Hamiltonian mechanics basics

⊙ From Lagrangian to Hamiltonian.

- $L(q^i, \dot{q}^i; t)$
- $p_i := \frac{\partial L}{\partial \dot{q}^i}$
- $H := p_i \dot{q}^i - L \quad \rightarrow \quad H(q^i, p_i)$

$$dH = dp_i\, \dot{q}^i + p_i\, d\dot{q}^i - \frac{\partial L}{\partial q^i} dq^i - \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i$$

$$\Rightarrow \begin{cases} \dot{q}^i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = + \frac{\partial L}{\partial q^i} = - \frac{\partial H}{\partial q^i} \end{cases} \quad \text{(Hamiltonian EOM)}$$

Lagrangian EOM.

⊙ 1st order action

$$S = \int L\, dt = \int [p_i \dot{q}^i - H]\, dt = \int (p_i\, dq_i - H\, dt)$$

$$\delta S = \int \left(\delta p_i\, \dot{q}^i + p_i\, \delta \dot{q}^i - \frac{\partial H}{\partial q^i} \delta q^i - \frac{\partial H}{\partial p_i} \delta p_i \right) dt$$

(integration by parts)

$$= \int \left[\delta p_i \left[\dot{q}^i - \frac{\partial H}{\partial p_i} \right] - \delta q^i \left[\dot{p}_i + \frac{\partial H}{\partial q^i} \right] \right] dt$$

$$+ (\text{total derivative})$$

$\therefore$ We recover the Hamiltonian EOM.

(Be careful with the "boundary condition")

ⓐ Poisson bracket

- Def $\{f, g\}_{P.B.} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i}$.

- $\{q^i, p_j\} = \delta^i_j$

[Digression: Dirac's quantization

$\{x^i, p_j\} \longrightarrow [\hat{x}^i, \hat{p}_j] \times \frac{1}{i\hbar}$]

- Properties

1) Anti-symm: $\{f, g\} = -\{g, f\}$

2) Leibniz rule

$\{fg, h\} = f\{g, h\} + \{f, h\}g$.

3) Jacobi identity

$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$.

★ Hamiltonian EOM.

$\boxed{\frac{df}{dt} = \{f, H\}}$ for any $f(q^i, p_i)$.

$$\dot{f} = \dot{q}^i \frac{\partial f}{\partial q^i} + \dot{p}_i \frac{\partial f}{\partial p_i}$$

$$= \frac{\partial f}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q^i}\frac{\partial f}{\partial p_i}$$

$$= \{f, H\} \qquad (\checkmark).$$

Symmetry in Hamiltonian mech.

. Ex) Translation: $\vec{x}(t) \to \vec{x}(t) + s\hat{n}$

Rotation : $\delta\vec{r}(t) \approx s\,\hat{n} \times \vec{r}(t)$ $(s \ll 1)$

· **Claim**

The conserved quantity F from Noether's thm generate the "symm orbit" via

$$\frac{dq^i}{ds} = \{q^i, F\}$$

· **Check**

a) In general, for time-indep. symm.

$$F = p_i \frac{\partial Q^i}{\partial s}\bigg|_{s=0}$$

$$\left\{q^i, p_j \frac{\partial Q^j}{\partial s}\right\} = \{q^i, p_j\}\frac{\partial Q^j}{\partial s} + p_j \underbrace{\left\{q^i, \frac{\partial Q^j}{\partial s}\right\}}_{=0}$$

$$= \frac{\partial Q^i}{\partial s}$$

b) In the two examples,

. Translation: $F = \vec{n} \cdot \vec{p}$

$$\frac{dx^i}{ds} = \{x^i, \vec{n}\cdot\vec{p}\} = n^i \qquad (\checkmark)$$

. Rotation: $F = \hat{n} \cdot (\vec{x} \times \vec{p})$

$$\frac{d\vec{r}}{ds} = \{\vec{r}, \hat{n}\cdot(\vec{r}\times\vec{p})\}$$

$$= \{\vec{r}, \vec{p}\cdot(\hat{n}\times\vec{r})\} = \hat{n}\times\vec{r} \qquad (\checkmark)$$

⊙ Symm. in H (cont'd)

* Hamiltonian treats q & p on an equal footing.

$$\frac{dq^i}{ds} = \{ q^i, F \}$$

can be generalized to

$$\frac{df}{ds} = \{ f, F \} \quad \text{for any } f(q^i, p_i)$$



. Translation

$$\frac{d\vec{P}}{ds} = \{ \vec{p}, \hat{n} \cdot \vec{p} \} = 0$$

. Rotation

$$\frac{d\vec{p}}{ds} = \{ \vec{p}, \hat{n} \cdot (\vec{r} \times \vec{p}) \} = \hat{n} \times \vec{p}$$

⊙ Conservation law revisited.

