\pi^2}{30} g_* T^4 \qquad\qquad p_R = \frac{1}{3}\rho_R = \frac{\pi^2}{90} g_* T^4$$

with effective degrees of freedom,

$$g_* = \sum_{bosons} g_i + \frac{7}{8} \sum_{fermions} g_i$$

$$g_* = 4 + 8 \times 2 + 2 + 2 \times 2 + 2 + \frac{7}{8}(72 + 12 + 6) = 106.75$$

Higgs doublet, gluons, Z-boson, W-boson, photon, quarks, charged leptons, neutrinos

$$1\ \mathrm{MeV} = 1.1605 \times 10^{10}\,\mathrm{K}$$

$T \gg M_h$ All SM particles are in the thermal eq. and relativistic

$$\rho_R = \frac{\pi^2}{30} g_* T^4 \quad \text{with} \quad g_* = 106.75$$

-(8/7)*12

$T \ll m_{top}$ top-quark decouples from relativistic particles.

$$g_* = 93.04$$

-1 -3

$T \ll M_Z$ Higgs and Z-boson becomes non-relativistic and the energy density is suppressed.

$$g_* = 89.04$$

-6

$T \ll M_W$ W-boson becomes non-relativistic

$$g_* = 83.04$$

and so on

For entropy density, $g_{*S} = g_*$ at high temperature before electron-positron annihilations.

At temperature $T \gtrsim 1$ MeV

$$1 \text{ MeV} = 1.1605 \times 10^{10} \text{ K}$$

In the Universe, photons, electrons, positrons, and neutrinos are relativistic, and small number of non-relativistic baryons (protons and neutrons) remains. They are in the thermal equilibrium by weak interactions.

$$g_* = 10.75 \qquad 2 + \frac{7}{8}(4 + 3 \times 2)$$

Soon the neutrinos are decoupled from the thermal plasma.

Interaction rate

$$\Gamma = n\sigma v \sim G_F^2 T^5$$

Expansion rate

$$H = \sqrt{\frac{\pi^2}{90}} g_*^{1/2} \frac{T^2}{M_P}$$

Both becomes similar at

$$T \simeq (M_P G_F^2)^{-1/3} \simeq 1 \text{ MeV} \qquad \text{or t=1 sec.}$$

Neutrinos are still relativistic, after freeze-out of neutrinos,

$$g_* = 10.75$$

After neutrino decoupling, there are two decoupled systems with same temperature.

photons, electrons, positrons, baryons

neutrinos

$T_\nu \propto R^{-1}$

"neutrino background"

In each system the comoving entropy is conserved.

The electrons and positrons annihilate into photons at $T \lesssim m_e$

$$g_{*S,before} T^3 = g_{*S,after} T_\gamma^3$$

$$g_{*S,before} = 2 + \frac{7}{8} \times 4 \qquad g_{*S,after} = 2$$

$$\longrightarrow \quad \frac{T_\gamma}{T_\nu} = \left(\frac{11}{4}\right)^{1/3} = 1.401$$

$T_\gamma = 2.725\,\mathrm{K}$

present neutrino temperature $T_\nu = 1.945\,\mathrm{K}$

Energy density after electron-positron annihilation

The energy density of the Universe was dominated for a long while by photons, and (anti-)neutrinos with all of them highly relativistic. The effective degrees of freedoms are given by

$$g_* = 2 + \frac{7}{8} \times 2 \times 3 \times \left(\frac{4}{11}\right)^{4/3} = 3.36$$

$$g_{*S} = 2 + \frac{7}{8} \times 2 \times 3 \times \left(\frac{4}{11}\right) = 3.91$$

In the present Universe, with $T = 2.75\,\mathrm{K}$

$$\rho_R = \frac{\pi^2}{30} g_* T^4 = 8.09 \times 10^{-34} \mathrm{g\ cm^{-3}}$$

$$s = \frac{2\pi^2}{45} g_{*S} T^3 \simeq 2970\ \mathrm{cm^{-3}}$$

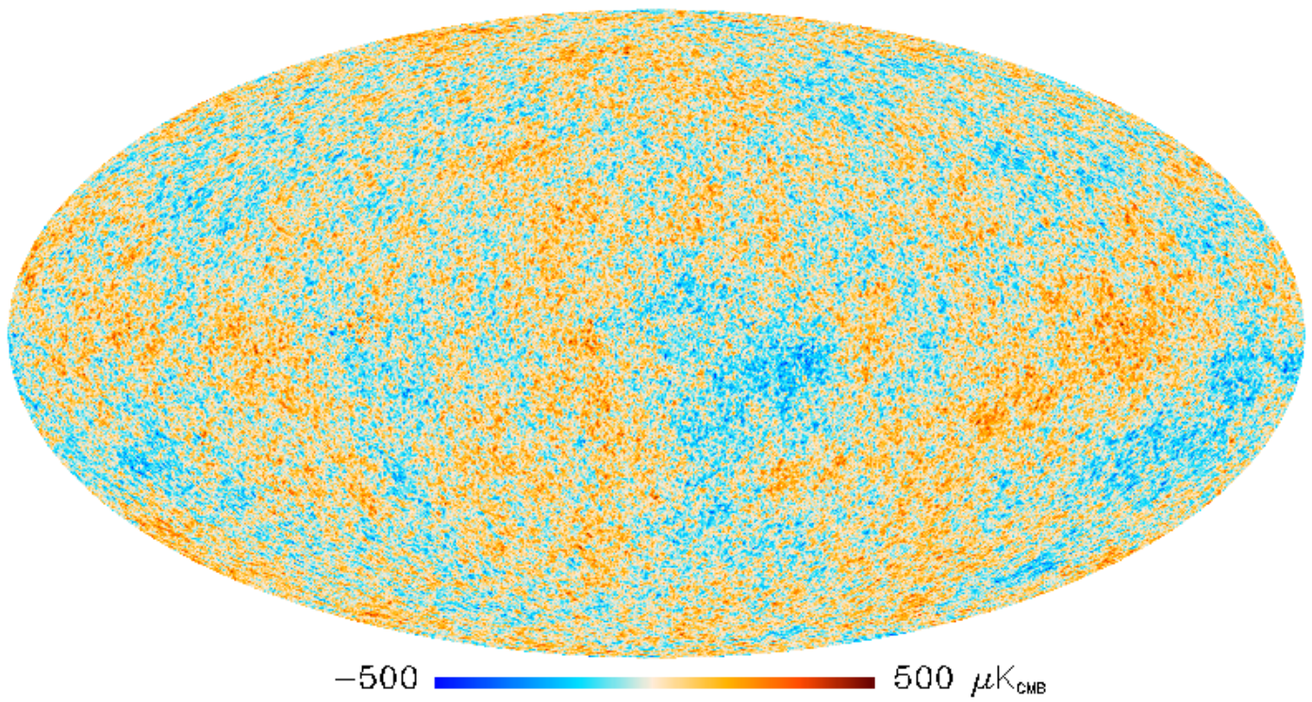
$$n_\gamma = \frac{2\zeta(3)}{\pi^2} T^3 = 422\ \mathrm{cm^{-3}}$$

Up to now

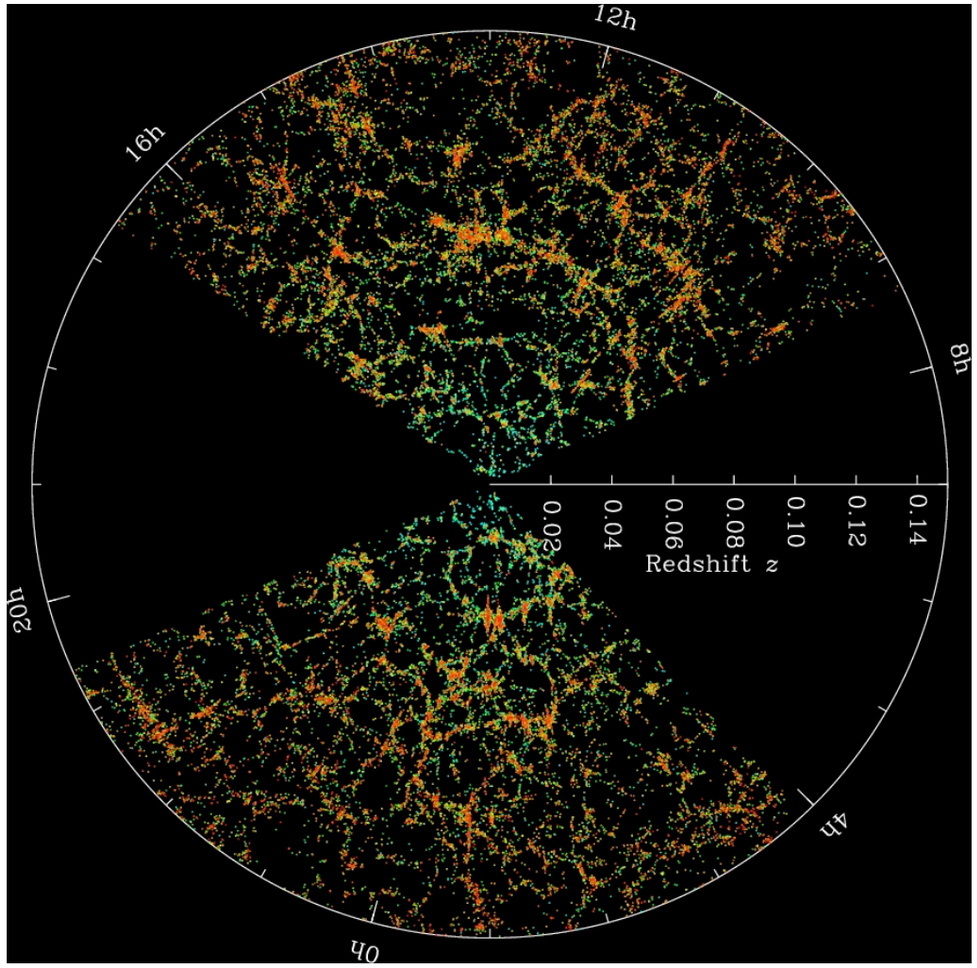
$$\rho(t, \mathbf{x}) = \rho(t), T(t, \mathbf{x}) = T(t), \cdots$$

Cosmological principle:
Homogeneous and Isotropic Universe

However



- CMB temperature anisotropy



- Large Scale Structure formation

We need In-homogeneous and
an-isotropic Universe

We need to consider

$$\rho(t, \mathbf{x}) = \rho_0(t) + \delta\rho(t, \mathbf{x}), T(t, \mathbf{x}) = T_0(t) + \delta T(t, \mathbf{x}), \cdots$$

Initial Small fluctuations to the background values in the early Universe

$$\frac{\delta\rho(t, \mathbf{x})}{\rho_0(t)} \sim \frac{\delta T(t, \mathbf{x})}{T_0(t)} \sim 10^{-5}$$

We can solve Einstein equation

Einstein equation

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

But with linear perturbation

$$ds^2 = (1+2\phi)dt^2 - a^2[(1-2\psi)\delta_{ij} - h_{ij}]dx^i dx^j$$

$$\rho(t,\mathbf{x}) = \rho_0(t) + \delta\rho(t,\mathbf{x}), T(t,\mathbf{x}) = T_0(t) + \delta T(t,\mathbf{x}), \cdots$$

we can solve the perturbed equations

$$G^{\mu\nu} = G_0^{\mu\nu}(t) + \delta G^{\mu\nu}(t,\mathbf{x})$$

→

$$T^{\mu\nu} = T_0^{\mu\nu}(t) + \delta T^{\mu\nu}(t,\mathbf{x})$$

In the 0-th order, background equation,

$$G_0^{\mu\nu}(t) = \frac{8\pi G}{c^4} T_0^{\mu\nu}(t)$$

In the 1st order (linear), perturbation equation,

$$\delta G^{\mu\nu}(t, \mathbf{x}) = \frac{8\pi G}{c^4} \delta T^{\mu\nu}(t, \mathbf{x})$$

Looks complicated!

- Linear growth of the perturbation

The growth of the primordial density perturbation in the expanding universe depends on the scale, type of matter and background matter.

Outside horizon : density perturbation $\frac{\delta\rho}{\rho}$ is constant
with adiabatic condition $p = p(\rho)$

Inside horizon : density perturbation grows

$\frac{\delta\rho}{\rho} \propto$	Radiation dom.	Matter dom.
Non-Rel. matter DM	$\frac{\delta\rho}{\rho} \propto \log a \propto \log t$	$\frac{\delta\rho}{\rho} \propto a \propto t^{2/3}$
Rel. matter photon	oscillating	oscillating

- Notations

density $\rho = \rho(t, \mathbf{x}) = \rho_{\mathbf{0}}(\mathbf{t}) + \delta\rho(\mathbf{t}, \mathbf{x})$

$\rho_0 \gg \delta\rho$

density contrast $\delta(\mathbf{t}, \mathbf{x}) \equiv \dfrac{\delta\rho(\mathbf{t}, \mathbf{x})}{\rho_{\mathbf{0}}} = \dfrac{\rho(\mathbf{t}, \mathbf{x}) - \rho_{\mathbf{0}}}{\rho_{\mathbf{0}}}$

Fourier expansion $\delta(t, \mathbf{x}) = \dfrac{\mathbf{1}}{(\mathbf{2\pi})^{\mathbf{3}}} \displaystyle\int \delta_{\mathbf{k}}(\mathbf{t}) \exp(-\mathbf{ik} \cdot \mathbf{x}) \mathbf{d^3 k}$

$$\delta_{\mathbf{k}} = \int \delta(\mathbf{x}) \exp(\mathbf{ik} \cdot \mathbf{x}) \mathbf{d^3 x}$$

Physical distance and comoving distance $dx_{\rm phys} = R(t) dx,$

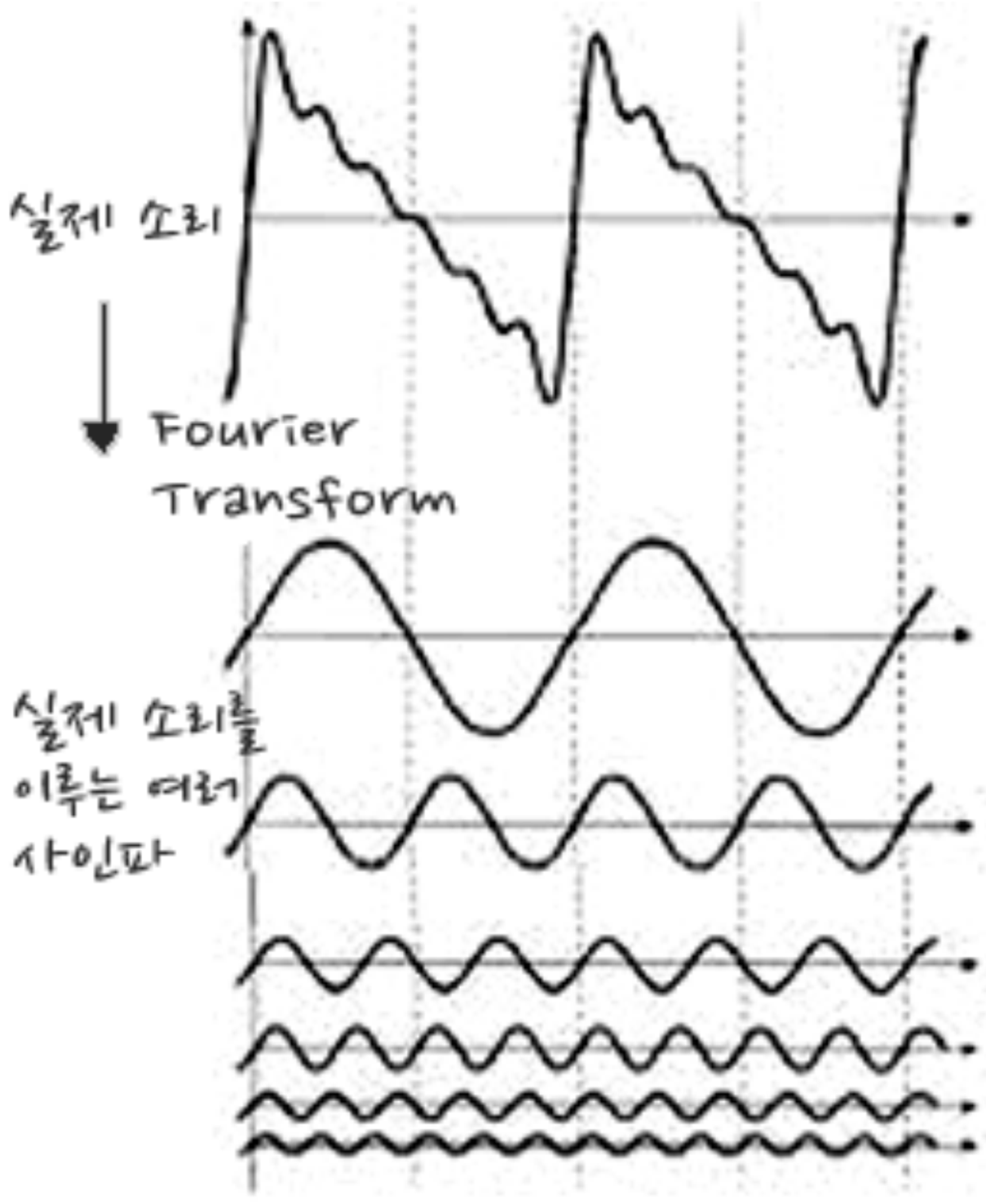
Physical wave number and comoving wave number $k_{\rm phys} = \dfrac{k}{R(t)}$

