Nonequilibrium Statistical Physics of Complex Systems

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Scaling relations of excess work when quenching a system across phase transitions

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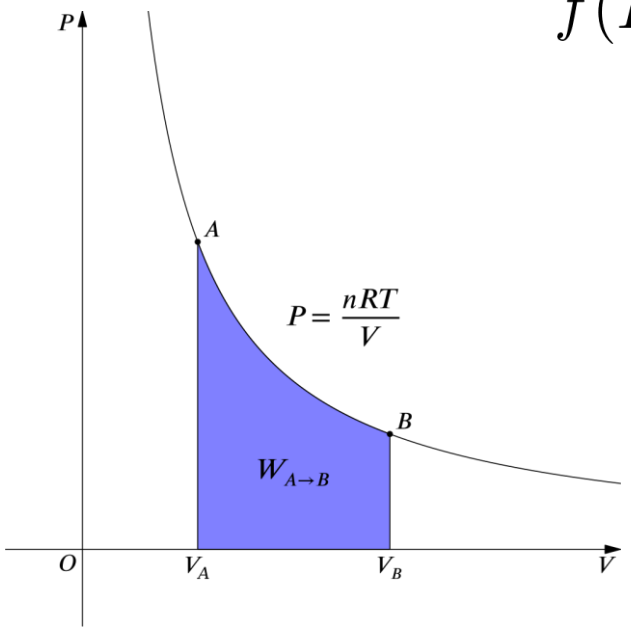
4. Crossovers of scaling behaviors in small systems

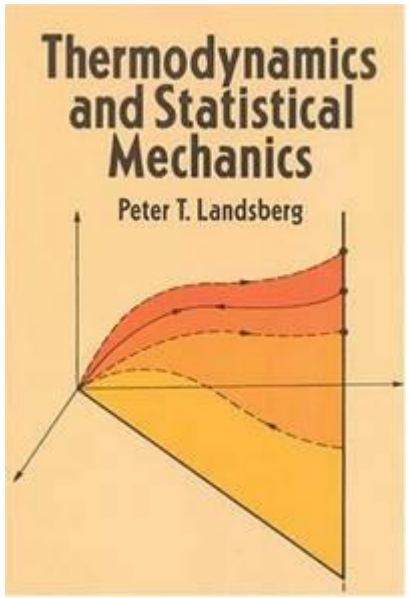
5. Summary

Quasi-static process

seeks to establish relationship the macroscopic variables, which specify initial and final equilibrium states of a system

$$f(P, V, N, ...) = 0$$



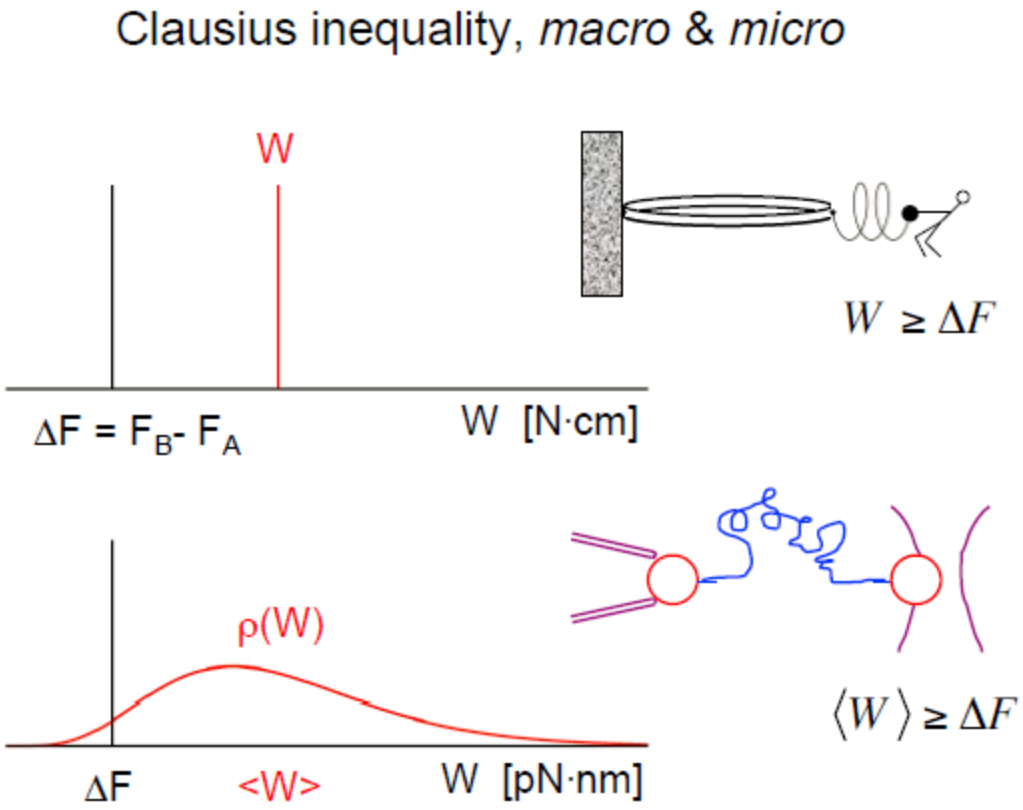


Isothermal process

This quasi-static process takes infinitely long time

Time does not appear in thermodynamics.

