Point particle in special relativity

⊙ Minkowski spacetime ($c = 1$)

- $ds^2 = -dt^2 + d\vec{x}^2$ $\qquad x^\mu = (t, \vec{x}) = (x^0, x^1, x^2, x^3)$.

  $= \eta_{\mu\nu} dx^\mu dx^\nu$ $\qquad \eta_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$

- ds^2 is invariant under Lorentz transf.

$$x^\mu \to (x')^\mu = L^\mu{}_\nu x^\nu$$

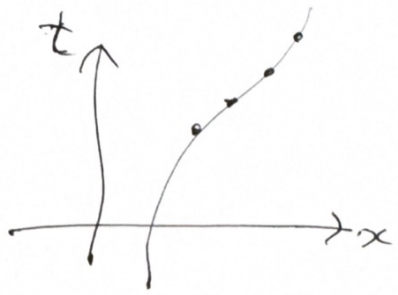
$$\eta_{\mu\nu} L^\mu{}_\rho L^\nu{}_\sigma = \eta_{\rho\sigma} \qquad L \in SO(1,3).$$

$$(L^T \cdot \eta \cdot L = \eta)$$

- Proper time

$$d\tau = \sqrt{dt^2 - d\vec{x}^2}$$

$$= \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}$$



  - Time measured by the moving body
  - Lorentz invariant.

- 4-velocity

$$u^\mu = \frac{dx^\mu}{d\tau} : \qquad u \cdot u = \eta_{\mu\nu} u^\mu u^\nu = -1 .$$

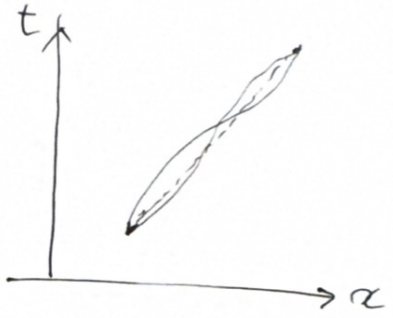
- 4-momentum

$$p^\mu = m u^\mu : \qquad p \cdot p = -m^2 \quad \text{or} \quad p^2 + m^2 = 0$$

(mass-shell condition)

(p^0 = rest mass energy + kinetic energy)

⊙ Lagrangian.



· Free EOM → straight line

· what is the action ?

Ans) $S = -m\int d\tau$

$= -m\int \sqrt{dt^2 - d\vec{x}^2}$

$= -m\int \sqrt{1-\vec{v}^2}\, dt$

$= -m\int \left[1 - \frac{1}{2}\vec{v}^2 - \frac{1}{8}(\vec{v}^2)^2 + \cdots\right]$

const. ↑ Newtonian ↑ correction.

· Variation.

$S = -m\int \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\, ds$ ← Auxiliary time-like variable.

$\left(\dot{x}^\mu := \frac{dx^\mu}{ds}\right)$

$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = m\eta_{\mu\nu}\frac{\dot{x}^\nu}{\sqrt{\cdots}} = m\eta_{\mu\nu}\frac{dx^\nu}{d\tau}$

EOM: $\frac{d}{ds}p_\mu = 0 \Rightarrow \frac{dp_\mu}{d\tau} = 0$ (Lorentz cov.)

· Reparametrization.

$s \to s' = f(s)$

does not change physics as long as $f'(s) > 0$

$\left(\frac{ds}{d\tau} > 0\right)$

⊙ Hamiltonian — I. Non-cov.

· $S = -m \int \sqrt{1-\vec{v}^2}\, dt = \int L^{n.c.}\, dt$

· $\vec{p}^{\,n.c.} = \frac{\partial L^{n.c.}}{\partial \vec{v}} = m \frac{\vec{v}}{\sqrt{1-\vec{v}^2}} \left(= m \frac{d\vec{x}}{d\tau} \right)$

· Hamiltonian

$$H^{n.c.} = \vec{p} \cdot \vec{v} - L^{n.c.}$$

$$= m \frac{\vec{v}^2}{\sqrt{1-\vec{v}^2}} + m\sqrt{1-\vec{v}^2}$$

$$= \frac{m}{\sqrt{1-\vec{v}^2}} = p^0 \quad (!?)$$

⊙ Hamiltonian — II. Cov.