Wavelength of the perturbation

$$\lambda \equiv \frac{2\pi}{k}, \quad \lambda_{\rm phys} = R(t)\lambda$$

If we set the scale factor today as $R(t_0) = 1$ then, today,

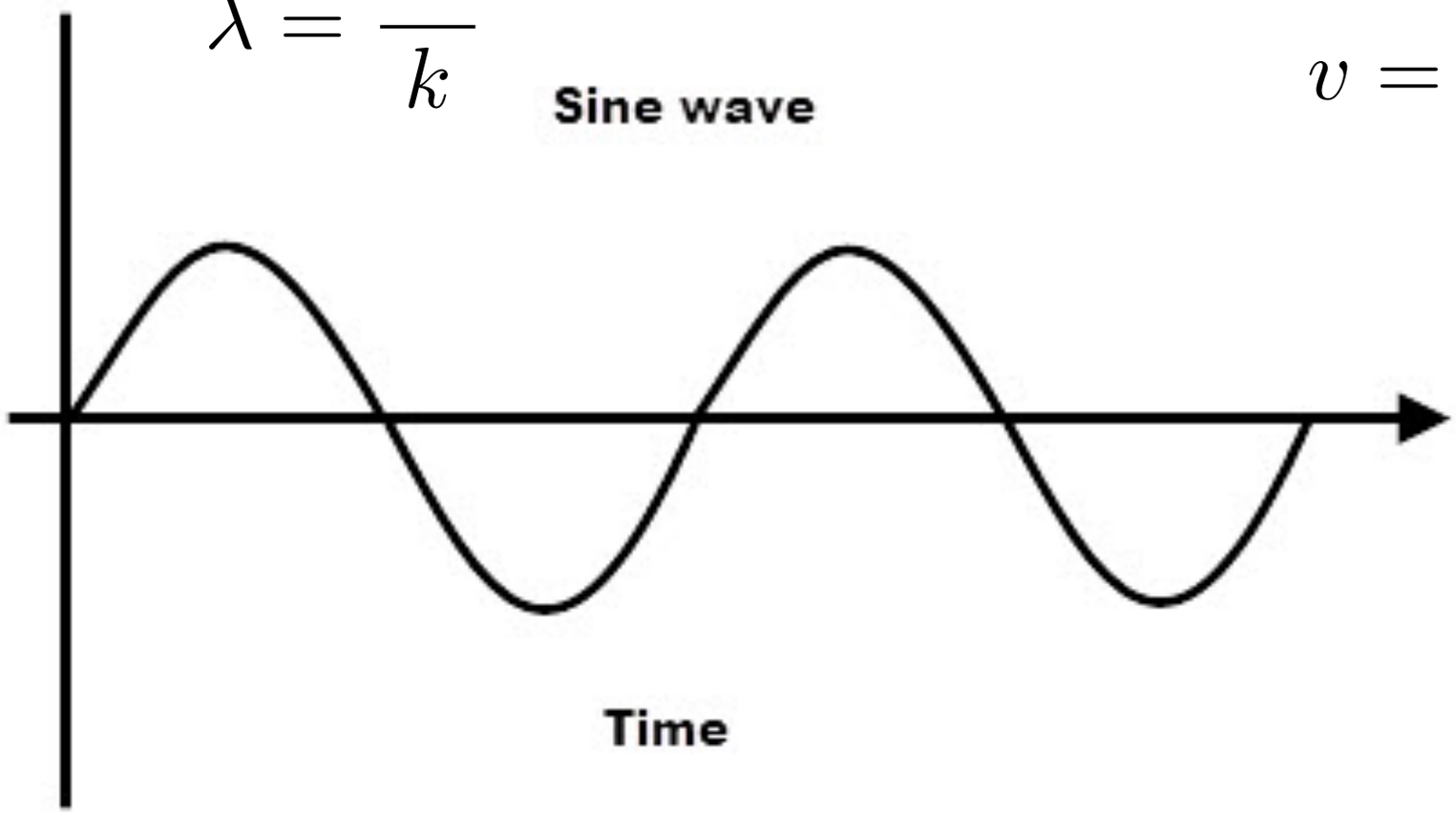
$$k_{\rm phys} = k, \quad \lambda_{\rm phys} = \lambda$$


실제 소리
Fourier
Transform
실제 소리를
이루는 여러
사인파

$$\sin(kx - \omega t)$$

$$\lambda = \frac{2\pi}{k}$$

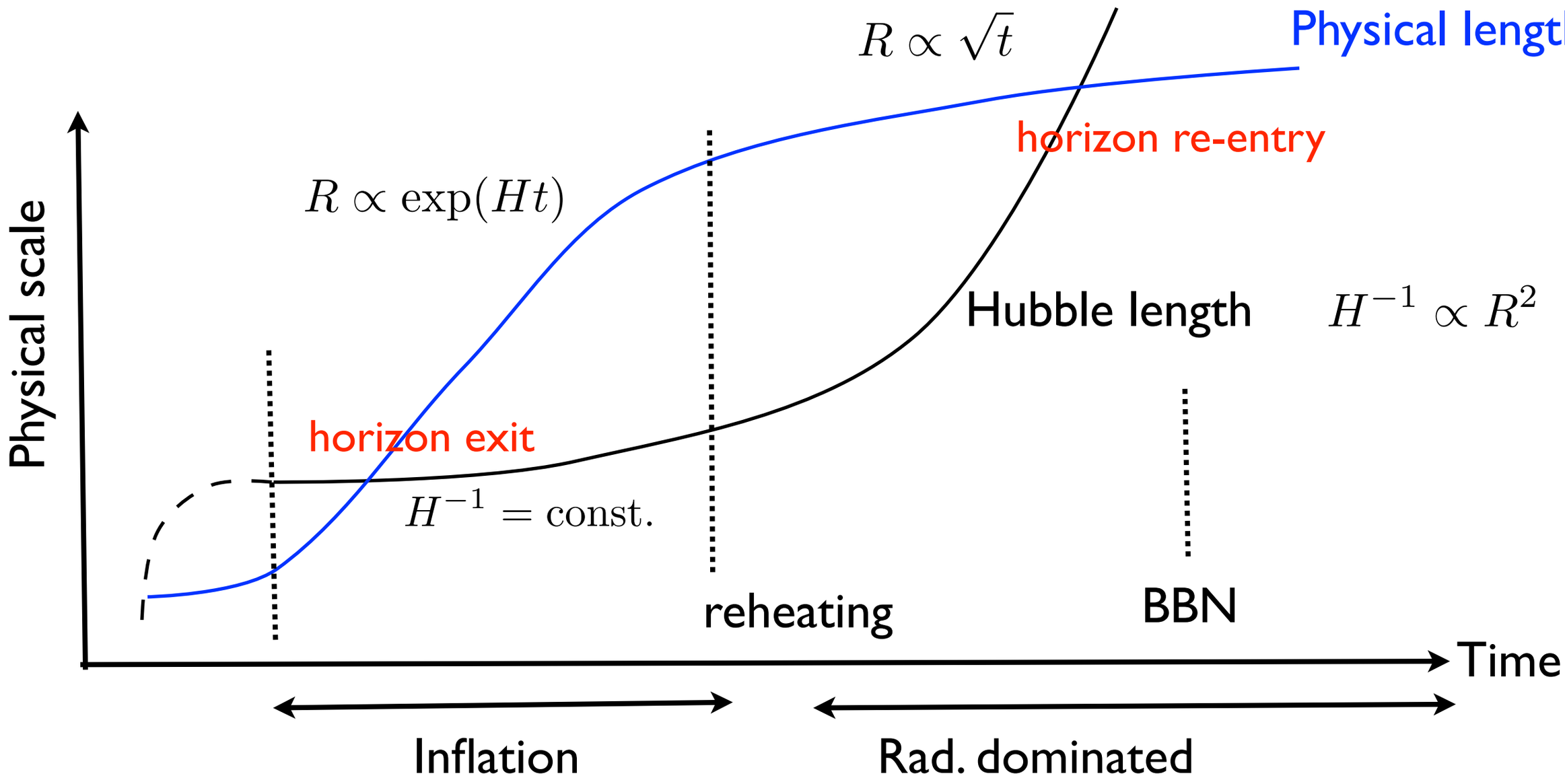
$$v = \frac{\omega}{k}$$



Horizon exit and reentry

$\lambda_{\rm phys}$ vs H^{-1}

$$\lambda \equiv \frac{2\pi}{k}, \quad \lambda_{\rm phys} = R(t)\lambda$$



Baryon-Jeans scale

Baryon+photon plasma

$$\rho_0 = \rho_B + \rho_\gamma = m n_B + \frac{\pi^2}{15} T^4 \qquad c_s^2 = \frac{1}{3}$$

$$\lambda_{J,phys} = c_s \sqrt{\frac{\pi}{G\rho_0}} \quad > \quad H^{-1} = \sqrt{\frac{3}{8\pi G\rho_0}} \qquad \frac{\lambda_{J,phys}}{H^{-1}} = \frac{\sqrt{8\pi^2}}{3} \simeq 3$$

During RD, the baryon Jeans length is larger than the Hubble horizon.

Only the oscillating modes can be treated in the Newtonian analysis and the unstable mode cannot be treated.

After recombination

$$c_s^2 \simeq \frac{T_B}{m} = \frac{T^2}{m T_{\rm rec}}$$

Baryon is decoupled from the radiation.

Therefore the baryon Jeans length dramatically drops and become below the horizon scale. Only after the recombination the baryon fluctuation can grow in the subhorizon scale.

The energy density of the Universe was very smooth in the early Universe and they grew from the small inhomogeneities via gravitational instability into the large inhomogeneities we observe today.

In the matter-dominated epoch, the density can grow as $\frac{\delta\rho}{\rho} \propto R$ (linear regime).

Thus at most it can grow 3000 times during MD .

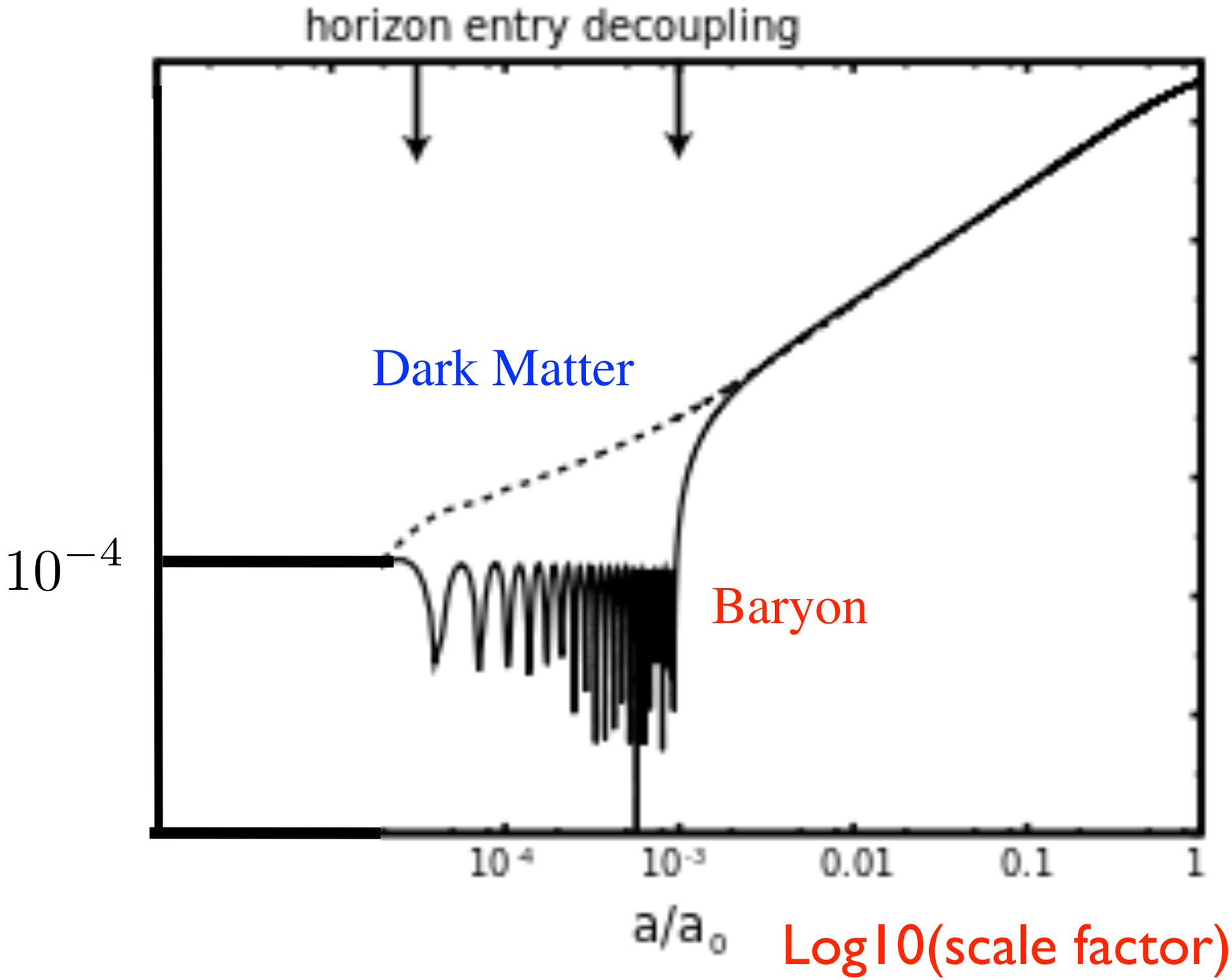
Baryons are coupled to photon until the decoupling moment and they cannot grow before and thus only can grow from 10^{-4} to 0.1.

This cannot explain the present very inhomogeneous Universe. This is one of the reasons we need dark matter, that does not interact with photons and thus their density can grow before MD.

Now we consider the epoch below cosmic temperature MeV.
We separate the matter into baryonic matter, dark matter, neutrino, and photon.

non-relativistic : baryonic matter, dark matter

relativistic : neutrino, photons


horizon entry decoupling
$\frac{\delta\rho}{\rho}$
Dark Matter
Baryon
10^{-4}
10^{-4}
10^{-3}
0.01
0.1
1
a/a_0
Log10(scale factor)

Evolution of density inhomogeneity in the General Relativity

Applicable for any scale and for rel and non-rel.

- Evolution of density inhomogeneity
  : Gravitational instability in General relativity

The wavelength size is much smaller than horizon, then the Newtonian analysis is sufficient. However in the super-horizon size, Newtonian analysis breaks down.

Einstein equation $G_{\mu\nu} = \dfrac{8\pi G}{c^4} T_{\mu\nu}$

But with linear perturbation

$$ds^2 = (1+2\phi)dt^2 - a^2[(1-2\psi)\delta_{ij} - h_{ij}]dx^i dx^j$$

$\delta\rho(t,\mathbf{x})$ and $\delta p(t,\mathbf{x})$

$$G^{\mu\nu} = G_0^{\mu\nu}(t) + \delta G^{\mu\nu}(t,\mathbf{x})$$

$$T^{\mu\nu} = T_0^{\mu\nu}(t) + \delta T^{\mu\nu}(t,\mathbf{x})$$

we can solve the Einstein equation order by order.

