# III. Path integral method

- Path integral for a single particle

Let's consider a matrix element of the evolution operator for a time interval $t_f - t_i$ governed by the Hamiltonian $\hat{H} = \hat{H}(\hat{x}, \hat{p})$ with $[\hat{x}, \hat{p}] = i\hbar$:

$$\mathcal{O}(x_f, t_f; x_i, t_i) = \langle x_f | e^{-\frac{i}{\hbar}(t_f - t_i)\hat{H}} | x_i \rangle$$

$$= \int \frac{dp}{2\pi\hbar} \langle x_f | p \rangle \langle p | e^{-\frac{i}{\hbar}(t_f - t_i)\hat{H}} | x_i \rangle$$

If $\hat{H}(\hat{x}, \hat{p})$ is normal ordered (i.e. all $\hat{p}$'s appear to the left of all $\hat{x}$'s), $\langle p | \hat{H}(\hat{x}, \hat{p}) | x \rangle = H(x, p) \langle p | x \rangle$ by replacing the operators $\hat{x}$ and $\hat{p}$ by the corresponding numbers $x$ and $p$, respectively. However, $e^{-\frac{i}{\hbar}(t_f - t_i)\hat{H}}$ is not a normal ordered operator even if $\hat{H}$ is normal ordered, thus we cannot simply replace the operators by numbers.

For an infinitesimal interval $\delta t$, however,

$$\langle p | e^{-\frac{i}{\hbar}\delta t \hat{H}(\hat{x}, \hat{p})} | x \rangle \approx \langle p | 1 - \frac{i}{\hbar}\delta t \hat{H}(\hat{x}, \hat{p}) | x \rangle \approx \langle p | e^{-\frac{i}{\hbar}\delta t H(x,p)} | x \rangle$$

($\hat{H}(\hat{x}, \hat{p}) \to H(x, p)$; $H(x,p)$: number)

Then we can replace the operators by numbers in each time slice. This means that the time slicing is necessary because the operators in general do not commute.

The basic idea of the path integral formulation is

① break the time interval into infinitesimal steps

② evaluate the evolution operator for each time step

③ Chain the matrix elements together to obtain the result for the finite interval.

① $O(x_f,t_f;x_i,t_i) = \langle x_f|\left(e^{-\frac{i}{\hbar}\Delta t \hat{H}}\right)^M|x_i\rangle \quad \Delta t = \frac{t_f-t_i}{M}$

$= \int dx_1 \cdots dx_{M-1} \langle x_f| e^{-\frac{i}{\hbar}\Delta t \hat{H}} |x_{M-1}\rangle \langle x_{M-1}| \cdots |x_1\rangle \langle x_1| e^{-\frac{i}{\hbar}\Delta t \hat{H}} |x_i\rangle$

② $\langle x_n| e^{-\frac{i}{\hbar}\Delta t \hat{H}(\hat{x},\hat{p})} |x_{n-1}\rangle = \int \frac{dp_n}{2\pi\hbar} \langle x_n|p_n\rangle \langle p_n| e^{-\frac{i}{\hbar}\Delta t \hat{H}(\hat{x},\hat{p})} |x_{n-1}\rangle$

$\approx \int \frac{dp_n}{2\pi\hbar} \langle x_n|p_n\rangle \langle p_n| 1 - \frac{i}{\hbar}\Delta t \hat{H}(\hat{x},\hat{p}) |x_{n-1}\rangle$

note) $\langle p_n|\hat{H}(\hat{x},\hat{p})|x_{n-1}\rangle \to H(x_{n-1},p_n)\langle p_n|x_{n-1}\rangle$ ← number

$\approx \int \frac{dp_n}{2\pi\hbar} e^{\frac{i}{\hbar}\Delta x_n p_n - \frac{i}{\hbar}\Delta t H(x_{n-1},p_n)}$

