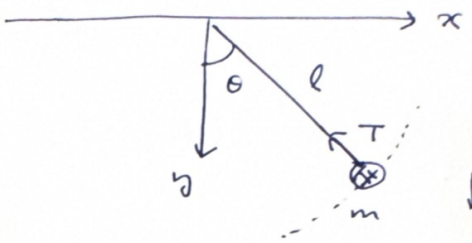
Lagrangian with constraints (Tong 2.3)

⊙ Example: Pendulum



- Constraint: $x^2 + y^2 = \ell^2$

- EOM

$$\begin{cases} m\ddot{x} = -T\sin\theta = -T\dfrac{x}{\ell} \\ m\ddot{y} = mg - T\cos\theta = mg - T\dfrac{y}{\ell} \end{cases}$$

(Newton mech so far)

* (x, y) to (r, θ)

- Answer:

$$\begin{cases} \ddot{\theta} = -\dfrac{g}{\ell}\sin\theta & \text{— a)} \\ T = m\ell\dot{\theta}^2 + mg\cos\theta & \text{— b)} \end{cases}$$

- Derivation

$$\vec{r} = (x, y) = \ell(\sin\theta, \cos\theta)$$
$$\dot{\vec{r}} = (\dot{x}, \dot{y}) = \ell(\cos\theta, -\sin\theta)\dot{\theta}$$
$$\ddot{\vec{r}} = (\ddot{x}, \ddot{y}) = -\ell(\sin\theta, \cos\theta)\dot{\theta}^2 + \ell(\cos\theta, -\sin\theta)\ddot{\theta}$$

a) $m(\dot{x}y - \ddot{y}x) = -mgx$

$$m\frac{d}{dt}(\dot{x}y - \dot{y}x)$$

$\dot{x}y - \dot{y}x$

$= \ell^2\dot{\theta}(c_\theta^2 + s_\theta^2) = \ell^2\dot{\theta}$

b) $\vec{r}^2 = \ell^2 \Rightarrow \vec{r}\cdot\dot{\vec{r}} = 0 \Rightarrow \dot{\vec{r}}^2 + \ddot{\vec{r}}\cdot\vec{r} = 0$

$$m\ddot{\vec{r}}\cdot\vec{r} = mgy - T\ell$$

$$-m\dot{\vec{r}}^2 = -m\ell^2\dot{\theta}^2$$

② Holonomic constraints

(free coord) : $3N \to n$

$$f^\alpha(x^A; t) = 0 \qquad (\alpha = 1, \cdots, 3N-n)$$

(P) velocity indep.

- "Solving" the constraints to find

$$x^A = x^A(q^1, \cdots, q^n; t)$$

is desirable but not necessarily easy.

- Ex) Pendulum. "$3N = 2$", $n = 1$.
  - $x^A = \{x, y\} \to q = \theta$
  - $f = x^2 + y^2 - \ell^2$
  - Solving the constr. $\begin{cases} x = \ell \sin\theta \\ y = \ell \cos\theta \end{cases}$

- Ex) Ptl on a sphere.
  - $x^A = \{x, y, z\} \to \vec{r}$
  - $f = \vec{r}^{\,2} - R^2$
  - Solve the constr

$$\vec{r} = R(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$L = \frac{1}{2}m(\dot\theta^2 + \sin^2\theta\,\dot\phi^2)$$

Drawback { not all symm manifest.
EOM becomes non-linear.

⊙ Lagrange multiplier

- $L' = L + \lambda_\alpha f^\alpha(x)$ (no $\dot{\lambda}$ terms)

$$\frac{\partial L'}{\partial \lambda_\alpha} = f^\alpha = 0$$ (Imposing constraints via Euler-Lagrange)

EOM for x^A

$$\dot{p}_A - \frac{\partial L}{\partial x^A} = \underbrace{\lambda_\alpha \frac{\partial f^\alpha}{\partial x^A}}_{\text{"constraint force"}}$$

- Ex) Pendulum

$$L' = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + mgy + \frac{1}{2}\lambda (x^2 + y^2 - \ell^2)$$

$$\Rightarrow \begin{cases} m\ddot{x} = \lambda x \\ m\ddot{y} = \lambda y + mg \end{cases}$$

* Comparison with Newton's eq. $\Rightarrow \lambda = -T/\ell$.