Finite-time process



The 2nd law of thermodynamics:
The maximum work principle

$W \geq \Delta F = W_{qs}$: quasi−static work

$W_{ex} = W - \Delta F \geq 0$

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The excess work done in a finite-rate quench with respect to the quasi-static process is non-negative

Linear response theory

The predicted v scaling behaviors for isothermal process are widely used in the theoretical investigations,

$$W_{ex} = W - \Delta F \propto v$$

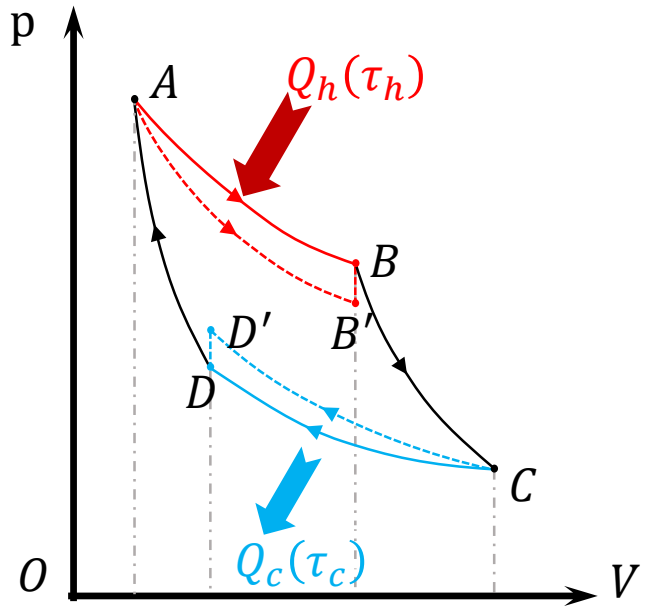
P. Salamon, et.al, Phys. Rev. A 21, 2115 (1980)

T. Schmiedl and U. Seifert, Europhys. Lett. 81, 20003 (2008)

M. Esposito, et.al, Phys. Rev. Lett. 105, 150603 (2010)

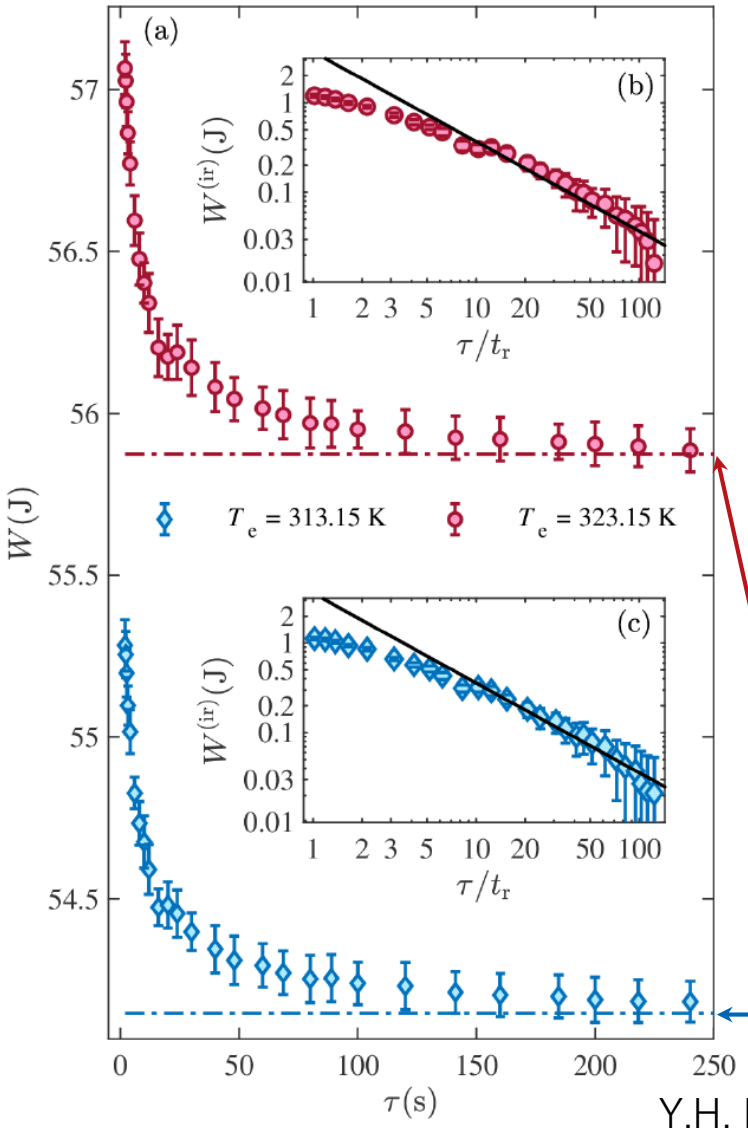
G. E. Crooks, Phys. Rev. Lett. 99, 100602 (2007)