$\langle x|p\rangle = e^{\frac{i}{\hbar}p\cdot x}$

where $\Delta x_n = x_n - x_{n-1}$ $(n=1,2\cdots,M)$ with $x_0 = x_i$, $x_M = x_f$

③ $$\mathcal{O}(x_f,t_f;x_i,t_i) = \int \prod_{n=1}^{M} dx_n \int \prod_{j=1}^{M} \frac{dp_n}{2\pi\hbar} e^{\frac{i\Delta t}{\hbar}\sum_{n=1}^{M}\left[\frac{x_n - x_{n-1}}{\Delta t}p_n - H(x_n,p_n)\right]}$$

$$\xrightarrow{M\to\infty} \int_{x_i,t_i}^{x_f,t_f} Dx \int Dp\, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\,[\dot{x}(t)p(t) - H(x,p)]}$$

Thus, the integration weighted by the Hamiltonian action extends over all possible paths through the classical phase space $(x,p)$ of the system which begin and end at the same configuration points $x(t_i)=x_i$ and $x(t_f)=x_f$.

For a Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(\hat{x})$, after integrating over $p$, we obtain

$$\mathcal{O}(x_f,t_f;x_i,t_i) = \int_{x_i,t_i}^{x_f,t_f} Dx\, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\, L(x,\dot{x},t)}$$

$L(x,\dot{x},t) = \frac{1}{2}m\dot{x}^2 - V(x)$ ← classical Lagrangian

The quantum transition amplitude is now expressed in a form that does not contain quantum mechanical operators, and each path is weighted by its classical action $S[x] = \int_{t_i}^{t_f} dt\, L(x,\dot{x},t)$.

Note that the integration extends not only over the classical path but over all paths. In the $\hbar \to 0$ or semiclassical limit, the classical path dominates which minimizes the action, while the fluctuations around it gives a quantum contribution to the action.

Next, consider the partition function of a single particle.

$$Z = \mathrm{Tr}[e^{-\beta \hat{H}}] = \int dx \langle x | e^{-\beta \hat{H}} | x \rangle .$$

which can be considered as the time evolution evaluated at an imaginary time interval $t \to -i\beta\hbar$.

The matrix element of the imaginary time evolution operator is given by

$$\mathcal{O}(x_f, t_f; x_i, t_i) = \langle x_f | e^{-\frac{1}{\hbar}(t_f - t_i)\hat{H}} | x_i \rangle \qquad (\dot{x} = \frac{dx}{dt})$$

$$= \int_{x_i, t_i}^{x_f, t_f} Dx Dp \, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt [p\dot{x} - H(x,p)]} \xrightarrow{H = \frac{p^2}{2m} + V(x)} \int Dx \, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \left[\frac{1}{2}m\dot{x}^2 + V(x)\right]}$$

$$\to Z = \mathcal{O}(x, \beta\hbar, x, 0) = \int_{x(\beta\hbar) = x(0)} Dx \, e^{\int_0^{\beta\hbar} dt \left[\frac{1}{2}m\dot{x}^2 + V(x)\right]}$$

Further reading: Sakurai, ch. 2.5.; Negele & Orland, ch. 2.2

- Coherent states

Motivation : $\langle p|\hat{A}(\hat{p},\hat{x})|x\rangle = A(p,x)\langle p|x\rangle$

$\langle\phi|\hat{A}(\hat{a}^\dagger,\hat{a})|\phi\rangle = A(\phi^*,\phi)\langle\phi|\phi\rangle$ ?

For a system composed of many identical particles, it is useful to define operators in terms of creation and annihilation operators. Just as the position states are defined as eigenstates of the position operator, the coherent states are defined as eigenstates of the annihilation operators, $\hat{a}_n|\phi\rangle = \phi_n|\phi\rangle$

First, consider a coherent state of a single bosonic operator $\hat{a}$ defined by $\hat{a}|\phi\rangle = \phi|\phi\rangle$.

For $|\phi\rangle = \sum_n C_n|n\rangle$, $\sum_n C_n\sqrt{n}|n-1\rangle = \sum_n C_n\phi|n\rangle$.