. $0 = \frac{dF}{dt} = \{ F, H \}$ "F should Poisson-commute with H"

. If F_1 & F_2 are both conserved,

so does $\{ F_1, F_2 \}$

$$\frac{d}{dt} \{ F_1, F_2 \} = \{ \{ F_1, F_2 \}, H \}$$ (Jacobi & Anti-symm)

$$= - \{ \{ F_2, H \}, F_1 \} + \{ \{ F_1, H \}, F_2 \}$$

$$= 0$$

Ex) $\begin{cases} L_z = x p_y - y p_x \\ L_x = y p_z - z p_y \\ L_y = z p_x - x p_z \end{cases}$ $\{ L_x, L_y \} = L_z$, etc.

Hamiltonian with constraints

⊙ Recall $L' = L + \lambda_\alpha f^\alpha$

[All within variational principle]

$$\begin{cases} \dfrac{\partial L'}{\partial \lambda_\alpha} = f^\alpha = 0 \\ \dot{p}_A - \dfrac{\partial L}{\partial x^A} = \lambda_\alpha \dfrac{\partial f^\alpha}{\partial x^A} \end{cases}$$

⊙ A first attempt for H.

$$S' = \int \left[p_A \dot{x}^A - H(x,p) + \lambda_\alpha f^\alpha(x) \right] dt .$$

* Try to treat p & x equally (if possible at all).

1) $f^\alpha(x)$ is a ftn of x only.
Generalize it to $f^\alpha(x,p)$?!

2) When it is easy enough to solve the constr. it would be nice to end up with

$$\tilde{S} = \int \left[p_i \dot{q}^i - H(q,p) \right] dt .$$

$$\begin{cases} 3N \to n & \text{(Lag)} \\ 6N \to 2n & \text{(Ham)} \end{cases}$$

Constraints should come in pairs ?

· Ex) Pendulum

$$L^{(1)} = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + mgy + \frac{1}{2}\lambda(x^2+y^2-\ell^2)$$

$$\begin{cases} x = r\sin\theta \\ y = r\cos\theta \end{cases}$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta + \frac{1}{2}\lambda(r^2-\ell^2)$$

$$\begin{cases} p_r = m\dot{r} \\ p_\theta = mr^2\dot{\theta} \end{cases}$$

$$L^{(2)} = p_r\dot{r} + p_\theta\dot{\theta} - \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + mgr\cos\theta + \frac{1}{2}\lambda(r^2-\ell^2)$$

· $\frac{\partial L}{\partial \lambda} = 0 \Rightarrow r = \ell$

$$\begin{cases} \dot{p}_r = \frac{p_\theta^2}{mr^3} + mg\cos\theta + \lambda r \\ \dot{p}_\theta = -mgr\sin\theta \end{cases}$$

* Lagrange multiplier λ "acts twice"

① $\frac{\partial L}{\partial \lambda} = 0 \Rightarrow f(x) = 0$ (kill q)

② Adjust the value of λ (kill p)

[In the example above,

$\begin{cases} p_r = m\dot{r} \\ r = 0 \end{cases}$ require that $\dot{p}_r = 0$]

(?) r and pr are <u>not</u> treated equally.

"Gauging" a symm. in H.

Ex 1) Translation

$$L = \frac{1}{2} m \left[(\dot{x} - A)^2 + \dot{y}^2 \right] - V(y)$$

$$\begin{cases} p_x = m(\dot{x} - A) \\ p_y = m\dot{y} \\ p_A = 0 \end{cases} \quad \text{(strange. But, let's proceed anyway)}$$

$$H = p_x \dot{x} + p_y \dot{y} + p_A \dot{A} - L$$

$$= p_x \left(\frac{p_x}{m} + A \right) + \cdots$$

$$= \frac{1}{2m} (p_x^2 + p_y^2) + V(y) + A p_x$$

looks more like Lagrange mult enforcing $p_x = 0$ (?!)

Ex 2) Rotation

$$L = \frac{1}{2} m \left[(\dot{x} + Ay)^2 + (\dot{y} - Ax)^2 \right] - V(r)$$

$$\begin{cases} p_x = m(\dot{x} + Ay) \rightarrow \dot{x} = \frac{p_x}{m} - Ay \\ p_y = m(\dot{y} - Ax) \rightarrow \dot{y} = \frac{p_y}{m} + Ax \end{cases}$$

$$H = p_x \dot{x} + p_y \dot{y} + p_A \dot{A} - L$$

$$= \frac{1}{2m} (p_x^2 + p_y^2) + V(r) + A(x p_y - y p_x)$$

lag. mult. enforcing

$$L_z = x p_y - y p_x = 0$$

(?!)

⊙ Gauging multiple symm (abelian)

- M symm generators, mutually commuting.