- Two strategies.

{ A. Derive the EOM first. Then solve the constraints
B. Solve the constraints first. Then, derive the EOM.

Derive EOM → ; ← Solve constr.

Typically, B is much quicker.

But, we may lose some info.

(For pendulum, T as a ftn of $\theta, \dot{\theta}$)

- Ex) Pendulum

$$L' = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + mgy + \frac{1}{2} \lambda (x^2 + y^2 - \ell^2)$$

$$\begin{cases} x = r \sin\theta \\ y = r \cos\theta \end{cases}$$

$$L' = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + mgr\cos\theta + \frac{1}{2} \lambda (r^2 - \ell^2)$$

$$\tilde{L} := L'|_{r=\ell} = \frac{1}{2} m \ell^2 \dot{\theta}^2 + mg\ell\cos\theta$$

EOM for θ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = m\ell^2 \ddot{\theta} + mg\ell\sin\theta = 0$$

Continuous symm. & Noether's thm. (Tong 2.4)

⊙ Def Conserved quantity (a.k.a. const. of motion)

$F(q^i, \dot{q}^i, t)$ s.t.

$$\text{EOM} \Rightarrow \frac{d}{dt}F = \frac{\partial F}{\partial q^i}\dot{q}^i + \frac{\partial F}{\partial \dot{q}^i}\ddot{q}^i + \frac{\partial F}{\partial t}$$

$$= 0.$$

· Ex) Hamiltonian ← L with $\frac{\partial L}{\partial t} = 0$.

$$H := p_i \dot{q}^i - L$$

$$\frac{dH}{dt} = \dot{p}_i \dot{q}^i + p_i \ddot{q}^i - \frac{\partial L}{\partial q^i}\dot{q}^i - \frac{\partial L}{\partial \dot{q}^i}\ddot{q}^i$$

$$= \left(p_i - \frac{\partial L}{\partial \dot{q}^i}\right)\ddot{q}^i + \left(\dot{p}_i - \frac{\partial L}{\partial q^i}\right)\dot{q}^i = 0$$

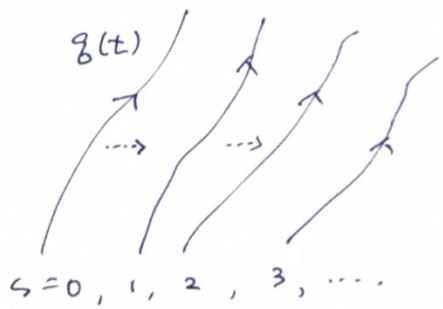
(def of p_i) (EOM)

⊙ Def Symmetry.

$q^i(t) \rightarrow Q^i(s,t)$

[one parameter family of paths]

s.t. $Q^i(0,t) = q^i(t)$



This transf is called a continuous symm. if

$$\frac{\partial}{\partial s} L(Q(s,t), \dot{Q}(s,t); t) = 0 \quad \text{for all } t.$$

· Remarks

a) No EOM involved

b) Time-indep. symm for now

c) Discrete symm deserves separate discussion.

⊙ Thm Noether

$$F := p_i \frac{\partial Q^i}{\partial s}\Big|_{s=0} \quad \text{is conserved}$$

along any solution to the EOM.