$\therefore C_n = \frac{\phi}{\sqrt{n}}C_{n-1}$ $(n=1,2,\cdots) \rightarrow C_n = \frac{\phi^n}{\sqrt{n!}}$ setting $C_0 = 1$.

$|\phi\rangle = \sum_n \frac{\phi^n}{\sqrt{n!}}|n\rangle = \sum_n \frac{(\phi\hat{a}^\dagger)^n}{n!}|0\rangle = e^{\phi\hat{a}^\dagger}|0\rangle$ $\left(|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle\right)$

$\rightarrow \langle\phi|\phi\rangle = \sum_n \frac{(\phi^*\phi)^n}{n!} = e^{\phi^*\phi}$ Not orthonormal

Although the coherent states are not orthonormal, importantly, they form a complete set with the following completeness relation:

$$\int \frac{d^2\phi}{\pi} e^{-\phi^*\phi} |\phi\rangle\langle\phi| = 1$$

The trace of an operator is given by

$$\mathrm{Tr}\hat{A} = \int \frac{d^2\phi}{\pi} e^{-\phi^*\phi} \langle\phi|\hat{A}|\phi\rangle$$

It is easy to generalize the coherent states to multiple bosonic operators $\{\hat{a}_i\}$

$$\hat{a}_i|\phi\rangle = \phi_i|\phi\rangle$$

$$\rightarrow \langle\phi|\hat{a}_{i_1}^\dagger \hat{a}_{i_2}^\dagger \cdots \hat{a}_{i_2}\hat{a}_{i_1}|\phi\rangle = \phi_{i_1}^*\phi_{i_2}^* \cdots \phi_{i_2}\phi_{i_1}\langle\phi|\phi\rangle$$

In general, if $\hat{A}(\hat{a}_i^\dagger, \hat{a}_i)$ is an operator in normal ordered form,

$$\langle\phi|\hat{A}(\hat{a}_i^\dagger, \hat{a}_i)|\phi\rangle = A(\phi_i^*, \phi_i)\langle\phi|\phi\rangle$$

Similarly as the bosonic case, suppose we constructed coherent states $|\theta\rangle$ for fermionic operators $\{\hat{a}_\alpha\}$: $\hat{a}_\alpha|\theta\rangle = \theta_\alpha|\theta\rangle$.

Note that because of anticommutativity of the fermionic operators $\{\hat{a}_\alpha, \hat{a}_\beta\} = 0$ with $\hat{a}_\alpha^2 = 0$, the eigenvalues also anticommute, $\{\theta_\alpha, \theta_\beta\} = 0$ with $\theta_\alpha^2 = 0$. Thus, in order to construct coherent states for fermions, it is necessary to use anticommuting numbers, called Grassman numbers.

Further reading: Negele and Orland, Ch. 1

- Coherent state functional integral method

Consider the partition function of a many particle system. First, consider a bosonic case.

$$\mathcal{Z} = \mathrm{Tr}[e^{-\beta K}] = \int \frac{d\phi^* d\phi}{\pi i} e^{-\phi^*\phi} \langle \phi | e^{\beta K} | \phi \rangle$$

where $K = H - \mu \hat{N}$

Similarly as we did in the path integral, we first break the interval $(0,\beta\hbar)$ into $M$ steps of size $\Delta\tau=\frac{\beta\hbar}{M}$. The matrix element of the evolution operator between $n+1$ and $n$th coherent states is

$$e^{-\phi_n^*\phi_n}\langle\phi_n|e^{-\frac{\Delta\tau}{\hbar}\hat{K}}|\phi_{n-1}\rangle \approx e^{\phi_n^*\phi_{n-1}-\frac{\Delta\tau}{\hbar}K(\phi_n^*,\phi_{n-1})}$$

($1-\frac{\Delta\tau}{\hbar}K$; $K$: number)

where $\Delta\phi_n=\phi_n-\phi_{n-1}$ for $n=1,2\cdots M$ with $\phi_0=\phi_M$.