D. A. Sivak and G. E. Crooks, Phys. Rev. Lett. 108, 190602 (2012)

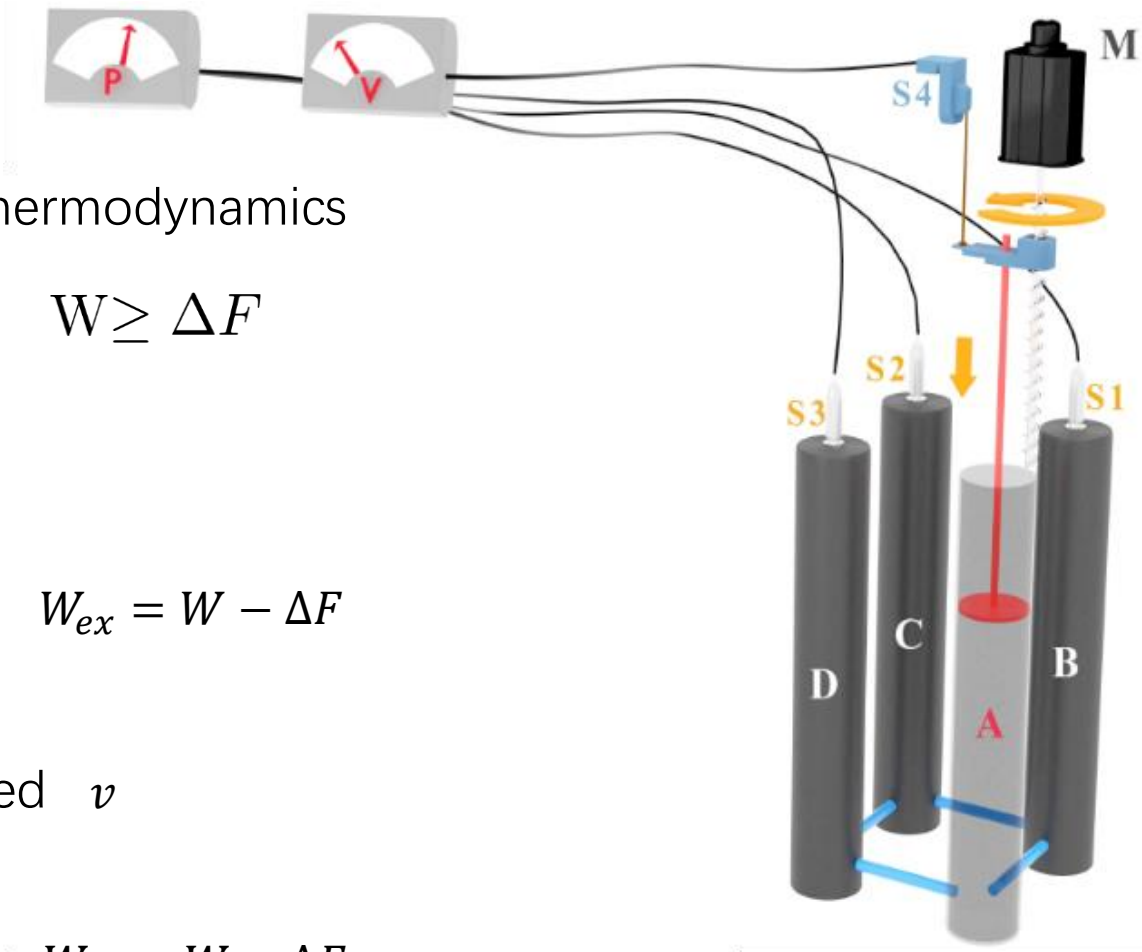


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Experimental verification



Free energy change



Traditional thermodynamics

$$W \geq \Delta F$$

Excess work

$$W_{ex} = W - \Delta F$$

Quench speed v

$$W_{ex} = W - \Delta F \propto v$$

Copyright © Hui Dong

Y.H. Ma, R.X. Zhai, J.F. Chen, C.P. Sun, H. Dong, Phys.Rev.Lett. 125, 210601 (2020)

Adiabatic perturbation theory

C. P. Sun, Journal of Physics A: Mathematical and General 21(7), 1595 (1988).
G. Rigolin, G. Ortiz, and V. H. Ponce, Phys. Rev. A 78(5), 052508 (2008).

- Regular and analytic adiabatic perturbation theory

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}(\lambda(t))|\psi(t)\rangle \qquad \hat{H}(\lambda)|E_m^\lambda\rangle = E_n^\lambda|E_m^\lambda\rangle \;\; [t_0,t_1],\,[\lambda_0,\lambda_1]$$

$$|\psi(t)\rangle = \sum_m e^{-i\Theta_m(t)}\alpha_m(t)|E_m^{\lambda(t)}\rangle$$

Dynamical phase and Berry phase

$$\Theta_m(t) = \int_{t_0}^{t} \mathrm{d}s E_m^{\lambda(s)} - i\int_{t_0}^{t} \mathrm{d}s\langle E_m^{\lambda(s)}|\partial_s|E_m^{\lambda(s)}\rangle$$

$$\alpha_n = -\int_{\lambda_0}^{\lambda_1}\mathrm{d}\lambda\sum_m \alpha_m(\lambda)\langle E_n^\lambda|\partial_\lambda|E_m^\lambda\rangle e^{iv^{-1}\int_{\lambda_0}^{\lambda}\mathrm{d}\lambda'(E_n^{\lambda'}-E_m^{\lambda'})} \qquad \alpha_n(t_0) = \delta_{mn}$$

Transition probability $(m \neq n)$

$$P_{m,n} = v^2\left[\frac{|\langle E_n^{\lambda_0}|\partial_{\lambda_0}|E_m^{\lambda_0}\rangle|^2}{(E_n^{\lambda_0}-E_m^{\lambda_0})^2} + \frac{|\langle E_n^{\lambda_1}|\partial_{\lambda_1}|E_m^{\lambda_1}\rangle|^2}{(E_n^{\lambda_1}-E_m^{\lambda_1})^2} - 2\cos(\Theta_{mn})\frac{\langle E_n^{\lambda_0}|\partial_{\lambda_0}|E_m^{\lambda_0}\rangle}{E_n^{\lambda_0}-E_m^{\lambda_0}}\frac{\langle E_n^{\lambda_1}|\partial_{\lambda_1}|E_m^{\lambda_1}\rangle}{E_n^{\lambda_1}-E_m^{\lambda_1}}\right]$$

Adiabatic condition

$$v\frac{\langle E_n^\lambda|\partial_\lambda|E_m^\lambda\rangle}{E_n^\lambda - E_m^\lambda} \leq 1$$