· Proof

$$\frac{\partial L}{\partial s} = \frac{\partial L}{\partial Q^i}\frac{\partial Q^i}{\partial s} + \frac{\partial L}{\partial \dot{Q}^i}\frac{\partial \dot{Q}^i}{\partial s}$$

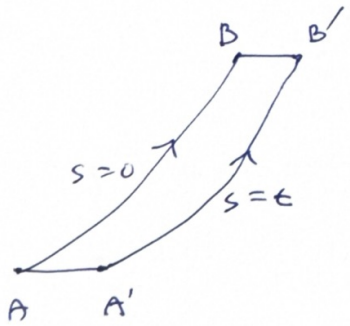
$$0 = \frac{\partial L}{\partial s}\Big|_{s=0} \qquad (\text{Def. of symm})$$

$$= \frac{\partial L}{\partial q^i}\frac{\partial Q^i}{\partial s}\Big|_{s=0} + \frac{\partial L}{\partial \dot{q}^i}\frac{\partial \dot{Q}^i}{\partial s}\Big|_{s=0} \quad \Big\} \text{Def. of } p_i,\ \text{EOM}$$

$$= \dot{p}_i \frac{\partial Q^i}{\partial s}\Big|_{s=0} + p_i \frac{\partial \dot{Q}^i}{\partial s}\Big|_{s=0}$$

$$= \frac{d}{dt}\left[p_i \frac{\partial Q^i}{\partial s}\Big|_{s=0} \right] \quad \Big(\frac{\partial^2}{\partial t \partial s} = \frac{\partial^2}{\partial s \partial t} \Big)$$

* Intuitive picture (Feynman)



$A \to B$: extremum.

$$S(A \to A' \to B' \to B) - S(A \to B) \simeq 0 \qquad \mathcal{O}(\epsilon^2)$$

$$S(AA'B'B) = S(AA') + S(B'B') + S(B'B)$$

$$\| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$$

$S(AB)$ by symm.

$$\Rightarrow \quad S(AA') = S(BB')$$

⊙ Examples

1) Translation symm.

$$L = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} - \sum_{i<j} V(\vec{r}_i - \vec{r}_j)$$

$$\vec{r}_i(t) \to \vec{r}_i(t) + s\vec{n}$$

($\vec{n}$ = arb. const. vector)

$$F = \sum_i \vec{p}_i \cdot \frac{\partial \vec{R}_i}{\partial s} = \left(\sum_i \vec{p}_i\right) \cdot \vec{n}$$

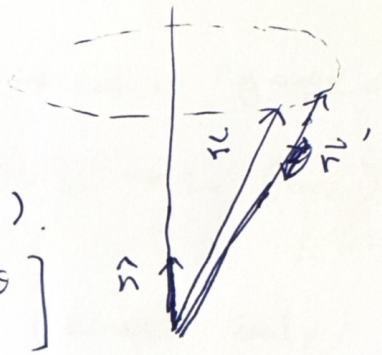
$\vec{n}$ arb. $\Rightarrow$ $\vec{P} = \sum_i \vec{p}_i$ is conserved.

2) Rotation symm.

$$\vec{R}(s,t) = \vec{r}(t) + \delta\vec{r}(s,t)$$

$$\delta\vec{r}(s,t) \simeq s\,\hat{n} \times \vec{r}(t) \quad (s \ll 1).$$

$$\left[\begin{array}{l} \sin\theta \simeq \theta \\ \cos\theta \simeq 1 \end{array}\right]$$



$$F = \sum_i \vec{p}_i \cdot (\hat{n} \times \vec{r}_i)$$

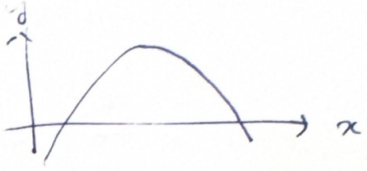
$$= \hat{n} \cdot \left(\sum_i \vec{r}_i \times \vec{p}_i\right) = \hat{n} \cdot \vec{L}$$

- 3d isotropy $\Rightarrow$ $\vec{L}$ conserved
- 2d " $\Rightarrow$ $\hat{n} \cdot \vec{L}$ "

"Gauging" a symm.

⊙ Idea: eliminate coord. for a "symm orbit"

Ex 1) Translation.