$$\rightarrow Z=\int D\phi^*D\phi\, e^{-\frac{1}{\hbar}S[\phi^*,\phi]}$$

$$S[\phi^*,\phi]=\int_0^{\beta\hbar}d\tau\left[\hbar\phi^*\frac{\partial\phi}{\partial\tau}+K(\phi^*,\phi)\right]$$

Note that the integral is done under the periodic boundary condition $\phi(\beta\hbar)=\phi(0)$ for bosons, whereas under the antiperiodic boundary condition $\phi(\beta\hbar)=-\phi(0)$ for fermions. From the boundary condition, we can interpret $\phi(\tau)$ as a periodic (antiperiodic) function with period $\beta\hbar$ satisfying $\phi(\tau+\beta\hbar)=\pm\phi(\tau)$ for bosons (fermions).

Thus, $\phi(\tau)$ can be expanded in a Fourier series as

$$\phi(\tau) = \frac{1}{\beta\hbar} \sum_n e^{-i\omega_n \tau} \phi(\omega_n)$$

$$\phi(\omega_n) = \int_0^{\beta\hbar} d\tau\, e^{i\omega_n \tau} \phi(\tau)$$

where $\omega_n$ is called the Matsubara frequency.

From $\phi(\beta\hbar) = \pm\phi(0)$, we find that $\omega_n = \frac{2n\pi}{\beta\hbar}$ for bosons while $\omega_n = \frac{(2n+1)\pi}{\beta\hbar}$ for fermions ($n$: integer).

Further reading: Negele and Orland, Ch. 2.2

- Hubbard-Stratonovich transformation

First, consider the following Gaussian integral.

$$\int dx\, e^{-\frac{1}{2}ax^2 + bx} = \sqrt{\frac{2\pi}{a}}\, e^{\frac{b^2}{2a}}$$

Note that the two body $b^2$ term can be expressed as an integral over the auxiliary term $x$ which couples linearly to $b$. In the path integral method, a similar technique can be employed to decouple a two-body interaction by introducing an auxiliary field, which is called the Hubbard-Stratonovich transformation.

Let's consider a fermionic system containing a two-body interaction. For simplicity, assume that Hamiltonian has the following form neglecting the momentum dependence:

$$\hat{K} = \sum_{\sigma} \xi \hat{a}_{\sigma}^{\dagger} \hat{a}_{\sigma} + \frac{v_0}{2} \sum_{\sigma,\sigma'} \hat{a}_{\sigma}^{\dagger} \hat{a}_{\sigma'}^{\dagger} \hat{a}_{\sigma'} \hat{a}_{\sigma}$$

The partition function can be expressed as a coherent state functional integral for fermions:

$$Z = \int D\psi^* D\psi \, e^{-S[\psi^*,\psi]} \qquad (\rho = \sum_{\sigma} \psi_{\sigma}^* \psi_{\sigma})$$

$$S[\psi^*,\psi] = \int_0^{\beta\hbar} d\tau \left[ \sum_{\sigma} \psi_{\sigma}^* (\hbar\partial_\tau + \xi) \psi_{\sigma} + \frac{1}{2} \rho V_0 \rho \right]$$

One problem is that the Hamiltonian contains a two-body term of $\rho$ (or a quartic term of $\psi^*$ and $\psi$) so the path integral cannot be performed. We can introduce a bosonic auxiliary field so that the two-body interaction term can be expressed as an integral over the auxiliary field that couples linearly to the one-body term $\rho$.