In the 0-th order, background equation,

$$G_0^{\mu\nu}(t) = \frac{8\pi G}{c^4} T_0^{\mu\nu}(t)$$

In the 1st order (linear), perturbation equation,

$$\delta G^{\mu\nu}(t, \mathbf{x}) = \frac{8\pi G}{c^4} \delta T^{\mu\nu}(t, \mathbf{x})$$

Looks complicated!

Metric perturbations

$$ds^2 = \left[{}^{(0)}g_{\alpha\beta} + \delta g_{\alpha\beta}(x^\gamma)\right] dx^\alpha dx^\beta,$$

0-th order with conformal time $dt = a(t)d\eta$

$${}^{(0)}g_{\alpha\beta}dx^\alpha dx^\beta = a^2(\eta)\left(d\eta^2 - \delta_{ij}dx^i dx^j\right).$$

1-st order metric tensor is symmetric 4x4 matrix : 10 components

classify them based on the symmetry of translation and rotation

S	V	T	
1			$\delta g_{00} = 2a^2\phi,$
1	2		$\delta g_{0i} = a^2\left(B_{,i} + S_i\right).$ with constraint $S^i_{,i} = 0$
2	2	2	$\delta g_{ij} = a^2\left(2\psi\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij}\right).$
4	+ 4 +	2	=10

traceless, transverse $h^i_i = 0, \quad h^i_{j,i} = 0,$

zero divergence $F^i_{,i} = 0$

Scalar perturbations ϕ, B, ψ, E

They are induced by the energy density inhomogeneities.
Related to the gravitational instability.

Vector perturbations S_i and F_i

They are related to the rotational motion of the fluid. They decay very quickly and are not interesting in the linear perturbation.

Tensor perturbations h_{ij}

They describe gravitational waves.

They are decoupled in the Einstein equations and thus can be studied separately.

General coordinate transformation

$$x^\alpha \to \tilde{x}^\alpha = x^\alpha + \xi^\alpha,$$

$$\xi^\alpha = \xi^\alpha(\mathbf{x}, t)$$

infinitely small functions of space and time

Scalar $\quad \tilde{\phi}(\tilde{x}) = \phi(x)$

Vector $\quad \tilde{V}_\alpha(\tilde{x}) = \dfrac{\partial x^\gamma}{\partial \tilde{x}^\alpha} V_\gamma(x)$

Tensor $\quad \tilde{g}_{\alpha\beta}(\tilde{x}^\rho) = \dfrac{\partial x^\nu}{\partial \tilde{x}^\alpha} \dfrac{\partial x^\delta}{\partial \tilde{x}^\beta} g_{\gamma\delta}(x^\rho)$

$$\frac{\partial x^\gamma}{\partial \tilde{x}^\alpha} = \frac{\partial(\tilde{x}^\gamma - \xi^\gamma)}{\partial \tilde{x}^\alpha} = \delta^\gamma_\alpha - \xi^\gamma_{,\alpha}$$

$$\approx {}^{(0)}g_{a\beta}(x^\rho) + \delta g_{\alpha\beta} - {}^{(0)}g_{\alpha\delta}\xi^\delta_{,\beta} - {}^{(0)}g_{\gamma\beta}\xi^\gamma_{,\alpha},$$

Using $\quad \tilde{g}_{\alpha\beta}(\tilde{x}^\rho) = {}^{(0)}g_{a\beta}(\tilde{x}^\rho) + \delta\tilde{g}_{a\beta}, \quad$ and $\quad {}^{(0)}g_{a\beta}(x^\rho) \approx {}^{(0)}g_{a\beta}(\tilde{x}^\rho) - {}^{(0)}g_{a\beta,\gamma}\xi^\gamma,$

we infer the gauge transformation law for tensor perturbation

$$\delta g_{\alpha\beta} \to \delta\tilde{g}_{a\beta} = \delta g_{\alpha\beta} - {}^{(0)}g_{a\beta,\gamma}\xi^\gamma - {}^{(0)}g_{\gamma\beta}\xi^\gamma_{,\alpha} - {}^{(0)}g_{\alpha\delta}\xi^\delta_{,\beta}.$$

For scalar,

$$\boxed{\delta q \to \delta\tilde{q} = \delta q - {}^{(0)}q_{,\alpha}\xi^{\alpha}.}$$

For 4-vector,

$$\boxed{\delta u_{\alpha} \to \delta\tilde{u}_{\alpha} = \delta u_{\alpha} - {}^{(0)}u_{\alpha,\gamma}\xi^{\gamma} - {}^{(0)}u_{\gamma}\xi^{\gamma}_{,\alpha}.}$$

4-scalar is invariant under the coordinate transformation, but the background and the perturbation depends on the coordinates used.

Gauge transformation of metric

For coordinate transformation with infinitesimal vector

$$\xi^\alpha \equiv (\xi^0, \xi^i) \qquad \text{with} \quad \xi^i = \xi^i_\perp + \zeta^{,i}, \qquad \xi^i_{\perp,i} = 0$$

the gauge transformations are

$$\begin{aligned}
\delta\tilde{g}_{00} &= \delta g_{00} - 2a\left(a\xi^0\right)', \\
\delta\tilde{g}_{0i} &= \delta g_{0i} + a^2\left[\xi'_{\perp i} + \left(\zeta' - \xi^0\right)_{,i}\right], \\
\delta\tilde{g}_{ij} &= \delta g_{ij} + a^2\left[2\frac{a'}{a}\delta_{ij}\xi^0 + 2\zeta_{,ij} + \left(\xi_{\perp i,j} + \xi_{\perp j,i}\right)\right],
\end{aligned}$$

$$\boxed{\xi_{\perp i} \equiv \xi^i_\perp}$$

$$\boxed{\zeta'^i = g^{ij}\zeta_{,j} = -\frac{1}{a^2}\zeta_{,i}} \qquad \boxed{\xi^i_\perp = -\frac{1}{a^2}\xi_{\perp,i}}$$

Scalar perturbations

$$ds^2 = a^2\left[(1+2\phi)\,d\eta^2 + 2B_{,i}dx^i d\eta - \left((1-2\psi)\,\delta_{ij} - 2E_{,ij}\right)dx^i dx^j\right]$$

the gauge transformations are

$$\phi \to \tilde{\phi} = \phi - \frac{1}{a}\left(a\xi^0\right)', \quad B \to \tilde{B} = B + \zeta' - \xi^0,$$

$$\psi \to \tilde{\psi} = \psi + \frac{a'}{a}\xi^0, \qquad E \to \tilde{E} = E + \zeta.$$

By choosing $\xi^0,\ \zeta$ we can eliminate two of the four functions ϕ, B, ψ, E and two combinations are physical.

(gauge fixing)

The simplest gauge invariant combinations are

$$\Phi \equiv \phi - \frac{1}{a}\left[a\left(B - E'\right)\right]', \quad \Psi \equiv \psi + \frac{a'}{a}\left(B - E'\right)$$

The following variable for energy density perturbation is gauge invariant

$$\overline{\delta\varepsilon} = \delta\varepsilon - \varepsilon_0'\left(B - E'\right)$$

Vector perturbations

$$ds^2 = a^2\left[d\eta^2 + 2S_i dx^i d\eta - \left(\delta_{ij} - F_{i,j} - F_{j,i}\right)dx^i dx^j\right]$$

the gauge transformations are

$$S_i \to \tilde{S}_i = S_i + \xi'_{\perp i}, \quad F_i \to \tilde{F}_i = F_i + \xi_{\perp i}.$$

two can be removed and two are physical variables. One gauge invariant combinations is

$$\boxed{\overline{V}_i = S_i - F'_i}$$

Tensor perturbations

$$ds^2 = a^2\left[d\eta^2 - \left(\delta_{ij} - h_{ij}\right) dx^i dx^j\right]$$

h_{ij} does not change under the coordinate transformations.
It is already gauge invariant

Perturbation equations with gauge invariant variables

Gauge invariant variables for energy-momentum tensor

$$\begin{aligned}
\overline{\delta T}^0_{\ 0} &= \delta T^0_{\ 0} - \left({}^{(0)}T^0_{\ 0}\right)'\left(B - E'\right), \\
\overline{\delta T}^0_{\ i} &= \delta T^0_{\ i} - \left({}^{(0)}T^0_{\ 0} - {}^{(0)}T^k_{\ k}/3\right)\left(B - E'\right)_{,i}, \\
\overline{\delta T}^i_{\ j} &= \delta T^i_{\ j} - \left({}^{(0)}T^i_{\ j}\right)'\left(B - E'\right),
\end{aligned}$$

similarly for $\overline{\delta G}^{\alpha}_{\ \beta}$

For perfect fluid with $T_{\alpha\beta} = (\varepsilon + p)u_\alpha u_\beta - p g_{\alpha\beta}$

the gauge invariant perturbations are

$$\overline{\delta T}^0_{\ 0} = \overline{\delta\varepsilon}, \quad \overline{\delta T}^0_{\ i} = \frac{1}{a}(\varepsilon_0 + p_0)\left(\overline{\delta u}_{\parallel i} + \delta u_{\perp i}\right), \quad \overline{\delta T}^i_{\ j} = -\overline{\delta p}\,\delta^i_{\ j},$$

$$\boxed{\overline{\delta\varepsilon} = \delta\varepsilon - \varepsilon_0'\left(B - E'\right)} \qquad \boxed{\overline{\delta u}_0 = \delta u_0 - \left[a\left(B - E'\right)\right]'} \qquad \boxed{\overline{\delta u}_i = \delta u_i - a\left(B - E'\right)_{,i}}$$

Then the linear order Einstein equation $\overline{\delta G}^{\alpha}_{\beta} = 8\pi G \overline{\delta T}^{\alpha}_{\beta}$. gives perturbation equations.

Scalar perturbations

$$\Delta\Psi - 3\mathcal{H}\left(\Psi' + \mathcal{H}\Phi\right) = 4\pi G a^2 \overline{\delta T}^{0}_{0},$$

$$\left(\Psi' + \mathcal{H}\Phi\right)_{,i} = 4\pi G a^2 \overline{\delta T}^{0}_{i},$$

$$\left[\Psi'' + \mathcal{H}(2\Psi + \Phi)' + \left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi + \tfrac{1}{2}\Delta(\Phi - \Psi)\right]\delta_{ij} - \tfrac{1}{2}(\Phi - \Psi)_{,ij} = -4\pi G a^2 \overline{\delta T}^{i}_{j}.$$

Vector perturbations

$$\Delta \overline{V}_i = 16\pi G a^2 \overline{\delta T}^{0}_{i(V)},$$

$$\left(\overline{V}_{i,j} + \overline{V}_{j,i}\right)' + 2\mathcal{H}\left(\overline{V}_{i,j} + \overline{V}_{j,i}\right) = -16\pi G a^2 \overline{\delta T}^{i}_{j(V)},$$

Tensor perturbations

$$\left(h''_{ij} + 2\mathcal{H}h'_{ij} - \Delta h_{ij}\right) = 16\pi G a^2 \overline{\delta T}^{i}_{j(T)},$$

Scalar perturbations

Since $\delta T^i_j = 0$ for $i \neq j$, $(\Phi - \Psi)_{,ij} = 0 \quad (i \neq j)$ and thus $\Psi = \Phi$.

to be compatible with perturbation.

The equations become

$$\Delta\Phi - 3\mathcal{H}\left(\Phi' + \mathcal{H}\Phi\right) = 4\pi G a^2 \overline{\delta\varepsilon},$$

$$(a\Phi)'_{,i} = 4\pi G a^2 (\varepsilon_0 + p_0)\, \overline{\delta u}_{\shortparallel i},$$

$$\Phi'' + 3\mathcal{H}\Phi' + \left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi = 4\pi G a^2 \overline{\delta p}.$$

In the sub-Hubble scale $\nabla^2\Phi \sim k^2\Phi \sim \frac{1}{\lambda^2}\Phi \gg \mathcal{H}^2\Phi$ $\boxed{\mathcal{H} = aH}$

the first eq. coincides with Poisson eq., thus Φ is interpreted as the relativistic generalization of the Newtonian gravitational potential.

With $\overline{\delta p} = c_s^2 \overline{\delta\epsilon}$

$$\Phi'' + 3\left(1 + c_s^2\right)\mathcal{H}\Phi' - c_s^2\Delta\Phi + \left(2\mathcal{H}' + \left(1 + 3c_s^2\right)\mathcal{H}^2\right)\Phi = 0$$

$$\Phi'' + 3\left(1 + c_s^2\right)\mathcal{H}\Phi' - c_s^2\Delta\Phi + \left(2\mathcal{H}' + \left(1 + 3c_s^2\right)\mathcal{H}^2\right)\Phi = 0$$

Nonrelativistic matter $(c_s^2 = 0)$ $\quad a \propto \eta^2,\ \mathcal{H} = \dfrac{2}{\eta}$ $\quad \displaystyle\int d\eta = \int \frac{dt}{a},\ a \propto t^{2/3}$

It becomes and the solution is $\mathcal{H} = aH$

$$\Phi'' + \frac{6}{\eta}\Phi' = 0,$$

$$\Phi = C_1(\mathbf{x}) + \frac{C_2(\mathbf{x})}{\eta^5},$$

always constant inside and outside of horizon

The density perturbations is

$$\frac{\overline{\delta\varepsilon}}{\varepsilon_0} = \frac{1}{6}\left[\left(\Delta C_1\eta^2 - 12C_1\right) + \left(\Delta C_2\eta^2 + 18C_2\right)\frac{1}{\eta^5}\right].$$

$\lambda_{ph} \sim a/k$

$H^{-1} \sim a\eta$

Superhorizon scale, $k\eta \ll 1$

$$\frac{\overline{\delta\varepsilon}}{\varepsilon_0} \simeq -2C_1 + \frac{3C_2}{\eta^5}.$$ remains constant and $\dfrac{\overline{\delta\epsilon_0}}{\epsilon_0} \simeq -2\Phi$

Subhorizon scale, $k\eta \gg 1,$

$$\frac{\overline{\delta\varepsilon}}{\varepsilon_0} \simeq -\frac{k^2}{6}\left(C_1\eta^2 + C_2\eta^{-3}\right) = \tilde{C}_1 t^{2/3} + \tilde{C}_2 t^{-1},$$ grows linearly

$$\Phi'' + 3\left(1 + c_s^2\right)\mathcal{H}\Phi' - c_s^2\Delta\Phi + \left(2\mathcal{H}' + \left(1 + 3c_s^2\right)\mathcal{H}^2\right)\Phi = 0$$

Relativistic matter $(p = \omega\epsilon)$ $\quad a \propto \eta^{2/(1+3\omega)}$ $\quad \mathcal{H} = \frac{2}{\eta(1+3\omega)}$ $\quad \int d\eta = \int \frac{dt}{a},\ a \propto t^{2/3(1+\omega)}$

It becomes

$$\Phi_{\mathbf{k}}'' + \frac{6(1+w)}{1+3w}\frac{1}{\eta}\Phi_{\mathbf{k}}' + wk^2\Phi_{\mathbf{k}} = 0.$$

$\mathcal{H} = aH$

$\triangle \to -k^2$

and the solution is

$$\Phi_{\mathbf{k}} = \eta^{-\nu}\left[C_1 J_\nu\left(\sqrt{w}k\eta\right) + C_2 Y_\nu\left(\sqrt{w}k\eta\right)\right], \quad \nu \equiv \frac{1}{2}\left(\frac{5+3w}{1+3w}\right),$$