· $S = -m \int \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}\, ds = \int L^{cov}\, ds \qquad \left(\dot{x}^\mu = \frac{dx^\mu}{ds} \right)$

· $p_\mu^{cov} = \frac{\partial L^{cov}}{\partial \dot{x}^\mu} = \eta_{\mu\nu} \left(m \frac{dx^\nu}{d\tau} \right)$

· Hamiltonian

$$H^{cov} = p_\mu \dot{x}^\mu - L^{cov}$$

$$= m \left[-\frac{(d\tau)^2}{d\tau\, ds} + m \frac{d\tau}{ds} \right]$$

$$= 0 \qquad (?!)$$

⊙ Lagrangian with auxiliary field.

· $S_0 = -m \int \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}\, ds$

· $S_1 = \int \left[\frac{1}{2\kappa} \dot{x}^2 - \frac{1}{2} \kappa m^2 \right] ds \qquad (\kappa > 0)$.

$$\delta_\kappa S_1 = 0 \Rightarrow -\frac{1}{\kappa^2} \dot{x}^2 - m^2 = 0$$

$$\Rightarrow \kappa^2 = \frac{-\dot{x}^2}{m^2} \Rightarrow \kappa = \frac{\sqrt{-\dot{x}^2}}{m} \equiv \kappa_*$$

$$S_1 \big|_{\kappa \to \kappa_*} = S_0 \quad (!)$$

· x-EoM from S_1

$$p_\mu = \eta_{\mu\nu} \frac{1}{\kappa} \dot{x}^\nu = m \cdot \frac{ds}{d\tau} \frac{dx^\nu}{ds} = \eta_{\mu\nu} \left(m \frac{dx^\nu}{d\tau} \right)$$

· Reparametrization invariance.

$s \to s' = f(s)$.

Redefine κ s.t. $\kappa(s)\, ds = \kappa'(s')\, ds'$.

⊙ Hamiltonian with auxiliary field.

· $S_1 = \int \left[\frac{1}{2\kappa} \dot{x}^2 - \frac{1}{2} \kappa m^2 \right] ds$

· $S_2 = \int \left[p_\mu \dot{x}^\mu - \frac{1}{2} \kappa (p^2 + m^2) \right] ds$.

$\delta_p S_2 = 0 \Rightarrow \dot{x}^\mu = \kappa p^\mu$ ——————— (*)

$\Rightarrow S_2 \big|_{p^\mu \to \dot{x}^\mu/\kappa} = S_1$ (!)

· But, S_2 looks very much like the 1st order action of a constrained Hamiltonian system

$$\left[S = \int \left[p_i \dot{x}^i - H^{(0)} - \kappa^A \phi_A - \dot{\kappa}_A \chi^A \right] dt \right]$$

with $\begin{cases} H^{(0)} = 0 \\ \phi = \frac{1}{2}(p^2 + m^2) \end{cases}$.

The time-evolution is generated by ϕ.

· A gauge fixing function ?

A nice (Lorentz-inv) choice is

$$\chi = \frac{1}{p^2}(x \cdot p) - \frac{\tau}{m} \Rightarrow \{\chi, \phi\}\big|_{\phi=0} = 1.$$

(*) ↓

$$\frac{d}{ds}\chi = \frac{1}{p^2}(\dot{x} \cdot p) - \frac{1}{m}\frac{d\tau}{ds} = \kappa - \frac{1}{m}\frac{d\tau}{ds} = 0$$

$$\left(p^\mu = m \frac{dx^\mu}{d\tau} \right)$$

as desired

$$\Rightarrow \boxed{\kappa = \frac{1}{m} \cdot \frac{d\tau}{ds}}$$

⊙ Spinning particle in SR.

https://arxiv.org/abs/2102.07063

- Disclaimer

There is no rigid body in SR/GR in a strict sense

Nevertheless, a rigid-body-like model to describe a star (or a black hole) can be a good approximation.