Let us introduce an auxiliary field $\phi$. Using the following Gaussian integral (up to a normalization factor)

$$\int D\phi\, e^{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \left(\frac{1}{2}\phi\mathcal{O}_0\phi + i\phi\rho\right)} = e^{\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \frac{1}{2}\rho\mathcal{O}_0^{-1}\rho}$$

Then the partition function can be rewritten as

$$Z = \int D\phi\, e^{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \frac{1}{2}\phi\mathcal{O}_0^{-1}\phi} Z_0[\phi]$$

Here $Z_0$ is the partition function for fermions with $\phi$ but without the two-body interaction term

$$Z_0[\phi] = \int D\psi^* D\psi\, e^{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \left[\sum_\sigma \psi_\sigma^*\left(\partial_\tau + \xi + i\phi\right)\psi_\sigma\right]}$$

$$= \left[\det\left(\partial_\tau + \omega + \frac{i}{\hbar}\phi\right)\right]^{g_s}$$

where $\omega = \xi/\hbar$ and $g_s = \sum_\sigma$ is the spin degeneracy factor. Note that the fermionic version of the Gaussian integral is given by $\int d\theta^* d\theta\, e^{-\theta^* a \theta} = a$ rather than its inverse.

By integrating out the fermions, we can obtain the effective action for the bosonic field $\phi$. cf) $S[\phi^*, \phi] \to S[\phi^*, \psi, \phi] \to S[\phi]$

- Random phase approximation

Let's consider the determinant of $M = \frac{\partial}{\partial\tau} + i\omega + \frac{i}{\hbar}\phi(\tau)$.

The determinant is not affected by the choice of basis so the matrix element can be expressed in $\tau$ or its Fourier component $\omega_n$:

$$M_{nn'} = \langle \omega_n | \frac{\partial}{\partial\tau} + i\omega + \frac{i}{\hbar}\phi(\tau) | \omega_{n'} \rangle$$

$$= -(i\omega_n - i\omega)\delta_{nn'} + \frac{i}{\hbar}\langle \omega_n | \phi | \omega_{n'} \rangle$$

$$\equiv -\mathcal{G}_0^{-1}(\omega_n)\delta_{nn'} + \frac{i}{\hbar}\phi(\omega_n - \omega_{n'})$$

$\mathcal{G}_0(\omega_n) = \frac{1}{i\omega_n - i\omega}$ is the noninteracting Green's function

Using the identity $\ln \mathrm{Det}\, M = \mathrm{Tr} \ln M$,

$$Z_0[\phi] = (\mathrm{Det}\, M)^{\mathcal{G}} = e^{\mathcal{G}\ln \mathrm{Det}\, M} = e^{\mathcal{G}\,\mathrm{Tr}\ln M}$$

Thus, by integrating out the fermionic degrees of freedom, we obtain the effective action for $\phi$ as $Z = \int \mathcal{D}\phi\, e^{-\frac{1}{\hbar}S[\phi]}$

$$S[\phi] = \int_0^{\beta\hbar} d\tau \frac{1}{2}\phi(\tau)\mathcal{D}_0^{-1}\phi(\tau) - \hbar\mathcal{G}\,\mathrm{Tr}\ln M$$

It can be shown that the saddle-point or mean-field solution $\frac{\partial S}{\partial \phi}=0$ gives $\phi=0$.

We now expand $S[\phi]$ in terms of $\phi$ (or the interaction strength) around $\phi=0$.

For $M=-g^{-1}$ where $g^{-1}=g_0^{-1}-\Sigma$ and $\Sigma=\frac{\partial^2}{\partial\phi^2}$ (Self-energy)

$$\mathrm{Tr}\ln(-g^{-1})=\mathrm{Tr}\ln(-g_0^{-1}+\Sigma)=\mathrm{Tr}\ln[(-g_0^{-1})(1-g_0\Sigma)]$$

$$=\mathrm{Tr}\ln(-g_0^{-1})-\mathrm{Tr}(g_0\Sigma)-\frac{1}{2}\mathrm{Tr}(g_0\Sigma)^2+\cdots$$

Here we used $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$ $(|x|<1)$.

$$S_0[\phi]\equiv -\hbar g\,\mathrm{Tr}\ln(-g^{-1})$$

$$=S_0[\phi=0]+\hbar g\,\mathrm{Tr}(g_0\Sigma)+\frac{\hbar g}{2}\mathrm{Tr}(g_0\Sigma)^2+\cdots$$

$$\equiv S_0^{(0)}[\phi]+S^{(1)}[\phi]+S^{(2)}(\phi)+\cdots$$

From now on, we set $\hbar=1$.