- $L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - V(y)$

$$\begin{cases} x(t) \to x(t) + s \quad (s \in \mathbb{R}) & \text{is a symm of } L. \\ x(t) \to x(t) + f(t) & \text{is } \underline{\text{not}} \text{ a symm of } L. \end{cases}$$

- "Gauged" action.

$$L' = \frac{1}{2} m \left((\dot{x} - A)^2 + \dot{y}^2 \right) - V(y)$$

A new (?) time-dep symm. → Call it "gauge symm"

$$\begin{cases} x(t) \to x(t) + f(t) \\ A(t) \to A(t) + \dot{f}(t) \end{cases} \Rightarrow L' \to L' \text{ (inv)}.$$

$$\begin{cases} x\text{-EOM}: & \frac{d}{dt}(\dot{x} - A) = 0 \quad \text{(kinetic only)} \\ A\text{-EOM}: & \dot{x} - A = 0 \quad \text{(potential only)} \end{cases}$$

- $x(t)$ can be anything: A "compensates" it.
- The x/A sector does not affect the dynamics of y
- Effectively, we have

$$L_{\text{eff}} = \frac{1}{2} m \dot{y}^2 - V(y).$$

Ex 2) Rotation

- $L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - V(r)$, $r = \sqrt{x^2+y^2}$.

* $\begin{cases} x \to x + s y \\ y \to y + s x \end{cases}$ $(|s| \ll 1)$ is a symm of L.

× $\begin{cases} \delta x = -f(t) y \\ \delta y = +f(t) x \end{cases}$ $(|f| \ll 1)$ is <u>not</u> a symm. of L.

- "Gauged" action.

$$L' = \frac{1}{2} m \left[(\dot{x} + A y)^2 + (\dot{y} - A x)^2 \right] - V(r)$$

$$= L - A m (x\dot{y} - y\dot{x}) + \frac{1}{2} m A^2 r^2 .$$

(looks like a Lagrange multiplier?)

"Gauge symm"

$$\begin{cases} \delta x = -f(t) y \\ \delta y = +f(t) x \\ \delta A = \dot{f}(t) \end{cases} \Rightarrow \delta L' = 0 + O(f^2)$$

(Exercise).

- Polar coord

$$(\dot{x} + A y)^2 + (\dot{y} - A x)^2 \qquad \begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}$$

$$= \left(\dot{r} c_\phi - r(\dot{\phi} - A) s_\phi \right)^2 + \left(\dot{r} s_\phi + r(\dot{\phi} - A) c_\phi \right)^2$$

$$= \dot{r}^2 + r^2 (\dot{\phi} - A)^2$$

Gauge symm: $\begin{cases} \phi \to \phi + f(t) \\ A \to A + \dot{f}(t) \end{cases}$ (no restriction on $|f|$)

Ex 2) Rotation (cont'd.)

$$\begin{cases} r\text{-EOM}: & \ddot{r} - r(\dot{\phi}-A)^2 + \frac{\partial V}{\partial r} = 0 \\ \phi\text{-EOM}: & \frac{d}{dt}\left[r^2(\dot{\phi}-A)\right] = 0 \\ A\text{-EOM}: & r^2(\dot{\phi}-A) = 0 \quad \text{——— (*)} \end{cases}$$

Upon imposing (*)

- ϕ-EOM becomes trivial (cf. x-EOM of Ex 1)
- r-EOM becomes $\ddot{r} + \frac{\partial V}{\partial r} = 0$.

$$L_{\text{eff}}(r, \dot{r}) = \frac{1}{2} m \dot{r}^2 - V(r)$$

as if ϕ had never been there.

(*) Questions for the next lecture

- What do we mean by "A-EOM" ?
- If A is regarded as a dynamical variable,

$$p_A = \frac{\partial L}{\partial \dot{A}} = 0 \quad \text{in both examples}$$

(It is true in general)

Can $p_A = 0$ cause any problem ?