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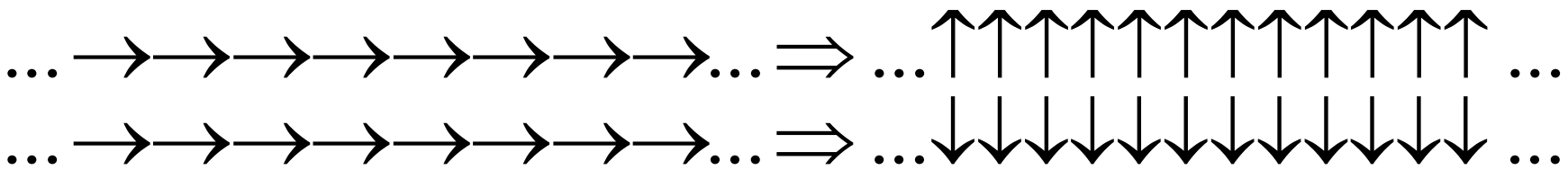
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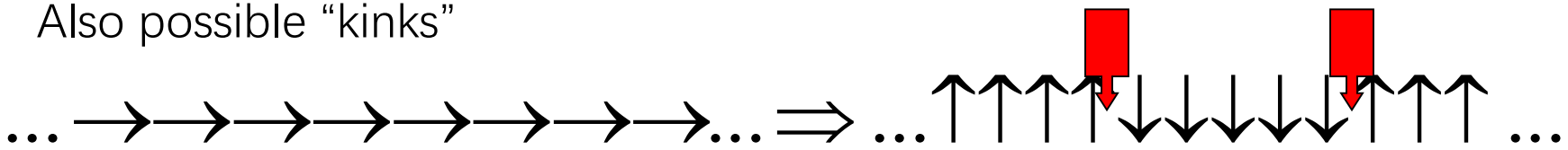
Symmetry breaking and defects. Kibble-Zurek mechanism

$$H = -J(t)\sum_l \sigma_l^x - W\sum_l \sigma_l^z \sigma_{l+1}^z \qquad J(t) = -vt$$

Broken symmetry states after the phase transition:



Also possible “kinks”

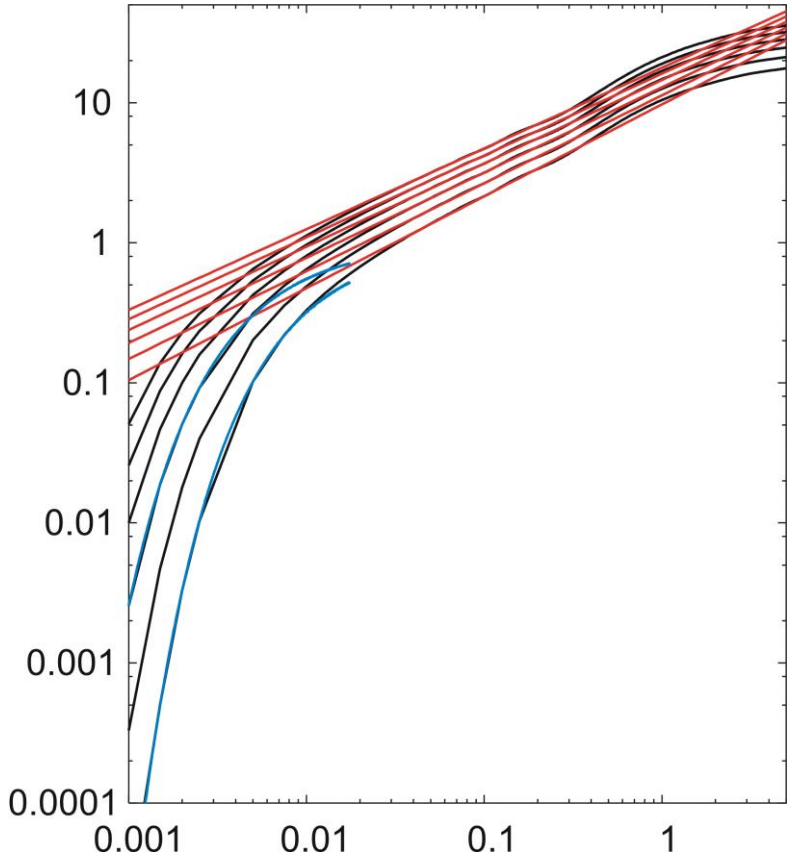


“kinks” density is decribed by Kibble-Zurek mechanism

W.H. Zurek, U. Dorner and P. Zoller, Phys. Rev. Lett. 95, 105701 (2005).

Key point: The density of topological defects (vortex, kinks) depends on the speed of quench

$$\hat{\xi} = \xi(\hat{g}) \sim \nu^{-\nu/(1+z\nu)}$$



Zurek, Dorner, Zoller, Phys. Rev. Lett. 95, 105701 (2005)

Kibble-Zurek Mechanism

Linear quench: $\varepsilon(t) = vt$

$$\tau \sim \Delta^{-1} \sim |\varepsilon|^{-\nu z}, \quad \xi \sim |\varepsilon|^{-\nu}$$

- Freeze-out time:

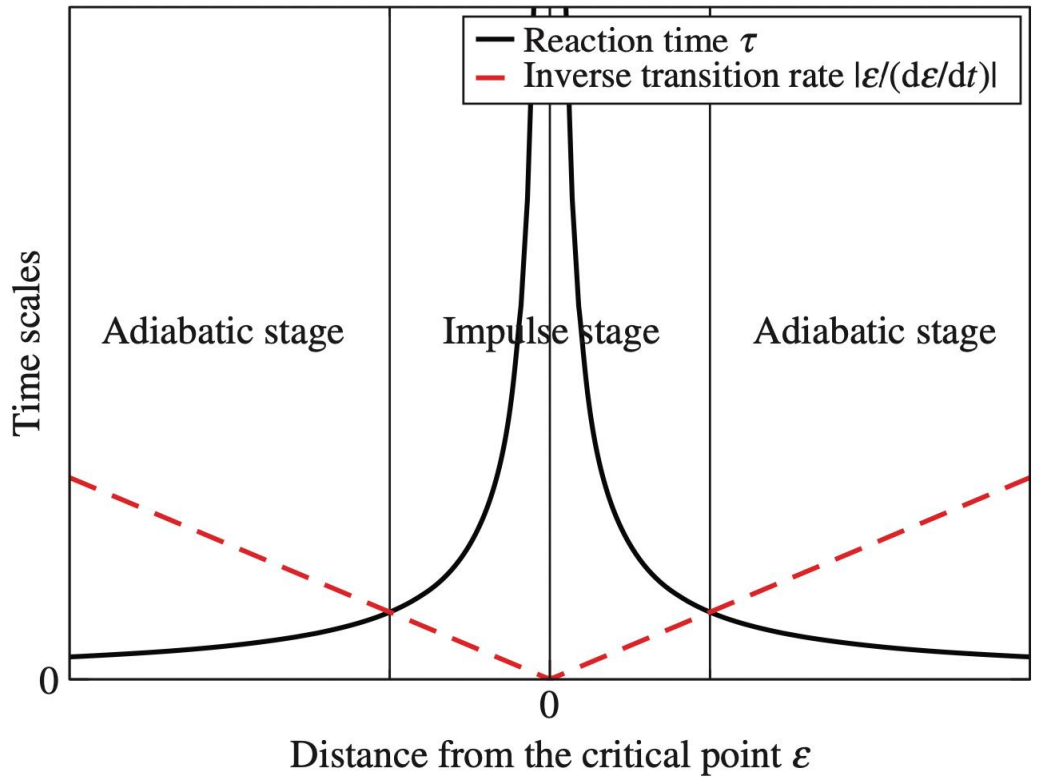
$$t^* \simeq \tau[\varepsilon(t)] \implies t^* \sim v^{-\nu z/(1+\nu z)}$$

- Characteristic length:

$$\xi^* = \xi[\varepsilon(t^*)] \implies \xi^* \sim v^{-\nu/(1+\nu z)}$$

- Average density of defects:

$$\langle n_{ex} \rangle \simeq \xi^{*-d} \sim v^{d\nu/(1+\nu z)}$$



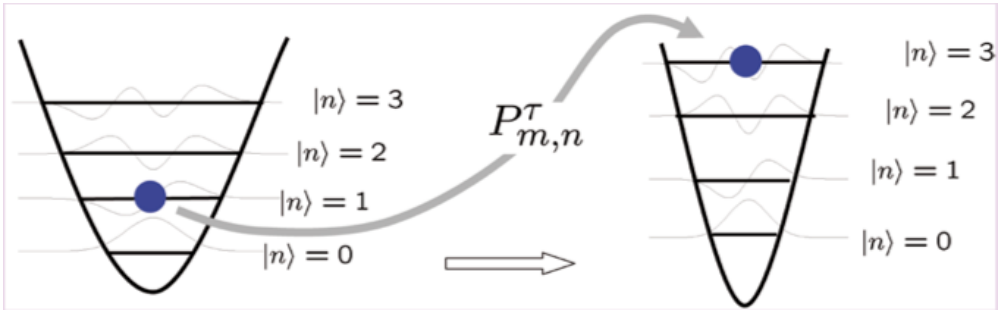
Zurek, Phys. Report, 276, 177 (1996)
Zurek, Dorner, Zoller, Phys. Rev. Lett. 95, 105701 (2005)
J. Dziarmaga, Advances in Physics, **59:6,** 1063-1189 (2010)

Work Statistics

H. Tasaki, arXiv:cond-mat/0009244, (2000)
J. Kurchan, arXiv:cond-mat/0007360 (2000)

- Two-point measurement scheme:

$\hat{H}(t_0)$



$\hat{H}(t_1)$

Work in single realization: $W_{mn} = E_m(t_1) - E_n(t_0)$

$$p^{t_1}_{mn} = \left| \left\langle m(t_1) \left| \hat{U}(t_1, t_0) \right| n(t_0) \right\rangle \right|^2$$
$$p^{t_0}_{n} = \left\langle n(t_0) \left| \hat{\rho}_0 \right| n(t_0) \right\rangle$$

- Probability distribution function (PDF) of work:

$$P(W) = \sum_{m,n} \delta(W - W_{mn}) p^{t_1}_{mn} p^{t_0}_{n}$$

- Characteristic function of work (CFW)

$$Z(\chi) = \int dW e^{i\chi W} P(W)$$
$$= \mathrm{Tr}\left[\hat{U}^{\dagger}(t_1, t_0) e^{i\chi \hat{H}(t_1)} \hat{U}(t_1, t_0) e^{-i\chi \hat{H}(t_0)} \hat{\rho}_0 \right]$$
$$= \mathrm{Tr}\left[e^{i\chi \hat{H}^H(t_1)} e^{-i\chi \hat{H}(t_0)} \hat{\rho}_0 \right]$$

1. The Conservation of energy

$$\langle w \rangle = \mathrm{Tr}\left[\hat{\rho}_0 \hat{H}^H(t_1)\right] - \mathrm{Tr}\left[\hat{\rho}_0 \hat{H}(t_0)\right]$$

2. Jarzynski's equality

$$\left\langle e^{-\beta w} \right\rangle = e^{-\beta \Delta F} \implies \langle w \rangle \geq \Delta F$$

P. Talkner et.al., Phys. Rev. E, **75,** 050102 (2007)

Scaling of Excess Work

- Cumulant CFW

$$\mathcal{G}(\chi) = \frac{1}{N} \ln Z(\chi) = \sum_{n=1}^{\infty} \frac{(i\chi)^n}{n!} \kappa_n$$

- Scaling behavior of average work?