① $S^{(1)}[\phi] = \frac{\beta}{2}\mathrm{Tr}(g_0\Sigma) = \frac{\beta}{2}\sum_n \langle i\omega_n | g_0 i\hat{\phi} | i\omega_n \rangle$

$= \left[\frac{\beta}{2}\sum_{i\omega_n} g_0(i\omega_n)\right] i\hat{\phi}(0) \to 0$

Note that $\hat{\phi}(0) = 0$ owing to the charge neutrality condition with the positively charged background (if we include the momentum or position dependence)

② The lowest contribution comes from $S^{(2)}[\phi]$:

$$S^{(2)}[\phi] = \frac{\beta}{2}\mathrm{Tr}(g_0\Sigma)^2$$

$$= \frac{\beta}{2}\sum_{i\omega_n} \langle i\omega_n | (g_0 i\hat{\phi})^2 | i\omega_n \rangle$$

$$= -\frac{\beta}{2}\sum_{i\omega_n, i\omega_{n'}} \langle i\omega_n | g_0 \hat{\phi} | i\omega_{n'} \rangle \langle i\omega_{n'} | g_0 \hat{\phi} | i\omega_n \rangle$$

$$= -\frac{\beta}{2}\sum_{i\omega_n, i\omega_{n'}} g_0(i\omega_n)\hat{\phi}(i\omega_n - i\omega_{n'}) g_0(i\omega_{n'})\hat{\phi}(i\omega_{n'} - i\omega_n)$$

$i\nu_m = i\omega_{n'} - i\omega_n$

$$= -\frac{1}{2\beta}\sum_{i\nu_m} \hat{\phi}(-i\nu_m)\hat{\phi}(i\nu_m)\left[\frac{\beta}{2}\sum_{i\omega_n} g_0(i\omega_n) g_0(i\omega_n + i\nu_m)\right]$$

where $i\omega_n, i\omega_{n'}$ are fermionic Matsubara frequencies and $i\nu_m$ is a bosonic Matsubara frequency.

Let us define the polarization function $\chi_0(\Omega_m)$ as (after recovering $\hbar$)

$$\hbar\chi_0(\Omega_m) = \frac{g}{\beta\hbar}\sum_{\omega_n} g_0(i\omega_n)\, g_0(i\omega_n + i\Omega_m)$$




Thus, the effective action for $\phi$ becomes

$$S[\phi] \approx S_0[\phi=0] + \frac{1}{2\beta\hbar}\sum_{\Omega_m} \phi(-\Omega_m)\, D^{-1}(\Omega_m)\, \phi(\Omega_m)$$

$$D^{-1}(\Omega_m) = D_0^{-1} - \chi_0(\Omega_m) \quad \text{or} \quad D(\Omega_m) = \frac{D_0}{1 - D_0\chi_0(\Omega_m)}$$

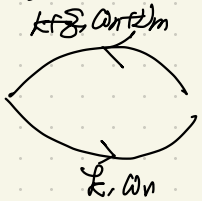
Writing $S[\phi]$ quadratic in $\phi$ and ignoring higher order terms corresponds to the RPA. Note that $D(\Omega_m)$ is an effective (screened) interaction between fermions within the RPA