$J_\nu\,, Y_\nu$: Bessel functions of order ν

$\lambda_{ph} \sim a/k$

$H^{-1} \sim a\eta$

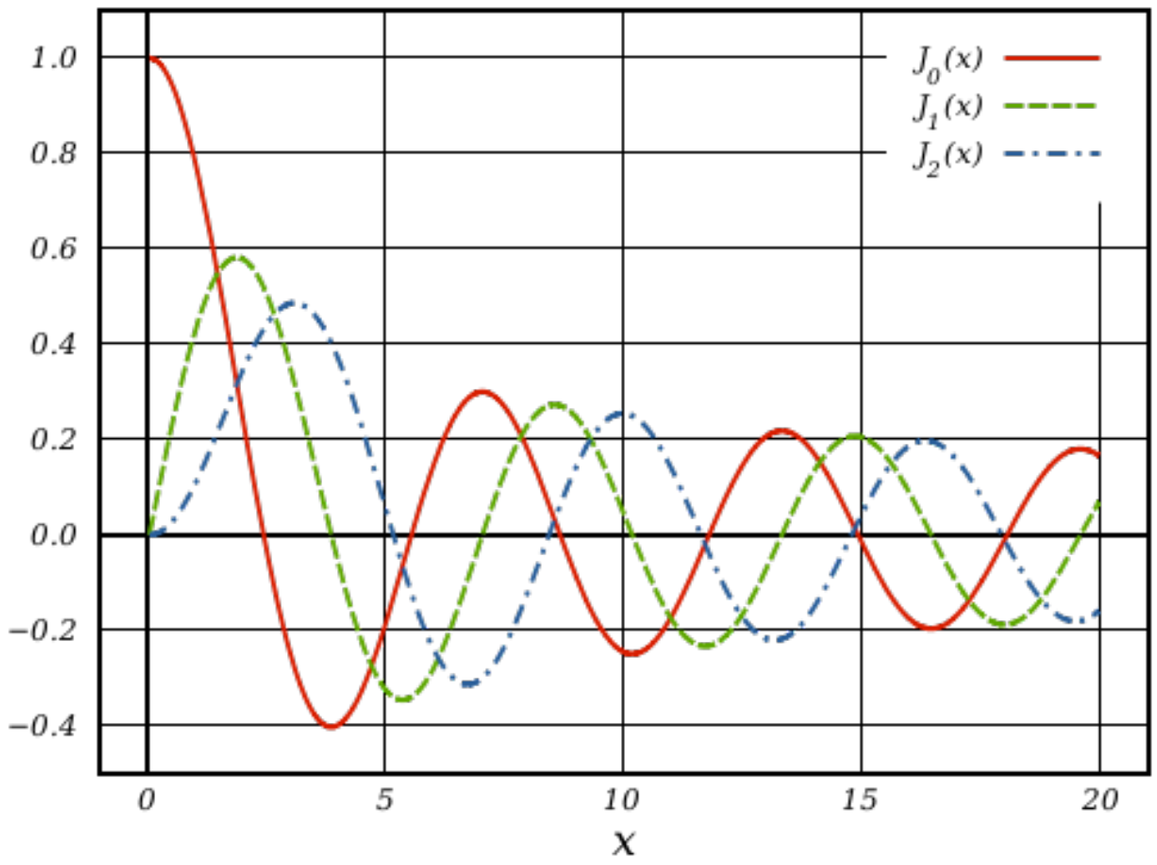
Long wavelength, $\quad \sqrt{w}k\eta \ll 1,$

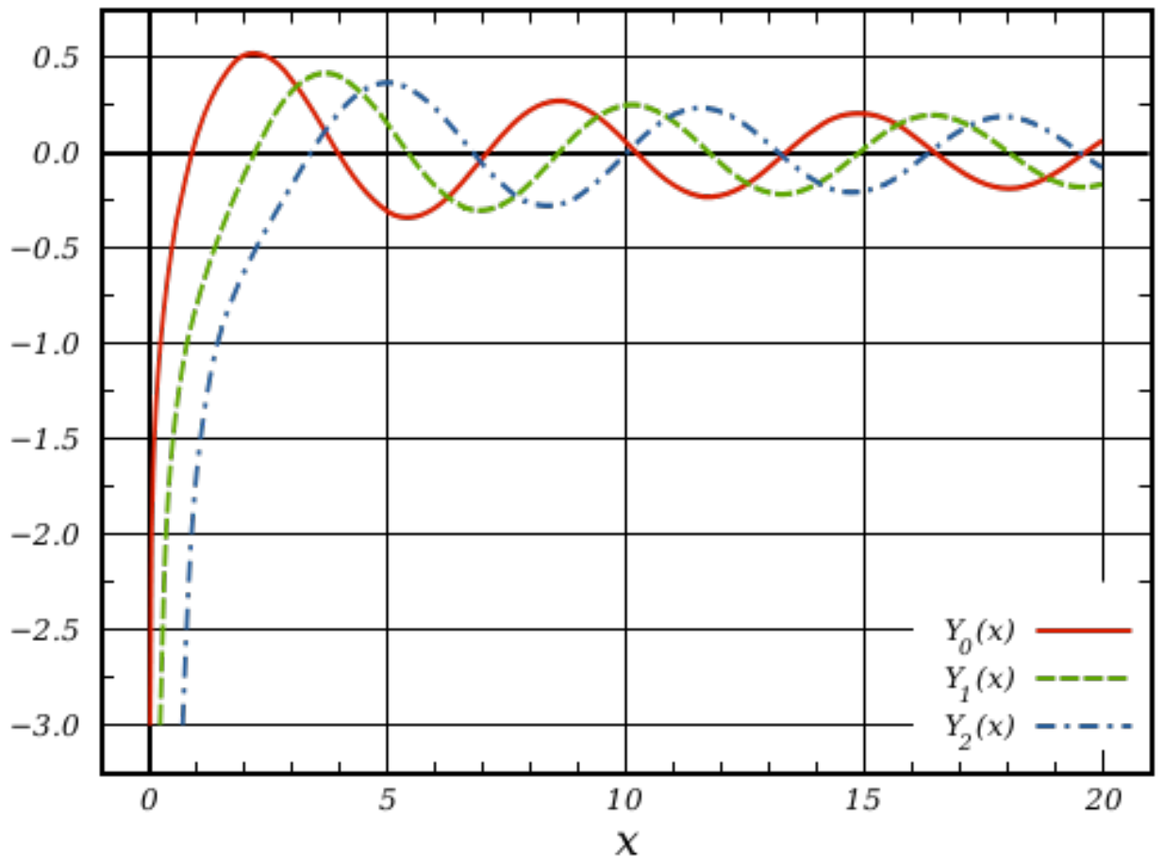
$\overline{\delta\varepsilon}/\varepsilon_0 \simeq -2\Phi$ $\quad$ remains constant

Small wavelength, $\quad \sqrt{w}k\eta \gg 1,$

$\Phi_{\mathbf{k}} \propto \eta^{-\nu-\frac{1}{2}} \exp\left(\pm i\sqrt{w}k\eta\right).$ $\quad$ oscillates with decaying amplitude

Bessel function





$$(0 \le x \ll \nu) \qquad\qquad (x \gg \nu)$$

$$J_\nu(x) \simeq \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu}, \qquad \left(\frac{2}{\pi x}\right)^{1/2}\cos(x-\frac{1}{2}\nu\pi-\frac{\pi}{4})$$

$$Y_\nu(x) \simeq \frac{-\Gamma(\nu)}{\pi}\left(\frac{x}{2}\right)^{-\nu}, \qquad \left(\frac{2}{\pi x}\right)^{1/2}\sin(x-\frac{1}{2}\nu\pi-\frac{\pi}{4})$$

Vector perturbations

$$\Delta \overline{V}_i = 16\pi G a^2 \overline{\delta T}^0_{i(V)},$$

$$\left(\overline{V}_{i,j} + \overline{V}_{j,i}\right)' + 2\mathcal{H}\left(\overline{V}_{i,j} + \overline{V}_{j,i}\right) = -16\pi G a^2 \overline{\delta T}^i_{j(V)},$$

only scalar component for perfect fluid

$$\overline{\delta T}^0_0 = \overline{\delta\varepsilon}, \quad \overline{\delta T}^0_i = \frac{1}{a}(\varepsilon_0 + p_0)\left(\overline{\delta u}_{\parallel i} + \delta u_{\perp i}\right), \quad \overline{\delta T}^i_j = -\overline{\delta p}\,\delta^i_j,$$

The equations become

$$\Delta \overline{V}_i = 16\pi G a(\varepsilon_0 + p_0)\,\delta u_{\perp i},$$

$$\left(\overline{V}_{i,j} + \overline{V}_{j,i}\right)' + 2\mathcal{H}\left(\overline{V}_{i,j} + \overline{V}_{j,i}\right) = 0.$$

The second one gives a solution

$$\overline{V}_i = \frac{C_{\perp i}}{a^2},$$

always decays and we can ignore it.

Tensor perturbations

$$\left(h''_{ij} + 2\mathcal{H}h'_{ij} - \Delta h_{ij}\right) = 16\pi G a^2 \overline{\delta T}^{i}_{j(T)}, \qquad = 0$$

for a perfect fluid

with $h_{ij} = \frac{v}{a} e_{ij}$, we can solve $v'' + \left(k^2 - \frac{a''}{a}\right) v = 0.$

then the solution is for RD $a \propto \eta$, and $v \propto \exp(\pm i k \eta)$

$$h_{ij} = \frac{1}{\eta}(C_1 \sin(k\eta) + C_2 \cos(k\eta))\, e_{ij}$$

In superhorizon, $(k\eta \ll 1)$ the amplitude is constant,

and in the subhorizon, it decays inverse proportional to the scale factor.

This is in general true independent on the equation of state.

Gauge fixing

Using the coordinate transformation we can eliminate the some of the metric perturbations and make the form simpler.

Longitudinal (conformal-Newtonian) gauge $\quad B_l = E_l = 0.$

$$ds^2 = a^2\left[(1+2\phi_l)d\eta^2 - (1-2\psi_l)\delta_{ij}dx^i dx^j\right]$$

$$\Phi \equiv \phi - \frac{1}{a}\left[a\left(B - E'\right)\right]', \quad \Psi \equiv \psi + \frac{a'}{a}\left(B - E'\right)$$

are the same in the longitudinal gauge

$$\overline{\delta\varepsilon} = \delta\varepsilon - \varepsilon_0'\left(B - E'\right)$$

Synchronous gauge $\quad \delta g_{0\alpha} = 0$ or $\phi_s = 0$ and $B_s = 0$

Comoving gauge $\quad \phi = 0, \quad \delta u^i_{\shortparallel} = -\frac{1}{a^2}\delta u_{\shortparallel i} + \frac{1}{a}B_{,i} = 0,$

- Linear growth of the perturbation

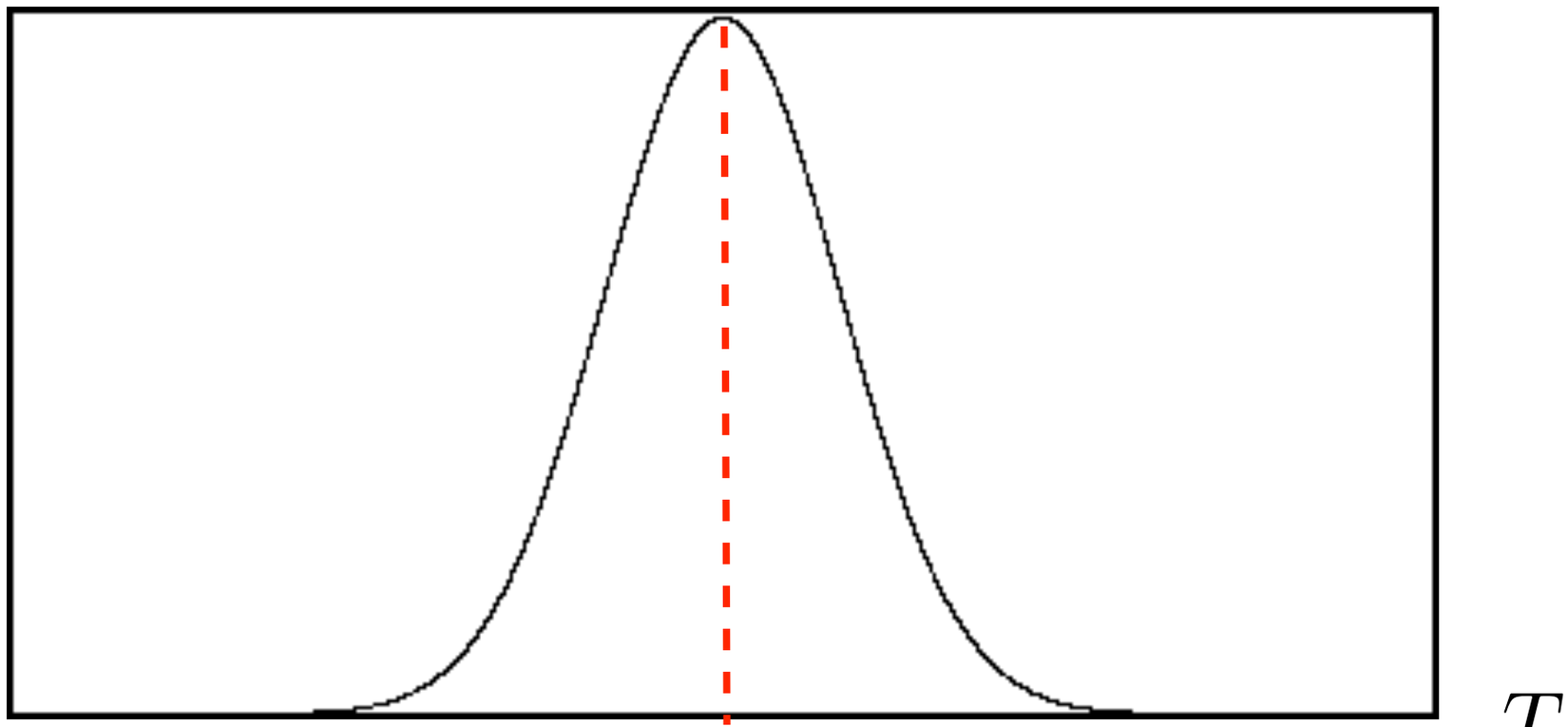
The growth of the primordial density perturbation in the expanding universe depends on the scale, type of matter and background matter.

Outside horizon : density perturbation $\frac{\delta\rho}{\rho}$ is constant
with adiabatic condition $p = p(\rho)$

Inside horizon : density perturbation grows

$\frac{\delta\rho}{\rho} \propto$	Radiation dom.	Matter dom.
Non-Rel. matter	$\log a \propto \log t$	$a \propto t^{2/3}$
Rel. matter	oscillating	oscillating

CMB temperature anisotropy and matter perturbation shows almost Gaussian distribution



$$\mathrm{T} = 2.725\,\mathrm{K}$$

$$f(x) = \frac{1}{2\pi\sigma^2}\exp\left[-\frac{(x-\langle x\rangle)^2}{2\sigma^2}\right]$$

$\langle x\rangle$: average, σ^2 : variance

- Probability distribution function of ζ : $f(\zeta)$

Moments of ζ for the given distribution function $f(\zeta)$

mean value : m(1)	$\langle\zeta\rangle = \int \zeta f(\zeta) d\zeta$
variance : m(2)	$\int (\zeta - \langle\zeta\rangle)^2 f(\zeta) d\zeta$
skewness : m(3)	$\int (\zeta - \langle\zeta\rangle)^3 f(\zeta) d\zeta$
kurtosis : m(4)	$\int (\zeta - \langle\zeta\rangle)^4 f(\zeta) d\zeta$
⋮	⋮

For Gaussian distribution $\quad f_G(\zeta) \equiv \dfrac{1}{\sqrt{2\pi m(2)}} e^{-(\zeta - m(1))^2 / 2m(2)}$

→ all m(n is odd >1) vanish

all m(n is even >2) are expressed only as products of variance

Power spectrum : size of perturbation

$$\langle \zeta(\mathbf{k_1})\zeta(\mathbf{k_2})\rangle = (2\pi)^3 P(k_1)\delta^3(\mathbf{k_1}+\mathbf{k_2})$$

$$= (2\pi)^3 \frac{2\pi^2}{k_1^3}\mathcal{P}(k_1)\delta^3(\mathbf{k_1}+\mathbf{k_2})$$

$$\int \frac{\mathrm{d}^3\mathrm{k}}{(2\pi)^3} P_\zeta(k) = \int \mathcal{P}_\zeta(k)\,\mathrm{d}\,\mathrm{logk}$$

Bispectrum : non-Gaussian distribution

$$\langle \zeta(\mathbf{k_1})\zeta(\mathbf{k_2})\zeta(\mathbf{k_3})\rangle = (2\pi)^3 B(k_1,k_2,k_3)\delta^3(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})$$