- Recall: Spherical top in Newtonian mechanics

$$S = \int \left[\frac{1}{2} m \dot{\vec{r}}^2 + \frac{1}{2} I \vec{\omega}^2 \right] dt$$

↳ $I_{ab}\omega^a\omega^b$ $(a=1,2,3)$ in general.

- Spherical top in SR

  - Long history
  - Hanson-Regge (1974)
  - Revival of interest due to gravitational wave detection. (2015~)

- Variables

(x^μ, p_μ) $\quad \{x^\mu, p_\nu\} = \delta^\mu{}_\nu$ (same as before)

$(\Lambda^\mu{}_A, S_{\mu\nu})$

$\Lambda^\mu{}_A \in SO(1,3)$ "Euler angle"

$S_{\mu\nu} \in so(1,3)$ "Angular velocity"

$$\{\Lambda^\rho{}_A, S_{\mu\nu}\} = -(\delta^\rho{}_\mu \Lambda_{\nu A} - \delta^\rho{}_\nu \Lambda_{\mu A})$$

$$\{S_{\mu\nu}, S_{\rho\sigma}\} = -(\eta_{\nu\rho} S_{\mu\sigma} - \eta_{\mu\rho} S_{\nu\sigma} - \eta_{\nu\sigma} S_{\mu\rho} + \eta_{\mu\sigma} S_{\nu\rho})$$

★ Constraints needed

. (x, p): $4+4 \to 3+3$

$$\begin{cases} \phi_0 = \frac{1}{2}(p^2 + m^2) \\ x^0 = \frac{1}{p^2}(x \cdot p) - \frac{\tau}{m} \end{cases}$$

. (Λ, S); $6+6 \longrightarrow 3+3$

Qu) $\begin{cases} \phi_a = \\ x^a = \end{cases}$? $(a = 1, 2, 3)$.

[Idea: keep rotation, discard "boost"

Obstacle; boost generators do not form a closed algebra.

$(\{K_i, K_j\} = -\epsilon_{ijk} J_k$.)]

Ans) $\begin{cases} \phi_a = \frac{1}{2}(\hat{p}^\mu + \Lambda^\mu{}_0)\, S_{\mu\nu}\, \Lambda^\nu{}_a \\ x^a = \hat{p}_\mu \Lambda^{\mu a} \end{cases}$ $\left(\hat{p}_\mu := \frac{p_\mu}{\sqrt{-p^2}}\right)$

. $\{\phi_0, \phi_a\} = 0 = \{\phi_a, \phi_b\}$ (Abelian!)

. $\{x^0, \phi_0\} \approx 1$ $\{x^0, \phi_a\} = 0$

$\{x^a, \phi_b\} \approx \delta^a{}_b$ $\{x^a, \phi_0\} = 0$

• Action (1st order)

$$S = \int ds \left[p_\mu \dot{x}^\mu + \frac{1}{2} S_{\mu\nu} \Omega^{\mu\nu} - \frac{\kappa^0}{2}(p^2 + m^2) - \frac{\kappa^a}{2}(\hat{p}^\mu + \lambda^\mu{}_0) S_{\mu\nu} \lambda^\nu{}_a \right]$$

↑ angular velocity

$$\Omega^{\mu\nu} = \lambda^\mu{}_A \frac{d\lambda^\nu{}_A}{ds}$$

Regge trajectory: $m(\tilde{S}^2)$

↑ spin-gauge-inv spin-squared.

$$m(\tilde{S}^2) = m_0 + \frac{1}{2I} \tilde{S}^2 + \cdots$$

↖ rotational inertia.

with $\kappa^0 = \frac{1}{m} \cdot \frac{d\tau}{ds}$ and $\kappa^a = 0$,

we obtain the expected EOM

($p_\mu = \text{const.}$, $S_{\mu\nu} = \text{const}$)

with the correct Newtonian limit.