Similarly, including the momentum dependence,

$$S[\phi] \approx S[\phi=0] + \frac{1}{2}\sum_{k,\omega_m}\int\frac{d^dq}{(2\pi)^d}\phi(q,\omega_m)\mathcal{O}(q,\omega_m)\phi(q,\omega_m)$$

$$\mathcal{O}^{-1}(q,\omega_m) = \mathcal{O}_0^{-1}(q) - \chi_0(q,\omega_m) \text{ or } \mathcal{O}(q,\omega_m) = \frac{\mathcal{O}_0(q)}{1-\mathcal{O}_0(q)\chi_0(q,\omega_m)}$$

$$\chi_0(q,\omega_m) = \frac{g}{\beta}\sum_{k,\omega_n}\int\frac{dk}{(2\pi)^d} G_0(k,\omega_n) G_0(k+q,\omega_n+\omega_m)$$




$$G(k,\omega_n) = \frac{1}{i\omega_n - \omega_k}$$

Note that $\frac{1}{\beta}\sum_{\omega_n}\frac{e^{i\omega_n 0^+}}{i\omega_n - \omega_k} = \mp\frac{1}{e^{\beta\omega_k}\mp 1}$

where $\omega_n = \frac{n\pi}{\beta}$ with even (odd) $n$ for bosons (fermions).

For fermions, using $\frac{1}{AB} = \frac{1}{B-A}\left(\frac{1}{A}-\frac{1}{B}\right)$,

$$\frac{1}{\beta}\sum_{\omega_n}\frac{1}{(i\omega_n-\omega)(i\omega_n+i\omega_m-\omega')} = \frac{f(\omega)-f(\omega')}{i\omega_m+\omega-\omega'}$$

where $f(\omega) = \frac{1}{e^{\beta\omega}+1}$.

After the frequency summation followed by the analytic continuation, $i\omega_m \to \omega + i0^+$,

$$\chi_0(q,\omega) = g\int\frac{dk}{(2\pi)^d}\frac{f(\xi_k)-f(\xi_{k+q})}{\omega+\xi_k-\xi_{k+q}+i0^+}$$

* Project: Many-body approach to superconductivity

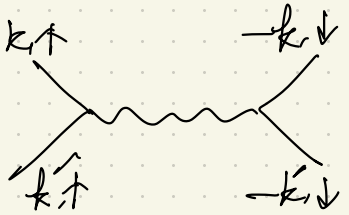
· Theoretical background

① BCS Hamiltonian

Due to phonon-mediated attractive interactions between electrons, bound pairs of electrons with opposite momenta and spins, called Cooper pairs, are formed leading to superconductivity when they are condensed. In this case, the most important interactions are those involving Cooper pairs, thus the effective Hamiltonian can be reduced so that only interactions for the opposite momenta and spins are retained:

$$\hat{H} = \sum_k \xi_k \left(\hat{c}^\dagger_{k\uparrow}\hat{c}_{k\uparrow} + \hat{c}^\dagger_{k\downarrow}\hat{c}_{k\downarrow}\right) - \sum_{kk'} O_{kk'}\, \hat{c}^\dagger_{k\uparrow}\hat{c}^\dagger_{-k\downarrow}\hat{c}_{-k'\downarrow}\hat{c}_{k'\uparrow}$$

$\xi_k = \varepsilon_k - \mu$ and $O_{kk'} = \langle k,-k|\hat{O}|k',-k'\rangle$




describing scattering of a pair of electrons from states $(k'\uparrow, -k'\downarrow)$ to $(k\uparrow, -k\downarrow)$.

Note $O_{kk'} > 0$ because of the attractive interactions.

⑤ Path integral formulation

In this project, we will construct the effective action for the superconducting order parameter using the path integral formulation. Starting from the BCS Hamiltonian, first introduce a bosonic field $\Delta$ to decouple the interaction, which will have the physical significance of the order parameter. After integrating over the fermionic field $\psi$, we obtain the quantum partition function as a path integral over the auxiliary bosonic field $\Delta$. Then, a variation of the action with respect to $\Delta$ generates a mean-field equation for $\Delta$, which gives the BCS gap equation. In the vicinity of the phase transition, we can perturbatively expand the action in $\Delta$, which gives the Ginzburg-Landau action.

Further reading: Tinkham. Ch.3; Coleman. Ch.14; Altland and Simons, Ch.6.4