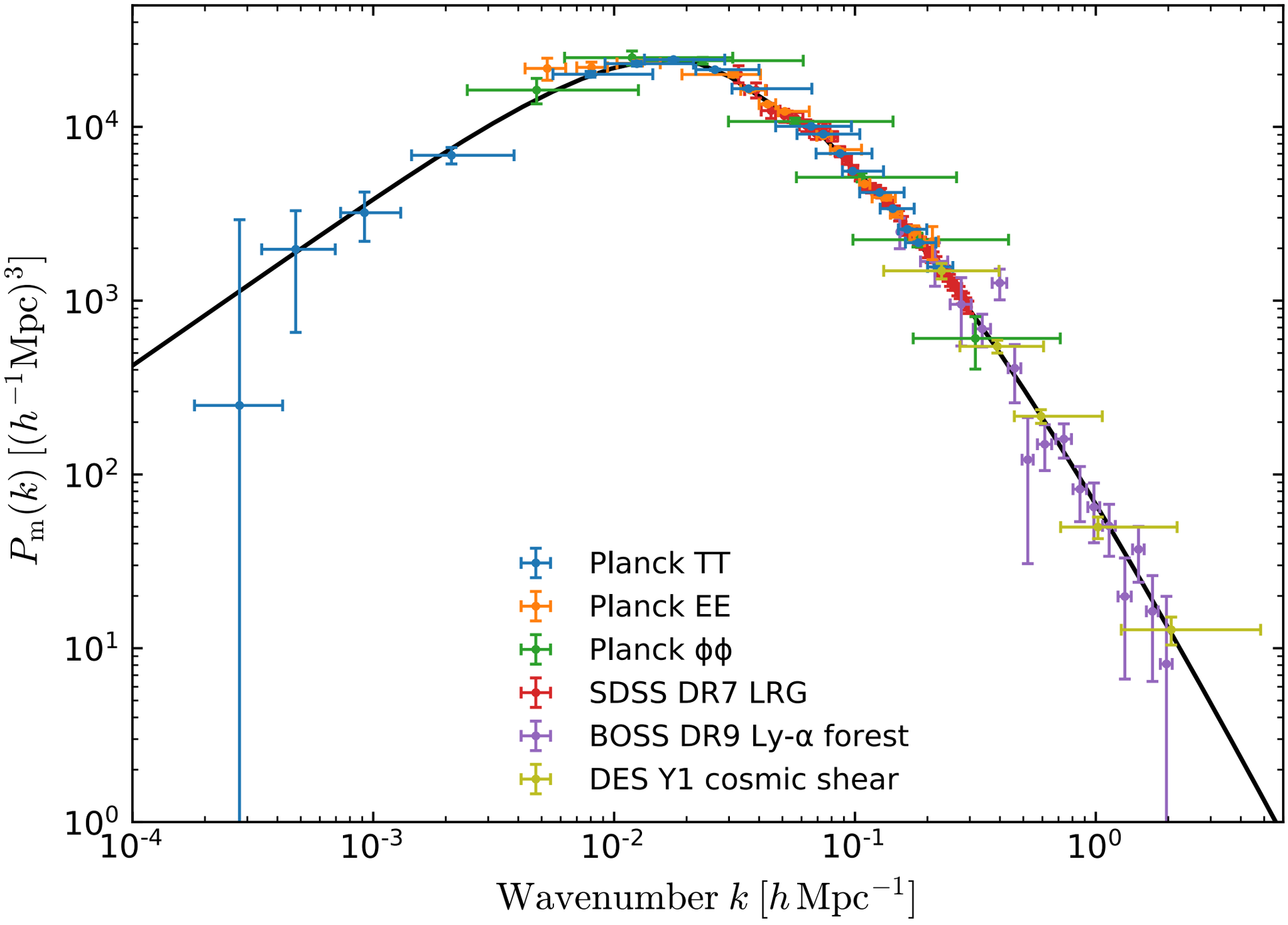
Non-vanishing bispecrum is the measure of the non-Gaussianity

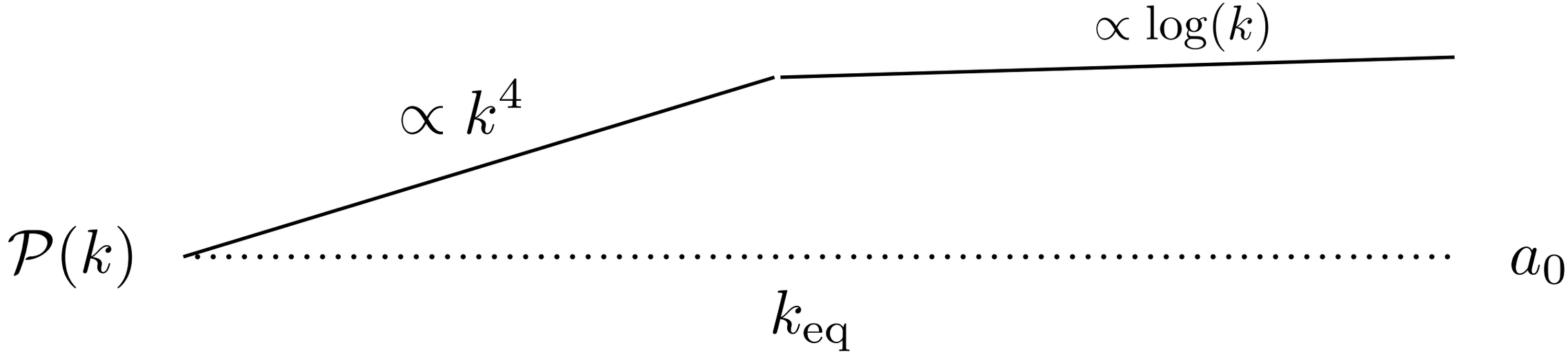
Non-linearity parameter f_{NL}

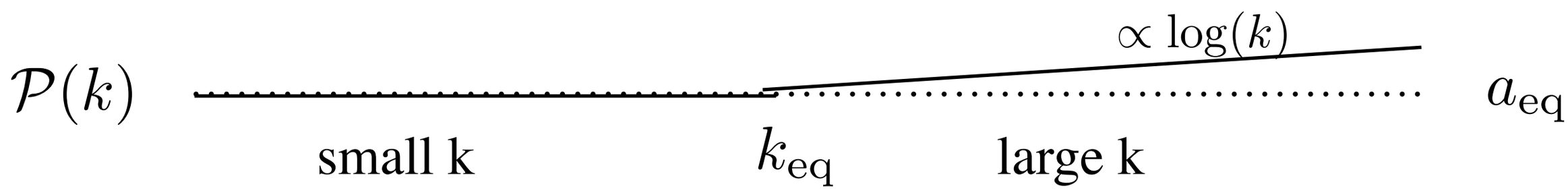
$$B(k_1,k_2,k_3) = (6/5) f_{\mathrm{NL}}[P(k_1)P(k_2) + 2 \text{ perm}]$$

Matter Power Spectrum

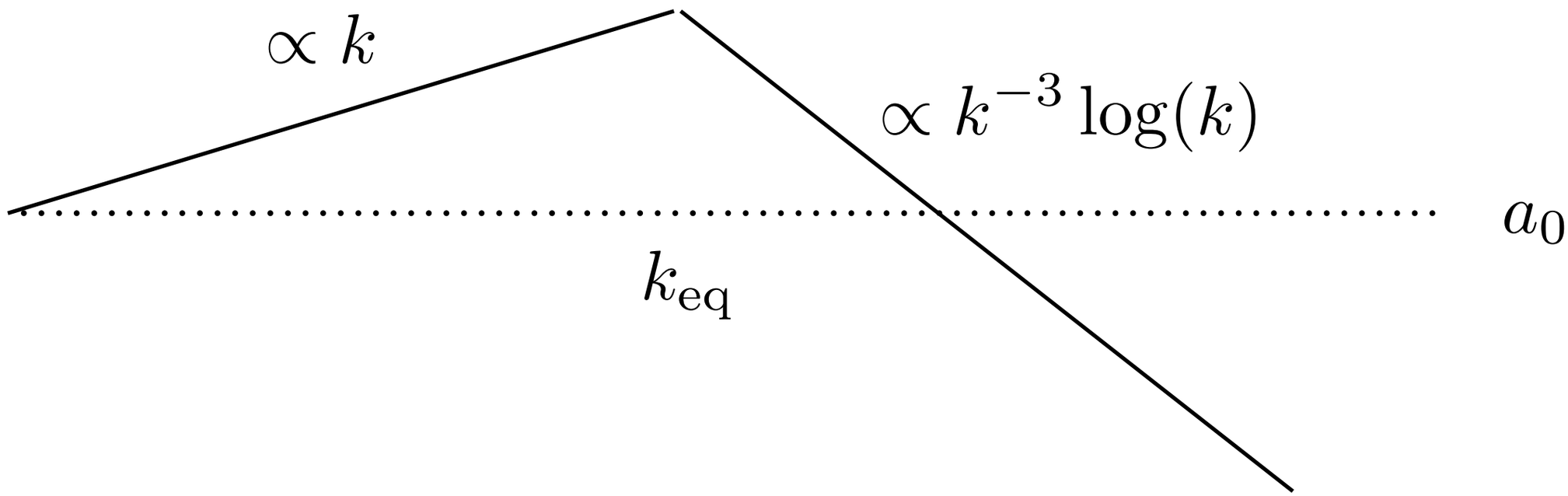
[Planck 1807.06205]




$\propto \log(k)$
$\propto k^4$
$\mathcal{P}(k)$
a_0
$k_{\rm eq}$


$\propto \log(k)$
$\mathcal{P}(k)$
$a_{\rm eq}$
small k
$k_{\rm eq}$
large k

$$P(k) = \frac{1}{k^3}\mathcal{P}(k)$$



CMB perturbations

- Linear growth of the perturbation

The growth of the primordial density perturbation in the expanding universe depends on the scale, type of matter and background matter.

Outside horizon : density perturbation $\frac{\delta\rho}{\rho}$ is constant
with adiabatic condition $p = p(\rho)$

Inside horizon : density perturbation grows

$\frac{\delta\rho}{\rho} \propto$	Radiation dom.	Matter dom.
Non-Rel. matter	$\log a \propto \log t$	$a \propto t^{2/3}$
Rel. matter	oscillating	oscillating

Three equations of (A), (B) and (C) give equations for three unknown variables of δ_d, δ_γ and Φ.

$$\delta_b = \frac{3}{4}\delta_\gamma .$$

$$\left(a(\delta_d - 3\Phi)'\right)' - a\Delta\Phi = 0. \qquad \text{(A)}$$

$$\left(\frac{\delta_\gamma'}{c_s^2}\right)' - \frac{3\eta}{\varepsilon_\gamma a}\Delta\delta_\gamma' - \Delta\delta_\gamma = \frac{4}{3c_s^2}\Delta\Phi + \left(\frac{4\Phi'}{c_s^2}\right)' - \frac{12\eta}{\varepsilon_\gamma a}\Delta\Phi', \qquad \text{(B)}$$

$$\begin{aligned}\Delta\Phi - 3\mathcal{H}\Phi' - 3\mathcal{H}^2\Phi &= 4\pi G a^2\left(\delta\varepsilon_d + \delta\varepsilon_b + \delta\varepsilon_\gamma\right) \\ &= 4\pi G a^2\left(\varepsilon_d\delta_d + \frac{1}{3c_s^2}\varepsilon_\gamma\delta_\gamma\right),\end{aligned} \qquad \text{(C)}$$

The approximate solutions at recombination epoch for different scales.

1. The scale which enters the horizon after equality, still outside horizon at recombination $\boxed{k^{-1} \gg \eta_r}$

$$\Phi = \frac{9}{10}\Phi^0 , \qquad \delta_\gamma = -\frac{8}{3}\Phi = -\frac{12}{5}\Phi^0 , \qquad \delta_d = -2\Phi = -\frac{9}{5}\Phi^0$$

2. The scale which enters the horizon between the equality and recombination $\boxed{\eta_r \gg k^{-1} \gg \eta_{eq}}$

$$\Phi = \frac{9}{10}\Phi^0 , \qquad \delta_\gamma(\eta) = \left[-\frac{4}{3c_s^2} + \frac{4\sqrt{c_s}}{3^{3/4}} \cos\left(k \int_0^\eta c_s d\tilde{\eta} \right) e^{-(k/k_D)^2} \right] \left(\frac{9}{10}\Phi_k^0 \right)$$

3. The scale which enters the horizon before equality $\boxed{\eta_{eq} \gg k^{-1}}$

$$\Phi_k(\eta > \eta_{eq}) \simeq \frac{\ln(0.15 k\eta_{eq})}{(0.27 k\eta_{eq})^2}\Phi_k^0 ,$$

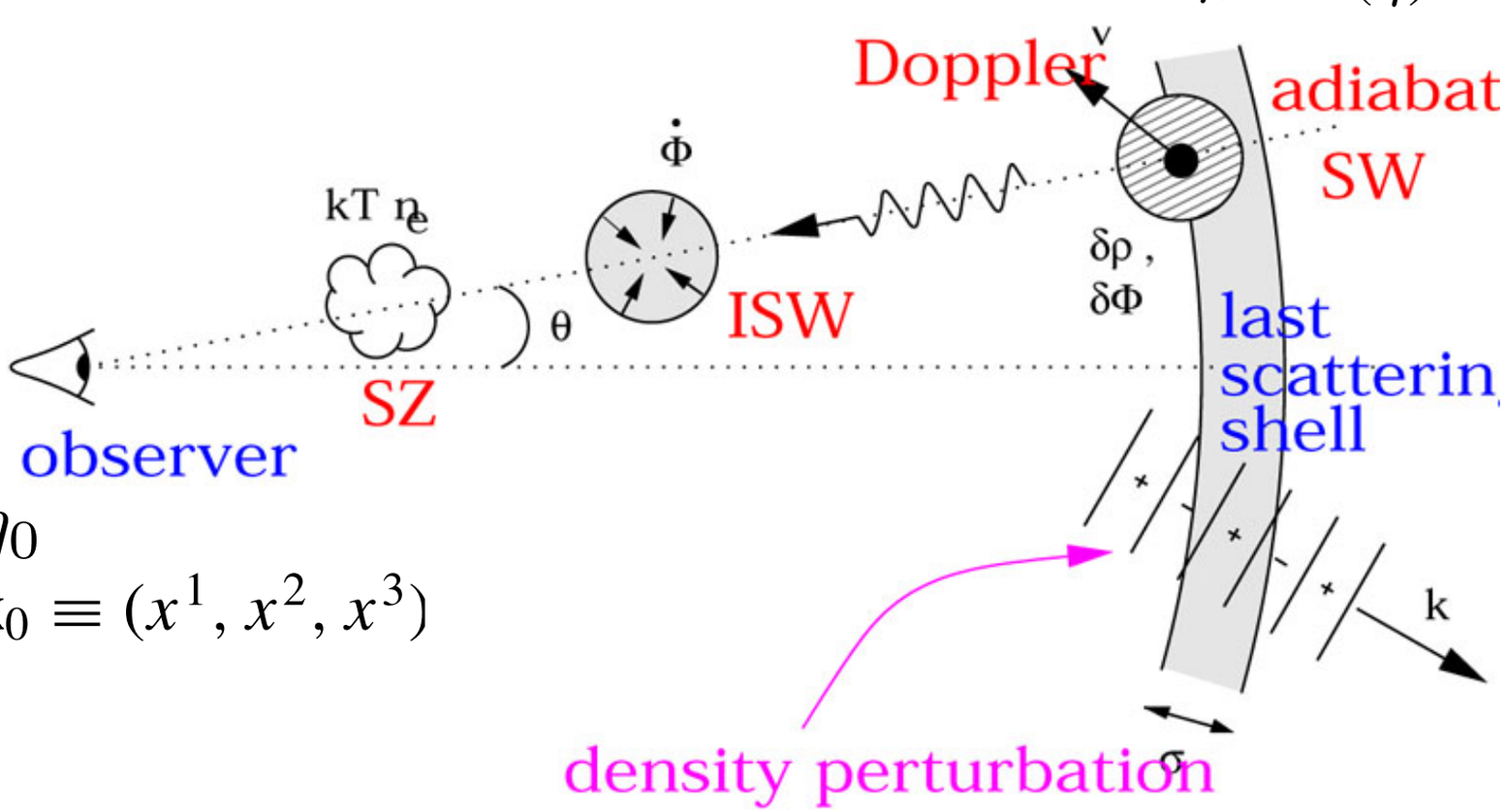
$$\delta_\gamma \simeq \left[-\frac{4}{3c_s^2}\frac{\ln(0.15 k\eta_{eq})}{(0.27 k\eta_{eq})^2} + 3^{5/4}\sqrt{4c_s} \cos\left(k \int_0^\eta c_s d\eta \right) e^{-(k/k_D)^2} \right] \Phi_k^0$$

Therefore with Fourier transformation, we obtain the final expression for the temperature perturbation in the direction $\mathbf{l} \equiv (l^1, l^2, l^3)$ as observed at location $\mathbf{x}_0 \equiv (x^1, x^2, x^3)$

$$\frac{\delta T}{T}(\eta_0, \mathbf{x}_0, \mathbf{l}) = \int \left[\left(\Phi + \frac{\delta}{4} \right)_{\mathbf{k}} - \frac{3\delta'_{\mathbf{k}}}{4k^2} \frac{\partial}{\partial \eta_0} \right]_{\eta_r} e^{i\mathbf{k}(\mathbf{x}_0 + \mathbf{l}(\eta_r - \eta_0))} \frac{d^3k}{(2\pi)^{3/2}},$$

where $k \equiv |\mathbf{k}|$, $\mathbf{k} \cdot \mathbf{l} \equiv k_m l^m$ and $\mathbf{k} \cdot \mathbf{x}_0 \equiv k_n x_0^n$.