A. Polkovnikov et.al., Rev. Mod. Phys., **83,** 863 (2011)

A. del Campo, Phys. Rev. Lett., **121,** 200601 (2018)

F. J. Go´mez-Ruiz, J. J. Mayo, and A. del Campo, Phys. Rev. Lett., **124,** 240602 (2020)

Adiabatic perturbation theory

- A generic Hamiltonian $\hat{H}\left[\varepsilon(t)\right], t \in [t_0, t_1], \varepsilon \in [\varepsilon_0, \varepsilon_1]$

$$|\psi(t)\rangle = \sum_n a_n(t) |u_n(t)\rangle \implies i\partial_t a_n + i\sum_m a_m(t) \langle u_n| \partial_t |u_m\rangle = E_n a_n(t)$$

- Gauge out the dynamical phase:

$$\partial_t \tilde{a}_n = -\sum_{m\neq n} \tilde{a}_m \langle u_n| \partial_t |u_m\rangle e^{i[\gamma_n(t)-\gamma_m(t)]}$$

$$\tilde{a}_n(t) = e^{-i\gamma_n(t)} a_n(t)$$

$$\gamma_n(t) = \int_{t_0}^{t} dt' E_n(t')$$

$$\tilde{\gamma}_n(\varepsilon) = \int_{\varepsilon_0}^{\varepsilon} d\varepsilon' \frac{E_n(\varepsilon')}{d\varepsilon'/dt}$$

- Formal solution:

$$\tilde{a}_n(t) = -\int_{t_0}^{t_1} dt' \sum_{m\neq n} \tilde{a}_m \langle u_n| \partial_t |u_m\rangle e^{i[\gamma_n(t)-\gamma_m(t)]}$$

$$= -\int_{\varepsilon_0}^{\varepsilon_1} d\varepsilon \sum_{m\neq n} \tilde{a}_m \langle u_n| \partial_\varepsilon |u_m\rangle e^{i[\tilde{\gamma}_n(\varepsilon)-\tilde{\gamma}_m(\varepsilon)]}$$

C. De Grandi and A. Polkovnikov, Quantum Quenching, Annealing and Computation, (pp. 75-114) (2010)

Adiabatic perturbation theory

- Assumptions: $H = -J(t)\sum_l \sigma_l^x - W\sum_l \sigma_l^z \sigma_{l+1}^z \qquad J(t) = -vt$

1. Quasiparticle description (denoted by mode k).
2. Low-energy dispersion at the QCP: $\omega_k = c|k|^z$.
3. Single-excitation approximation.

- For linear quench: $\varepsilon(t) = vt$
- Quasiparticle-excitation probability at mode k

$$p_k = |\tilde{a}_k|^2 \approx \left| \int_{\varepsilon_0}^{\varepsilon_1} d\varepsilon \left\langle 1_k^\varepsilon, 1_{-k}^\varepsilon \right| \partial_\varepsilon \left| 0_k^\varepsilon, 0_{-k}^\varepsilon \right\rangle e^{iv^{-1}\int_{\varepsilon_0}^{\varepsilon} d\varepsilon' 2\omega_k(\varepsilon')} \right|^2$$

$$\langle n_{ex} \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_k p_k = \int \frac{d^d k}{(2\pi)^d} p_k$$

C. De Grandi and A. Polkovnikov, Quantum Quenching, Annealing and Computation, (pp. 75-114) (2010)

Adiabatic perturbation theory

- Near QCP, the general scaling argument yields

$$2\omega_k(\varepsilon) = |\varepsilon|^{z\nu} F\left(\frac{k}{|\varepsilon|^{\nu}}\right) \qquad \left\langle 1_k^{\varepsilon}, 1_{-k}^{\varepsilon}\right| \partial_{\varepsilon} \left|0_k^{\varepsilon}, 0_{-k}^{\varepsilon}\right\rangle = \varepsilon^{-1} G\left(\frac{k}{|\varepsilon|^{\nu}}\right)$$

- Asymptotically, $F(x) \propto x^z$ and $G(x) \propto x^{-1/\nu}$ for $|x| \gg 1$.

- Change the variables

$$\varepsilon = \zeta v^{1/(z\nu+1)} \qquad k = \phi v^{\nu/(z\nu+1)}$$

- Scaling behavior in average density of quasiparticles

$$\langle n_{ex} \rangle = \int \frac{d^d k}{(2\pi)^d} p_k = v^{\frac{dv}{z\nu+1}} \int \frac{d^d \phi}{(2\pi)^d} K(\phi)$$

$$K(\phi) = \left| \int_{\zeta_0}^{\zeta_1} \frac{d\zeta}{\zeta} G\left(\frac{\phi}{|\zeta|^{\nu}}\right) \exp\left[i \int_{\zeta_0}^{\zeta} d\zeta' |\zeta'|^{z\nu} F\left(\frac{\phi}{|\zeta|^{\nu}}\right)\right]\right|^2$$