$\phi_{A=1,\cdots,M}$: $\{\phi_A, H^{(0)}\} = 0$

$\{\phi_A, \phi_B\} = 0$

(Generalization: $\{\phi_A, \phi_B\} = f_{AB}{}^C \phi_C$)

- M "gauge-fixing" fths

$\chi^{A=1,\cdots,M}$: $C^A{}_B : \{\chi^A, \phi_B\}$

$\det(C) \neq 0$

[Roughly speaking
ϕ_A are momenta
χ^A are positions]

- Gauged Hamiltonian. ~ (What we used to call "A")

$H^{(1)} = H^{(0)} + \kappa^A \phi_A + \bar{\kappa}_A \chi^A$

- 1st order action

$S = \int [\, p\cdot\dot{q}^i - H^{(0)} - \kappa^A \phi_A - \bar{\kappa}_A \chi^A \,]$

- EoM

$\delta_{\kappa,\bar{\kappa}} S = 0 \Rightarrow \phi_A = 0 = \chi^A$

$\delta_{p,q} S = 0 \Rightarrow \frac{df}{dt} = \{f, H^{(0)}\} + \kappa^A \{f, \phi_A\} + \bar{\kappa}_A \{f, \chi^A\}$

★ Consistency.

- $0 = \frac{d\phi_A}{dt} = \{\phi_A, H^{(0)}\} + \kappa^B \{\phi_A, \phi_B\} + \bar{\kappa}_B \{\phi_A, \chi^B\}$ ($\{\phi_A, H^{(0)}\} = 0$, $\{\phi_A, \phi_B\} = 0$, $\{\phi_A, \chi^B\} = C^B{}_A$)

$\Rightarrow \bar{\kappa}_B = 0$

- $0 = \frac{d\chi^A}{dt} = \{\chi^A, H^{(0)}\} + \kappa^B \{\chi^A, \phi_B\} + 0$.

$\Rightarrow \kappa^B = -(C^{-1})^B{}_A \{\chi^A, H^{(0)}\}$.

※

Ex 1) Translation

- $\phi = p_x$ $\Rightarrow$ $C = \{\chi, \phi\} = 1$.
- $\chi = x - x_0$ $(x_0 \in \mathbb{R})$

[We could consider $\chi = x - c(t)$
where $c(t)$ is an arb. smooth fcn.
But, we need to generalize the theory
to include "explicit time dep": $\frac{\partial f}{\partial t} \neq 0$]

- $\kappa = -C\{\chi, H^{(0)}\} = -\frac{p_x}{m} \approx 0$.

Ex 2) Rotation

- $\phi = x p_y - y p_x$
- $\chi = \theta = \arctan\left(\frac{y}{x}\right)$

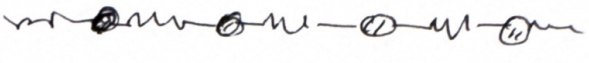
$C = \{\chi, \phi\} = ?$ (Exercise).

★ Appendix : ganging in field theory

(classical, non-relativistic)

· Particles to field

$q^i(t) \longrightarrow \phi(\vec{x}, t)$



· Simplest Lagrangians — I. scalar.

$$\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2} k_1 \phi^2 - \frac{1}{2} k_2 (\nabla \phi)^2$$

when $k_2 = 0$, $\phi(\vec{x}, t)$ & $\phi(\vec{y}, t)$ at $\vec{x} \neq \vec{y}$ never interact. → (boring!)

when $k_1 = 0$, we have field translation symm.

$$\phi \to \phi + s$$

We can imagine ganging it, but let's not do it.

· Simplest Lagranians — II. Vector.

$$\mathcal{L} = \frac{1}{2}(\partial_t \vec{V})^2 - \frac{1}{2} k_1 \vec{V}^2 - \frac{1}{2} k_2 (\nabla \cdot \vec{V})^2$$

Again, when $k_1 = 0$, ∃ field trans. symm.

$\vec{V} \to \vec{V} + s\vec{n}$ —— ①

One more thing! $\vec{V} \to \vec{V} + \nabla \times \vec{W}$ $\left(\frac{\partial \vec{W}}{\partial t} = 0\right)$ — ②

① & ②

· Another vector model ($k_1 = 0$)

$$\mathcal{L} = \frac{1}{2}(\partial_t \vec{A})^2 - \frac{1}{2}(\nabla \times \vec{A})^2$$

space dep. field trans. symm. $\vec{A} \to \vec{A} + \nabla \lambda$ $\left(\frac{\partial \lambda}{\partial t} = 0\right)$.

Let's gauge it! $\lambda(\vec{x}) \to \lambda(\vec{x}, t)$ (arbitrary).

$$\mathcal{L}' = \frac{1}{2}(\partial_t \vec{A} - \nabla A_0)^2 - \frac{1}{2}(\nabla \times \vec{A})^2 \text{ will do.}$$

$$\begin{cases} \vec{A} \to \vec{A} + \nabla \lambda \\ A_0 \to A_0 + \partial_t \lambda \end{cases}$$

… it looks familiar!