η_r $x^i(\eta) \simeq x_0^i + l^i(\eta - \eta_0)$



$\mathbf{l} \equiv (l^1, l^2, l^3)$

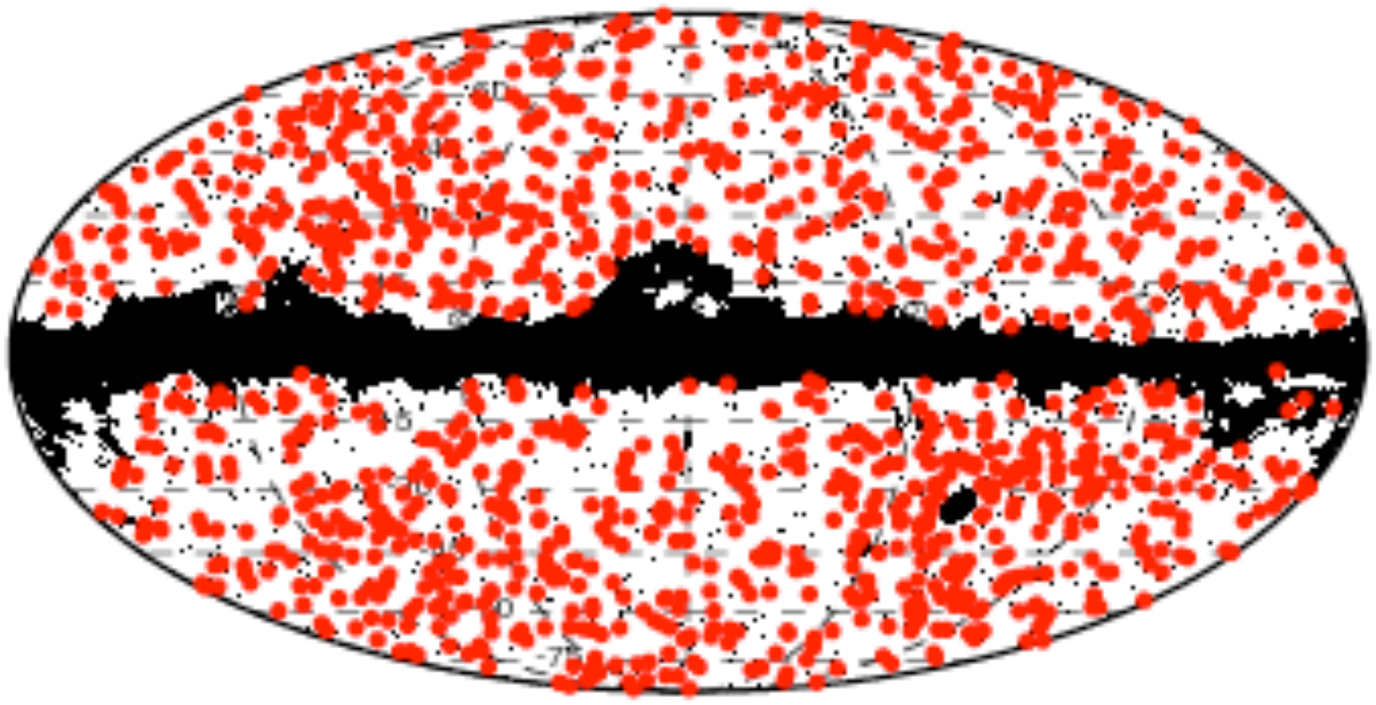
η_0

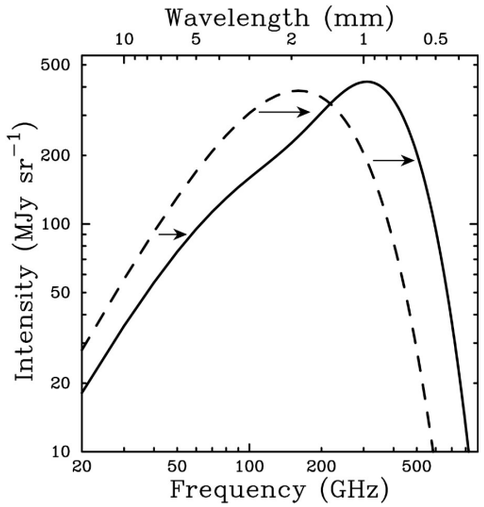
$\mathbf{x}_0 \equiv (x^1, x^2, x^3)$

$\frac{\eta_r}{\eta_0} < \frac{1}{30}$

$\eta_0 - \eta_r \simeq \eta_0$

Planck SZ sources





The distribution, shown in Mollweide projection with the Galactic plane horizontal and the Milky Way centre in the middle, of the 1227 \Planck\ clusters and candidates across the sky (red thick dots). The masked point-sources (black thin dots), the Magellanic clouds (large black areas) and the Galactic mask, covering a total of 16.3\% of the sky and used by the SZ-finder algorithms to detect SZ sources, are also shown.

Conformal Newtonian gauge

$$ds^2 = (1+2\Phi)dt^2 - a^2(t)(1+2\Psi)d\mathbf{x}^2,$$

The observed temperature perturbation of the CMB photons coming to us along the direction n from the point x where they have scattered is given by

$$\frac{\delta T}{T} = \frac{1}{4}\frac{\delta\rho_\gamma}{\rho_\gamma}(\mathbf{x}) + \Phi(\mathbf{x}) + \mathbf{n}\cdot\mathbf{v}_\gamma(\mathbf{x}) + \int_{t_r}^{t_0} dt\ (\dot{\Phi} - \dot{\Psi}),$$

Integrates Sachs-Wolfe effect due to the time-dep. gravitational potential (early ISW, ISW)

Doppler effect due to the velocity of the baryon-photon plasma

red(blue)-shift when they escape form the gravitational potential

temperature perturbation from the density perturbation of photons at last scattering

Sachs-Wolfe effect

Anisotropies on small angular scales

$$e^{i\mathbf{q}\cdot\mathbf{x}} = \sum_l (2l+1) i^l P_l(\hat{q}\cdot\hat{n}) j_l(qr)$$

$$\frac{\delta T}{T}(\eta_0, \mathbf{x}_0, \mathbf{l}) = \int \left[\left(\Phi + \frac{\delta}{4} \right)_{\mathbf{k}} - \frac{3\delta'_{\mathbf{k}}}{4k^2} \frac{\partial}{\partial \eta_0} \right]_{\eta_r} e^{i\mathbf{k}(\mathbf{x}_0 + \mathbf{l}(\eta_r - \eta_0))} \frac{d^3k}{(2\pi)^{3/2}},$$

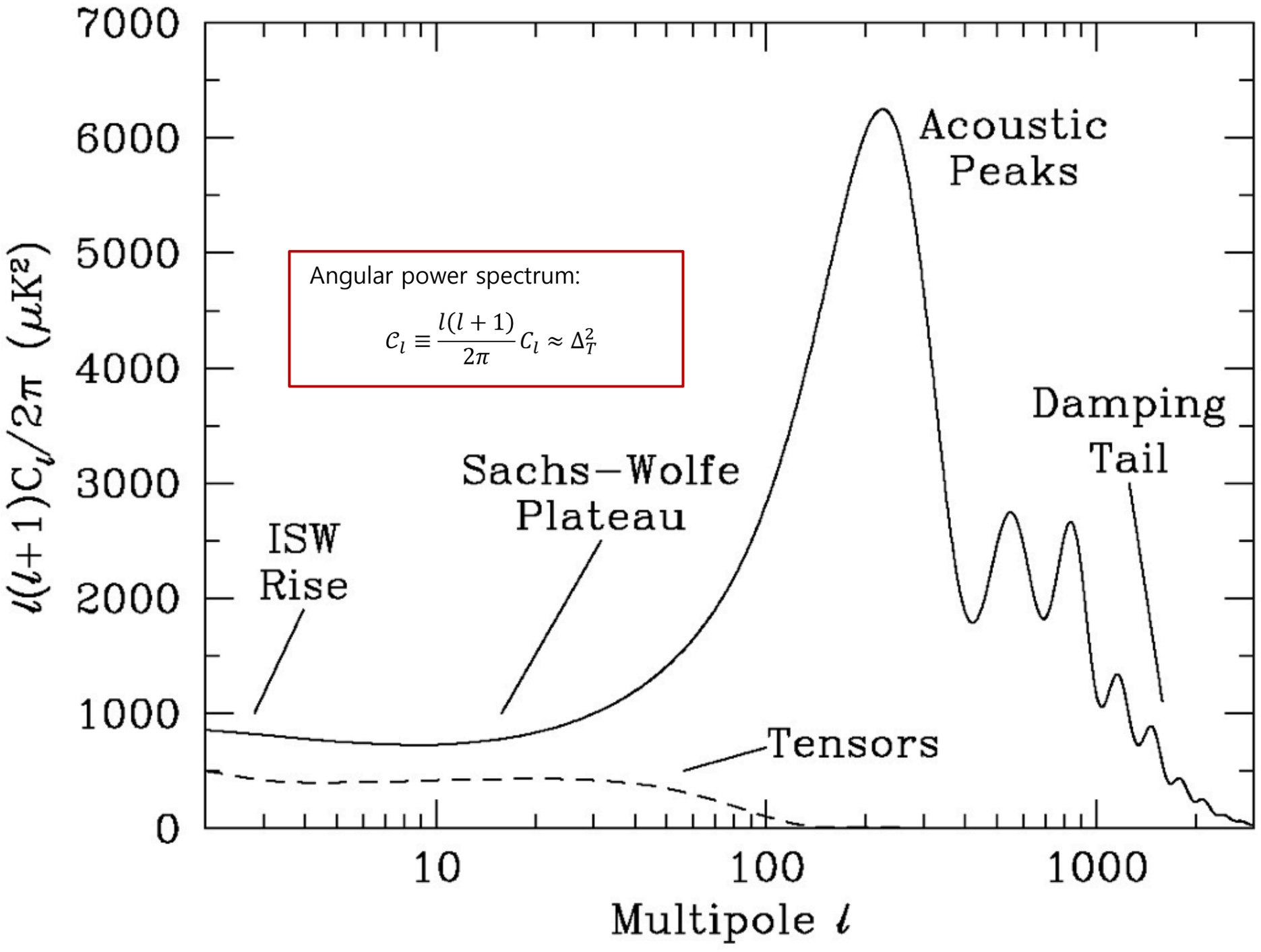
Similar to the SW plateau,

by comparing with $\langle \Delta T(\hat{n}) \Delta T(\hat{n}') \rangle = \sum_{\ell m} C_\ell Y_\ell^m(\hat{n}) Y_\ell^{-m}(\hat{n}') = \sum_\ell C_\ell \left(\frac{2\ell+1}{4\pi} \right) P_\ell(\hat{n}\cdot\hat{n}')$,

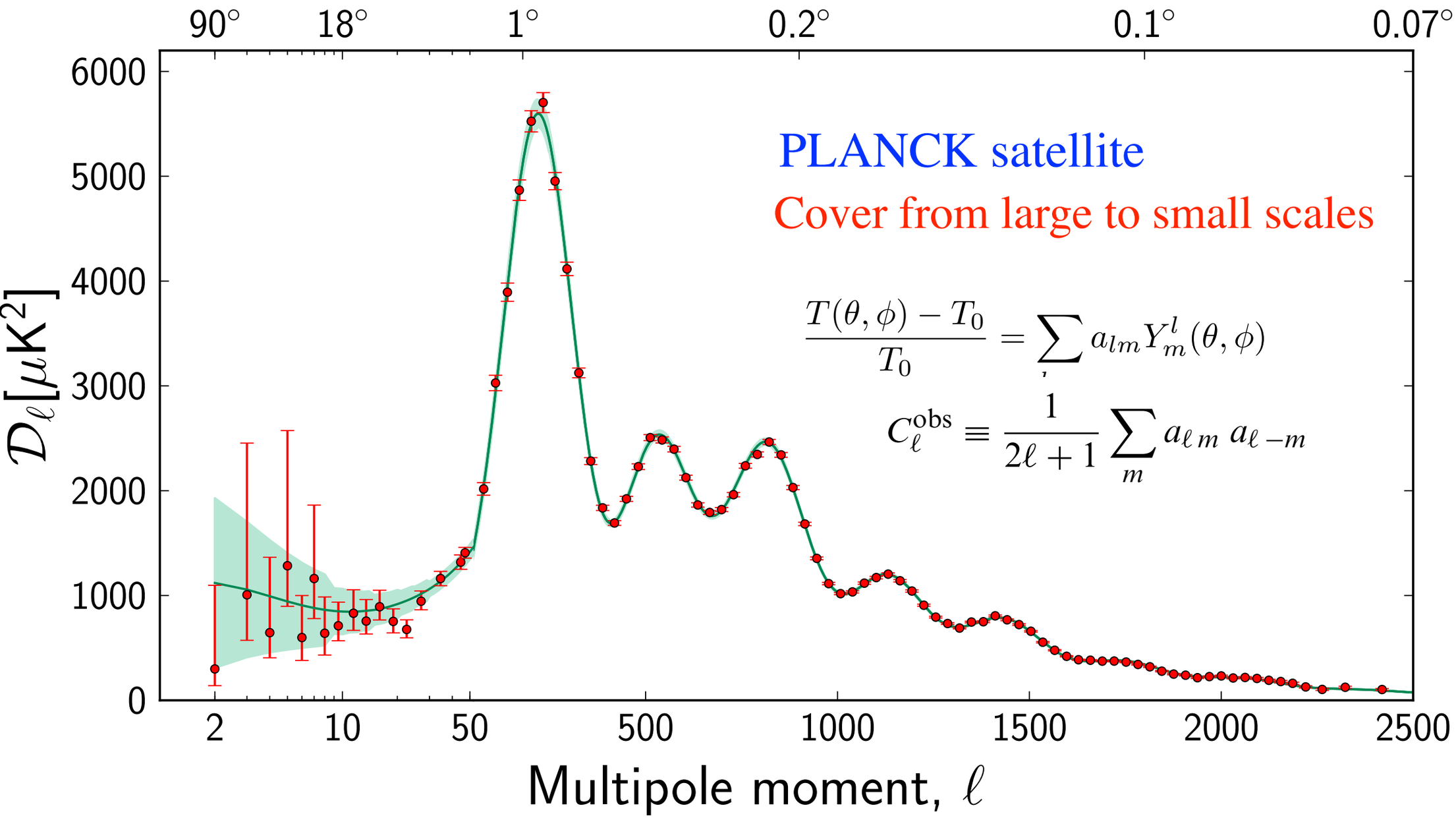
we obtain

$$C_l = \frac{2}{\pi} \int \left| \left(\Phi_k(\eta_r) + \frac{\delta_k(\eta_r)}{4} \right) j_l(k\eta_0) - \frac{3\delta'_k(\eta_r)}{4k} \frac{d j_l(k\eta_0)}{d(k\eta_0)} \right|^2 k^2 dk.$$