C. De Grandi and A. Polkovnikov, Quantum Quenching, Annealing and Computation, (pp. 75-114) (2010)

Adiabatic perturbation theory

- Convergence of $K(\phi)$: consider region $\bar{S} = \{\phi | \phi \gg 1\}$

$$F\left(\frac{\phi}{|\zeta|^{\nu}}\right) \propto \phi^{z}|\zeta|^{z\nu} \qquad G\left(\frac{\phi}{|\zeta|^{\nu}}\right) \propto \phi^{-1/\nu}|\zeta|$$

$$\int_{\bar{S}} \frac{d^{d}\phi}{(2\pi)^{d}} K(\phi) \propto \int_{\bar{S}} d^{d} \left| \int_{\zeta_0}^{\zeta_1} d\zeta \frac{\mathrm{sign}(\zeta)}{\phi^{1/\nu}} \exp\left[i \int_{\zeta_0}^{\zeta} d\zeta' \phi^{z} \right] \right|^{2} \sim \int_{\bar{S}} d^{d}\phi \frac{1}{\phi^{2(1/\nu+z)}}$$

- Critical dimension: $d = 2(1 + z\nu)/\nu$
- Scaling behavior in average density of quasiparticles

$$\langle n_{ex} \rangle = v^{\frac{dv}{z\nu+1}} \int \frac{d^{d}\phi}{(2\pi)^{d}} K(\phi) \sim \begin{cases} v^{d\nu/(z\nu+1)} & d\nu/(z\nu+1) < 2 \\ v^{2} \ln v & d\nu/(z\nu+1) = 2 \\ v^{2} & d\nu/(z\nu+1) > 2 \end{cases}$$

C. De Grandi and A. Polkovnikov, Quantum Quenching, Annealing and Computation, (pp. 75-114) (2010)

Scaling of Average Work

- The quasiparticle description and single-excitation approximation implies a specific form of CFW and cumulant CFW:

$$Z(\chi) \sim \prod_k (1 - p_k + p_k e^{i\chi w(k)}),$$

$$\mathcal{G}(\chi) \sim \sum_k \ln(1 - p_k + p_k e^{i\chi w(k)})$$

- Excess work:

$$\kappa_1 = \langle w \rangle \sim \mu + \sum_k p_k w(k)$$

A. del Campo, Phys. Rev. Lett., **121,** 200601 (2018)

F. J. Go´mez-Ruiz, J. J. Mayo, and A. del Campo, Phys. Rev. Lett., **124,** 240602 (2020)

F. Zhang and **H.T. Quan**, Physical Review E 105 (2), 024101 (2022).

Scaling of Average Work

- Scaling behavior of cumulants $\kappa_n \approx \sum_k p_k \omega_k^n$
- ε_1 is far away from the CP

Low energy mode: $\omega_k \sim \text{const}$

$$\kappa_n \sim \sum_k p_k \sim \begin{cases} v^{d\nu/(z\nu+1)} & d\nu/(z\nu+1) < 2 \\ v^2 \ln v & d\nu/(z\nu+1) = 2 \\ v^2 & d\nu/(z\nu+1) > 2 \end{cases}$$

n-independent
same scaling behavior of FCS
of quasiparticle excitation.

Z. Fei, N. Freitas, V. Cavina, **H. T. Quan**, and M. Esposito Phys. Rev. Lett., **124,** 170603 (2020)

Example: 1D transverse Ising chain

- Hamiltonian: $\hat{H}(\varepsilon) = -J\sum_{l=1}^{N}\left[\hat{\sigma}_l^z\hat{\sigma}_{l+1}^z + (\varepsilon-1)\hat{\sigma}_l^x\right]$

  Critical exponent: $d = z = \nu = 1$

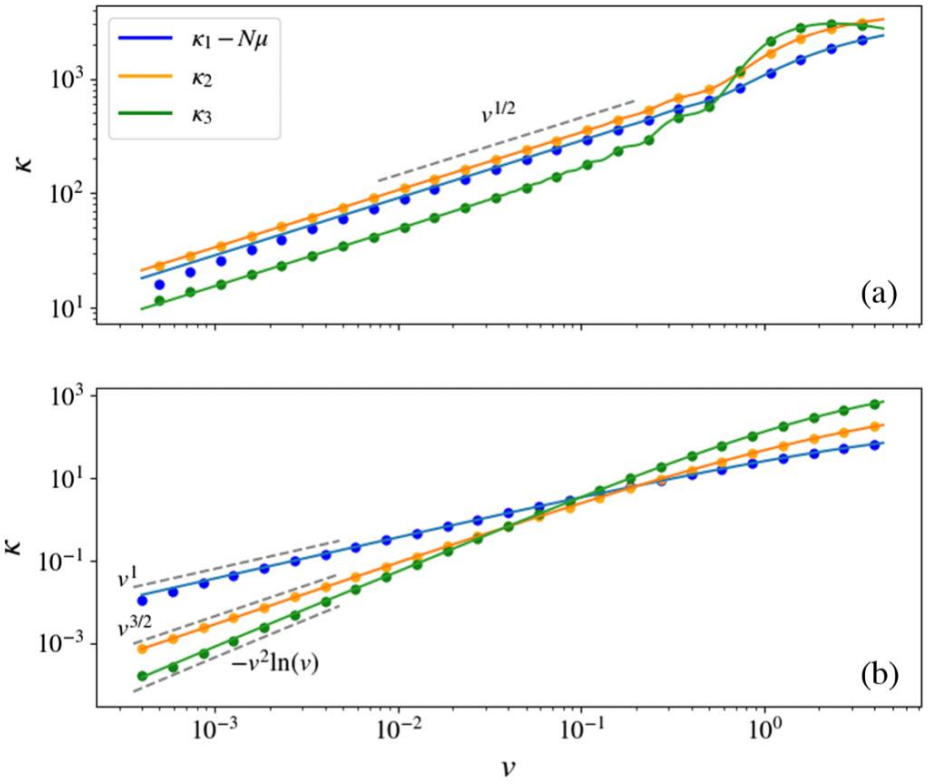
- (a) $\varepsilon_0, \varepsilon_1$ are far away from the CP