Cosmic Microwave Background Anisotropy acoustic peaks



우주배경복사의 온도 음향진동



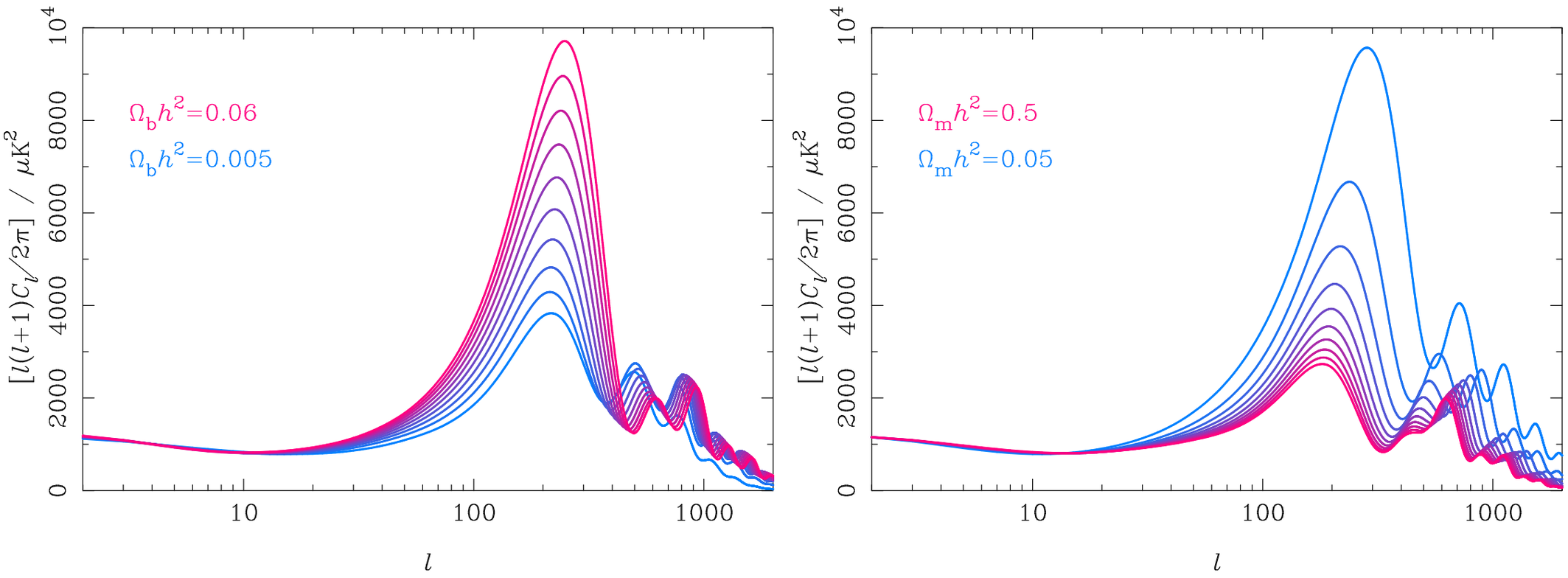


Fig. 9. Dependence of the temperature-anisotropy power spectrum on the physical density in baryons (left) and all non-relativistic matter (right). From top to bottom at the first peak, the baryon densities vary linearly in the range $\Omega_{\rm b}h^2 = 0.06$–0.005 (left) and the matter densities in $\Omega_{\rm m}h^2 = 0.05$–0.5 (right). The initial conditions are adiabatic.

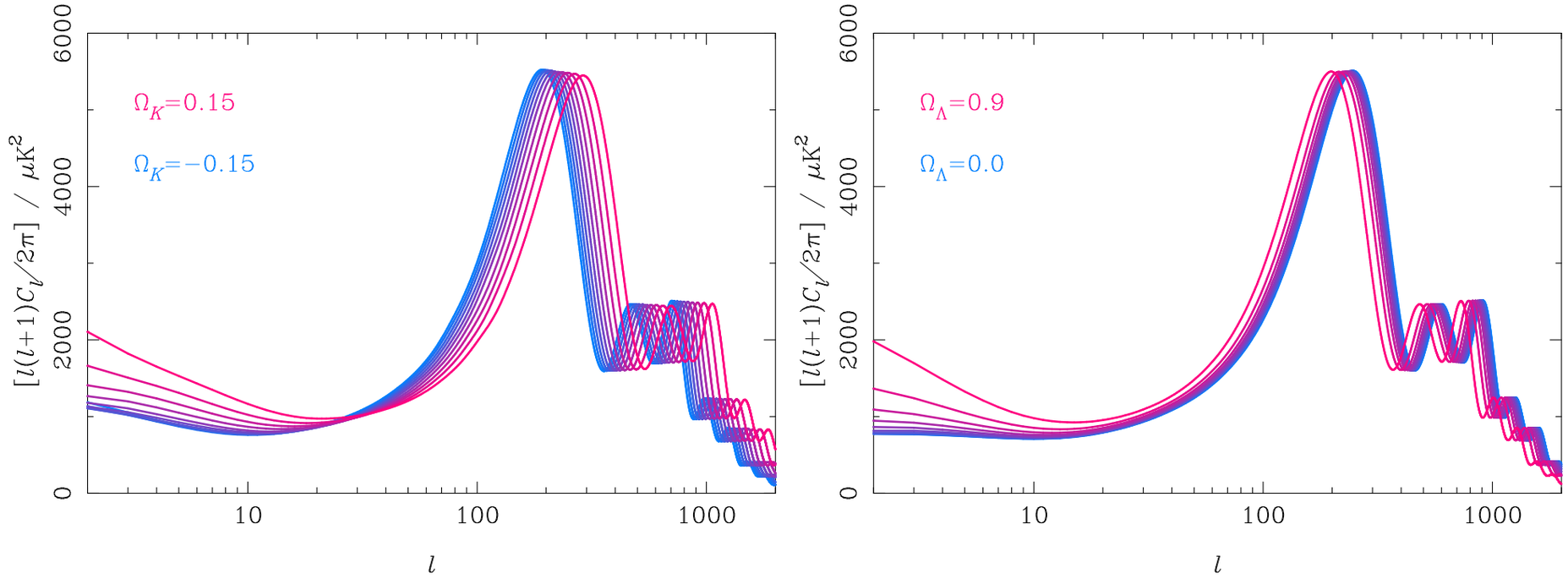


Fig. 10. Dependence of the temperature-anisotropy power spectrum on the curvature Ω_K (left) and cosmological constant Ω_Λ (right) in adiabatic models. In both cases, the physical densities in baryons and matter were held constant, thus preserving the conditions on the last scattering surface. The curvature varies (left to right) in the range -0.15–0.15 and the cosmological constant in the range 0.9–0.0.

Table 2. Parameter 68 % intervals for the base-LambdaCDM model from Planck CMB power spectra, in combination with CMB lensing reconstruction and BAO. [Planck 2018]

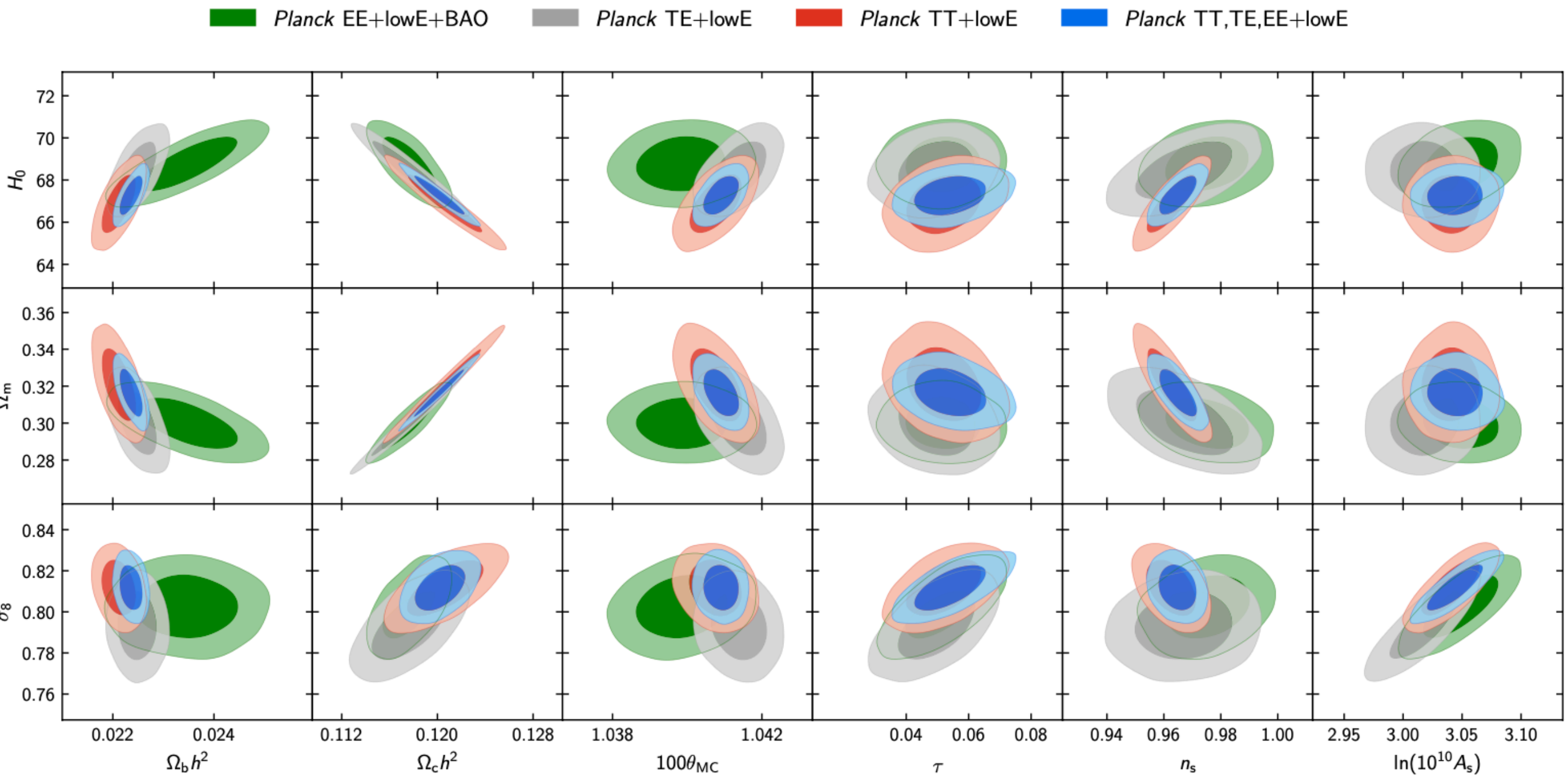
https://arxiv.org/abs/1807.06209

Parameter	TT+lowE 68% limits	TE+lowE 68% limits	EE+lowE 68% limits	TT,TE,EE+lowE 68% limits	TT,TE,EE+lowE+lensing 68% limits	TT,TE,EE+lowE+lensing+BAO 68% limits
$\Omega_b h^2$	0.02212 ± 0.00022	0.02249 ± 0.00025	0.0240 ± 0.0012	0.02236 ± 0.00015	0.02237 ± 0.00015	0.02242 ± 0.00014
$\Omega_c h^2$	0.1206 ± 0.0021	0.1177 ± 0.0020	0.1158 ± 0.0046	0.1202 ± 0.0014	0.1200 ± 0.0012	0.11933 ± 0.00091
$100\theta_{MC}$	1.04077 ± 0.00047	1.04139 ± 0.00049	1.03999 ± 0.00089	1.04090 ± 0.00031	1.04092 ± 0.00031	1.04101 ± 0.00029
τ	0.0522 ± 0.0080	0.0496 ± 0.0085	0.0527 ± 0.0090	$0.0544^{+0.0070}_{-0.0081}$	0.0544 ± 0.0073	0.0561 ± 0.0071
$\ln(10^{10} A_s)$	3.040 ± 0.016	$3.018^{+0.020}_{-0.018}$	3.052 ± 0.022	3.045 ± 0.016	3.044 ± 0.014	3.047 ± 0.014
n_s	0.9626 ± 0.0057	0.967 ± 0.011	0.980 ± 0.015	0.9649 ± 0.0044	0.9649 ± 0.0042	0.9665 ± 0.0038
H_0 [km s^{-1} Mpc^{-1}]	66.88 ± 0.92	68.44 ± 0.91	69.9 ± 2.7	67.27 ± 0.60	67.36 ± 0.54	67.66 ± 0.42
Ω_Λ	0.679 ± 0.013	0.699 ± 0.012	$0.711^{+0.033}_{-0.026}$	0.6834 ± 0.0084	0.6847 ± 0.0073	0.6889 ± 0.0056
Ω_m	0.321 ± 0.013	0.301 ± 0.012	$0.289^{+0.026}_{-0.033}$	0.3166 ± 0.0084	0.3153 ± 0.0073	0.3111 ± 0.0056
$\Omega_m h^2$	0.1434 ± 0.0020	0.1408 ± 0.0019	$0.1404^{+0.0034}_{-0.0039}$	0.1432 ± 0.0013	0.1430 ± 0.0011	0.14240 ± 0.00087
$\Omega_m h^3$	0.09589 ± 0.00046	0.09635 ± 0.00051	$0.0981^{+0.0016}_{-0.0018}$	0.09633 ± 0.00029	0.09633 ± 0.00030	0.09635 ± 0.00030
σ_8	0.8118 ± 0.0089	0.793 ± 0.011	0.796 ± 0.018	0.8120 ± 0.0073	0.8111 ± 0.0060	0.8102 ± 0.0060
$S_8 \equiv \sigma_8(\Omega_m/0.3)^{0.5}$	0.840 ± 0.024	0.794 ± 0.024	$0.781^{+0.052}_{-0.060}$	0.834 ± 0.016	0.832 ± 0.013	0.825 ± 0.011
$\sigma_8 \Omega_m^{0.25}$	0.611 ± 0.012	0.587 ± 0.012	0.583 ± 0.027	0.6090 ± 0.0081	0.6078 ± 0.0064	0.6051 ± 0.0058
z_{re}	7.50 ± 0.82	$7.11^{+0.91}_{-0.75}$	$7.10^{+0.87}_{-0.73}$	7.68 ± 0.79	7.67 ± 0.73	7.82 ± 0.71
$10^9 A_s$	2.092 ± 0.034	2.045 ± 0.041	2.116 ± 0.047	$2.101^{+0.031}_{-0.034}$	2.100 ± 0.030	2.105 ± 0.030
$10^9 A_s e^{-2\tau}$	1.884 ± 0.014	1.851 ± 0.018	1.904 ± 0.024	1.884 ± 0.012	1.883 ± 0.011	1.881 ± 0.010
Age [Gyr]	13.830 ± 0.037	13.761 ± 0.038	$13.64^{+0.16}_{-0.14}$	13.800 ± 0.024	13.797 ± 0.023	13.787 ± 0.020
z_*	1090.30 ± 0.41	1089.57 ± 0.42	$1087.8^{+1.6}_{-1.7}$	1089.95 ± 0.27	1089.92 ± 0.25	1089.80 ± 0.21
r_* [Mpc]	144.46 ± 0.48	144.95 ± 0.48	144.29 ± 0.64	144.39 ± 0.30	144.43 ± 0.26	144.57 ± 0.22
$100\theta_*$	1.04097 ± 0.00046	1.04156 ± 0.00049	1.04001 ± 0.00086	1.04109 ± 0.00030	1.04110 ± 0.00031	1.04119 ± 0.00029
z_{drag}	1059.39 ± 0.46	1060.03 ± 0.54	1063.2 ± 2.4	1059.93 ± 0.30	1059.94 ± 0.30	1060.01 ± 0.29
r_{drag} [Mpc]	147.21 ± 0.48	147.59 ± 0.49	146.46 ± 0.70	147.05 ± 0.30	147.09 ± 0.26	147.21 ± 0.23
k_D [Mpc^{-1}]	0.14054 ± 0.00052	0.14043 ± 0.00057	0.1426 ± 0.0012	0.14090 ± 0.00032	0.14087 ± 0.00030	0.14078 ± 0.00028
z_{eq}	3411 ± 48	3349 ± 46	3340^{+81}_{-92}	3407 ± 31	3402 ± 26	3387 ± 21
k_{eq} [Mpc^{-1}]	0.01041 ± 0.00014	0.01022 ± 0.00014	$0.01019^{+0.00025}_{-0.00028}$	0.010398 ± 0.000094	0.010384 ± 0.000081	0.010339 ± 0.000063
$100\theta_{s,eq}$	0.4483 ± 0.0046	0.4547 ± 0.0045	0.4562 ± 0.0092	0.4490 ± 0.0030	0.4494 ± 0.0026	0.4509 ± 0.0020
f_{2000}^{143}	31.2 ± 3.0			29.5 ± 2.7	29.6 ± 2.8	29.4 ± 2.7
$f_{2000}^{143\times217}$	33.6 ± 2.0			32.2 ± 1.9	32.3 ± 1.9	32.1 ± 1.9
f_{2000}^{217}	108.2 ± 1.9			107.0 ± 1.8	107.1 ± 1.8	106.9 ± 1.8

niversity, Korea

https://arxiv.org/abs/1807.06209

[Planck 2018]



Baryon acoustic oscillation (BAO)

Right after recombination, the density contrast of total matter is

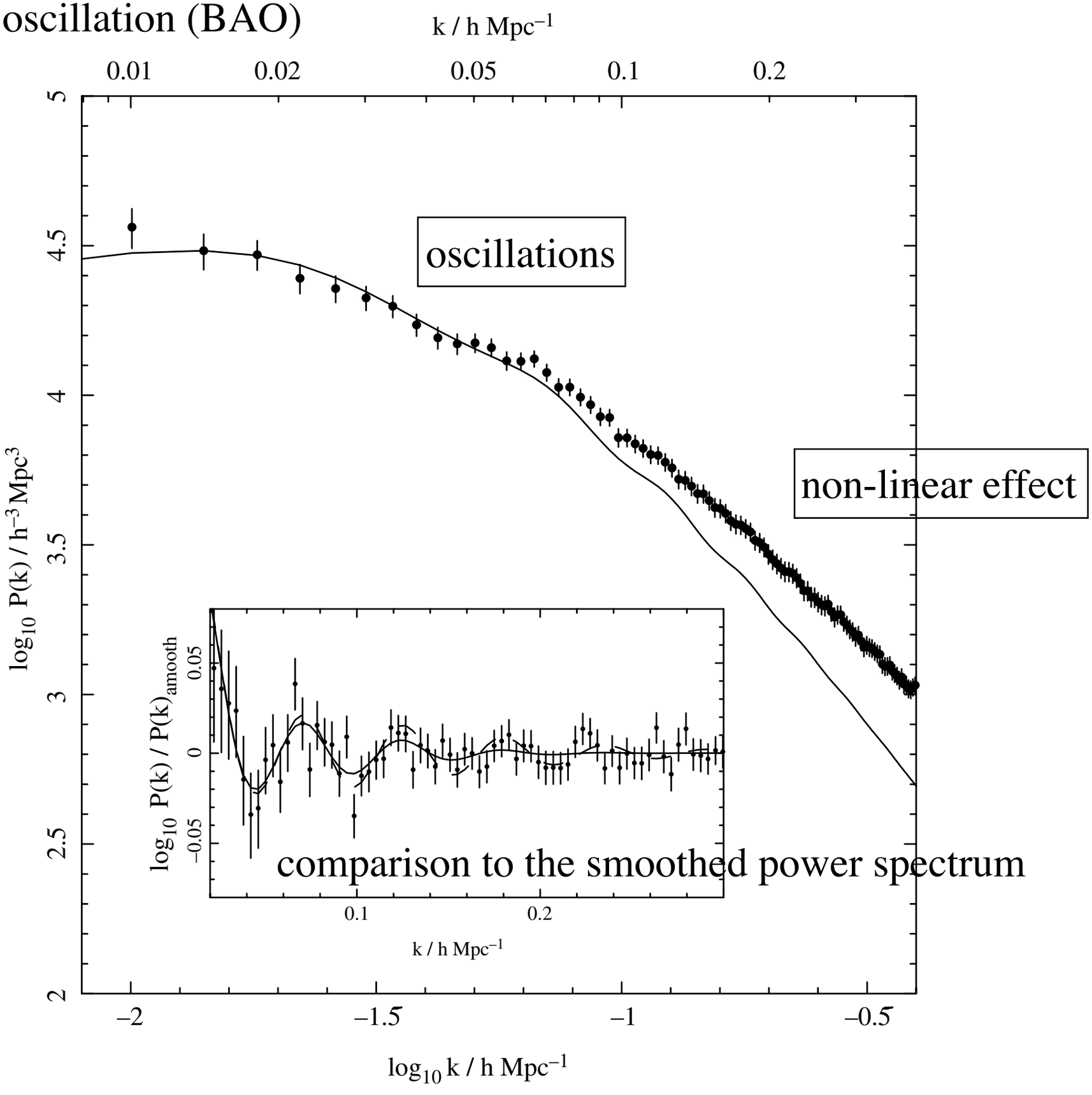
$$\frac{\delta\rho}{\rho} = \frac{\delta\rho_{\rm B} + \delta\rho_{\rm DM}}{\rho_{\rm B} + \rho_{\rm DM}} = \frac{\Omega_{\rm B}}{\Omega_{\rm M}}\frac{\delta\rho_{\rm B}}{\rho_{\rm B}} + \frac{\Omega_{\rm DM}}{\Omega_{\rm M}}\frac{\delta\rho_{\rm DM}}{\rho_{\rm DM}}$$

$\delta\rho_{\rm B}$ oscillates as a function of k, while $\delta\rho_{\rm DM}$ is a smooth function.

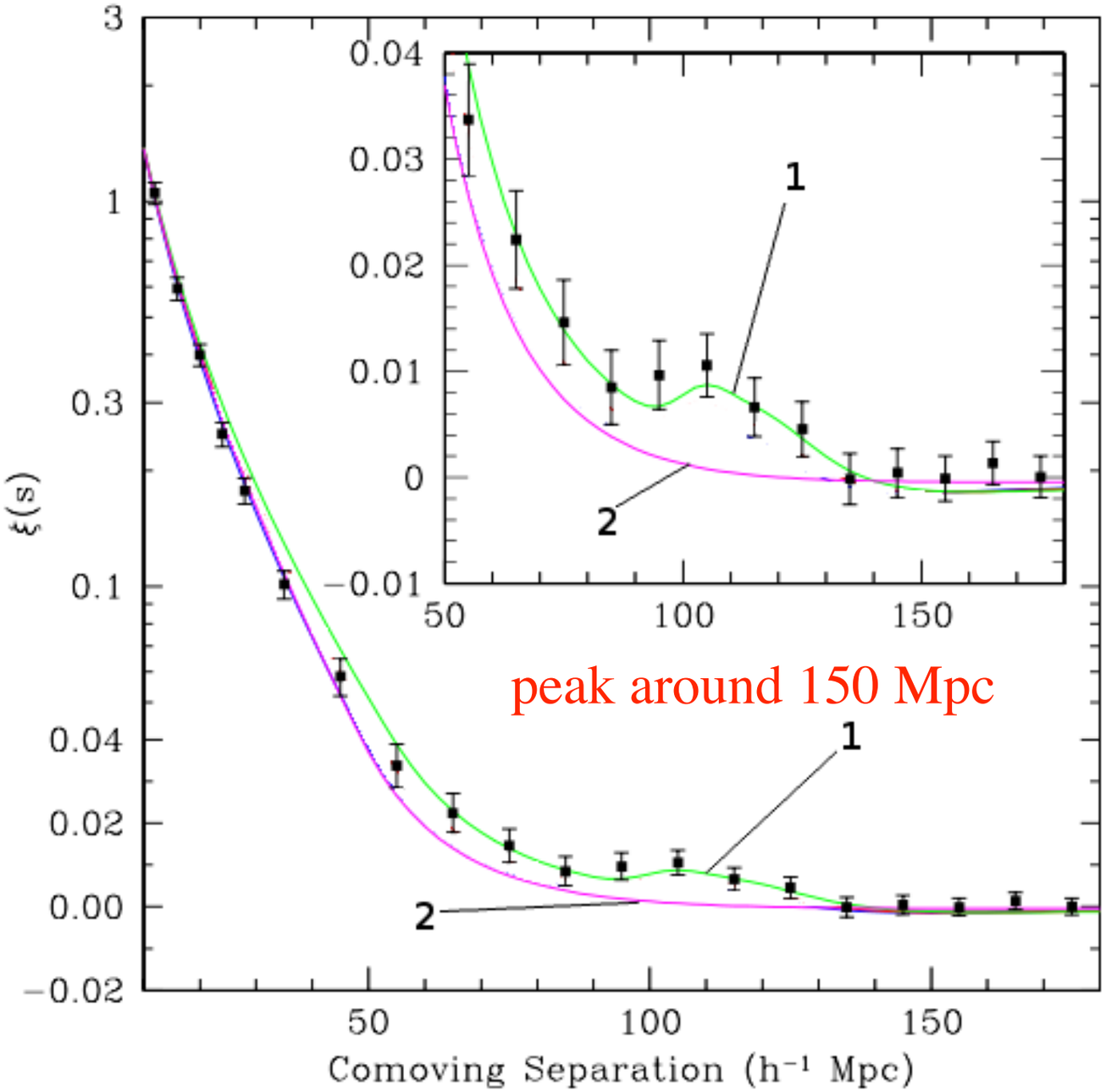
Therefore the power spectrum of the total matter density perturbation today has an oscillating part in k. This is called baryon acoustic oscillation (BAO).

Baryon acoustic oscillation (BAO)

matter power spectrum



Baryon acoustic oscillation (BAO) : The correlation function in the coordinates



1 : CDM + baryon

2 : pure CDM w/o baryon

BAO provides another standard ruler

- The properties of the primordial curvature perturbation

- Power spectrum $\langle \zeta_{\mathbf{k_1}} \zeta_{\mathbf{k_2}} \rangle \equiv (2\pi)^3 \delta^3(\mathbf{k_1} + \mathbf{k_2}) \frac{2\pi^2}{k^3} \mathcal{P}_\zeta(k)$ $\mathcal{P}_\zeta = 2.441^{+0.088}_{-0.092} \times 10^{-9}$

  : small perturbation

- Spectral index $n_\zeta - 1 \equiv \frac{d \log \mathcal{P}_\zeta}{d \log k}$ $n_s = 0.963 \pm 0.012\,(68\%\mathrm{CL})$

  : almost scale-invariant

- Tensor-to-scalar ratio $r \equiv \frac{\mathcal{P}_T}{\mathcal{P}_\zeta}$ $r_T < 0.36\,(95\%\,\mathrm{CL})$

  : unobserved yet

$$\mathcal{P}_T = 8\left(\frac{H_*}{2\pi}\right)^2$$

- primordial tensor perturbation determines the initial amplitude of gravitational waves, whose oscillation begins at horizon entry

- Non-Gaussianity $-10 < f_{\mathrm{NL}} < 74\,(95\%\ \mathrm{CL})$

  : No evidence of non-G yet

- Inflation

The small fluctuation in the density?

Inflation, also can explain the origin of the large scale structure of the Universe. The quantum fluctuations in the microscopic inflationary region is magnified to cosmic size and become the seeds of the growth of structure in the Universe.

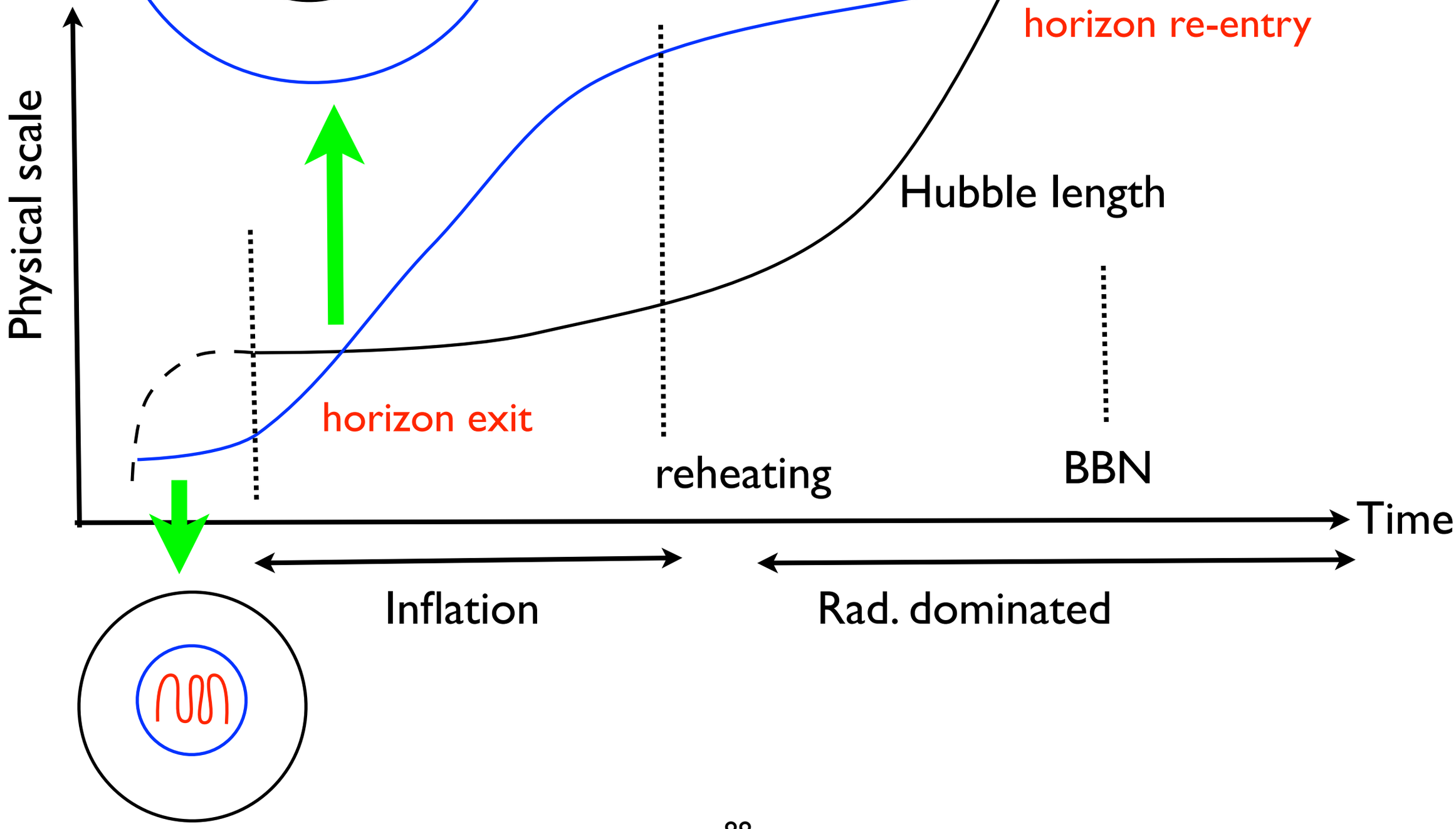
A massless scalar field acquires a perturbation and freezes in after horizon exit $\delta\phi_k = \frac{i}{\sqrt{2k}}\frac{H}{k}$ with almost Gaussian distribution. It gives a power spectrum

$$\mathcal{P}_{\delta\phi}(k) = \left(\frac{H_k}{2\pi}\right)^2$$ defined by $$\langle \delta\phi_{\mathrm{k}} \delta\phi_{\mathrm{k}'} \rangle = \frac{2\pi^2}{k^3}\mathcal{P}_{\delta\phi}(k)\delta^3(\mathbf{k}+\mathbf{k}')$$

[Mukhanov, Chibisov, 1981]

becomes classical

$$\delta\phi \sim \frac{H}{2\pi} \quad \text{and} \quad \zeta$$



- Origin of the primordial curvature perturbation : Inflation

Inflation → Radiation dominated era → CMB

$\delta\phi$

quantum fluctuation
of a light scalar field

ζ

curvature perturbation
: conserved if

$$p = p(\rho)$$

[Lyth, Malik, Sasaki, 2005]

If not, it can be changed
eg, multi-field

$$\frac{\Delta T}{T} \sim 10^{-5}$$

anisotropy in CMB

$$\frac{\Delta T}{T} = -\frac{1}{5}\zeta \quad \text{(SW limit)}$$

$$\zeta = -\psi - H\frac{\delta\rho}{\dot{\rho}}$$

$$\delta\rho \simeq \frac{dV}{d\phi}\delta\phi$$