$$\kappa_n \sim v^{1/2}$$

$$\kappa_1 = N\left[\mu + v^{1/2}J^{1/2}\varepsilon_1\sqrt{2}/(2\pi)\right]$$

$$\kappa_2 = Nv^{1/2}J^{3/2}\varepsilon_1^2(2\sqrt{2}-2)/\pi$$

$$\kappa_3 = Nv^{1/2}J^{3/2}\varepsilon_1^3 4(6\sqrt{2}-18+4\sqrt{6})/(3\pi)$$



$$\varepsilon_0 = -4, \varepsilon_1 = 1$$

Z. Fei, N. Freitas, V. Cavina, **H. T. Quan**, and M. Esposito Phys. Rev. Lett., **124,** 170603 (2020)

$$N = 4000$$

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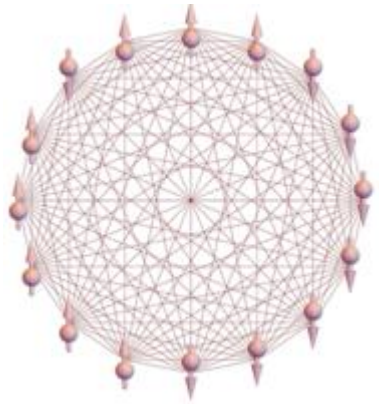
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4. Crossovers of scaling behaviors in small systems

5. Summary

The mean-field model



Curie-Weiss model

mean field theory $\mathcal{H} = -J\sum_{\langle ij\rangle}\sigma_i\sigma_j - H(t)\sum_i \sigma_i \qquad \sigma_i = \pm 1 \qquad i = 1, 2, \ldots N$

system state $M = \sum_i \sigma_i$

- Under thermal fluctuations $\sigma_i: \pm 1 \to \mp 1 \implies M \rightharpoonup M_\pm = M \pm 2$
- The probability distribution of state $P(M,t)$ evolves under

$$\frac{\partial P(M,t)}{\partial t} = \sum_{\eta=\pm}\Big[W_\eta(M_{-\eta}, H(t))P(M_{-\eta},t) - W_\eta(M, H(t))P(M,t)\Big],$$

where $W(M,H)$ stands for the transition rate $M \to M_\pm$ given by

$$W_\pm(M, H(t)) = \frac{N \mp M}{2\tau_0}\exp\left\{\pm\beta\left[\frac{J}{N}(M \pm 1) + H(t)\right]\right\}$$

obey the detailed balance condition

- The equilibrium distribution reads

$$P^{eq}(M,H) = e^{-\beta F_H(M)}/Z_H$$

with the free energy

$$F_H(M) = -JM^2/2N - HM - \beta^{-1}S(M)$$

interaction strength J

inverse temperature β

external field $H(t)$

microscopic relaxation time τ_0

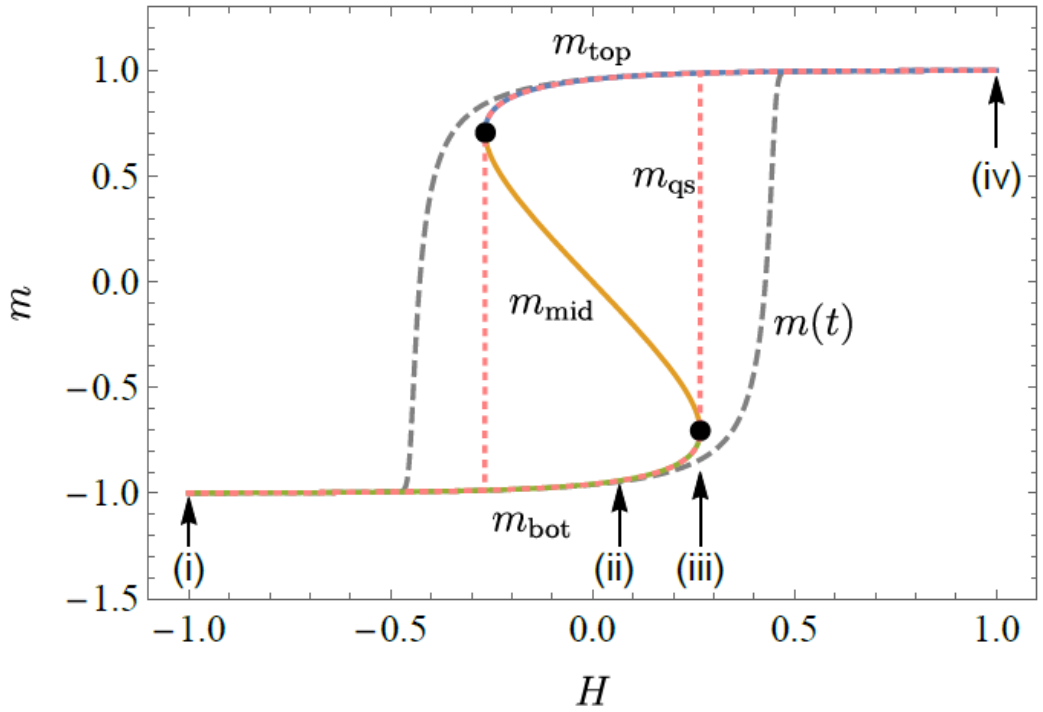
J. Meibohm and M. Esposito, Phys. Rev. Lett. 128, 110603 (2022)

The mean-field model

- Let $m = M/N$. In the limit $N \to \infty$, the equation of motion is

$$\frac{dm(t)}{dt} = f(m,t) = \frac{2}{\tau_0}\sinh[\beta J m + \beta H(t)] - \frac{2m}{\tau_0}\cosh[\beta J m + \beta H(t)].$$

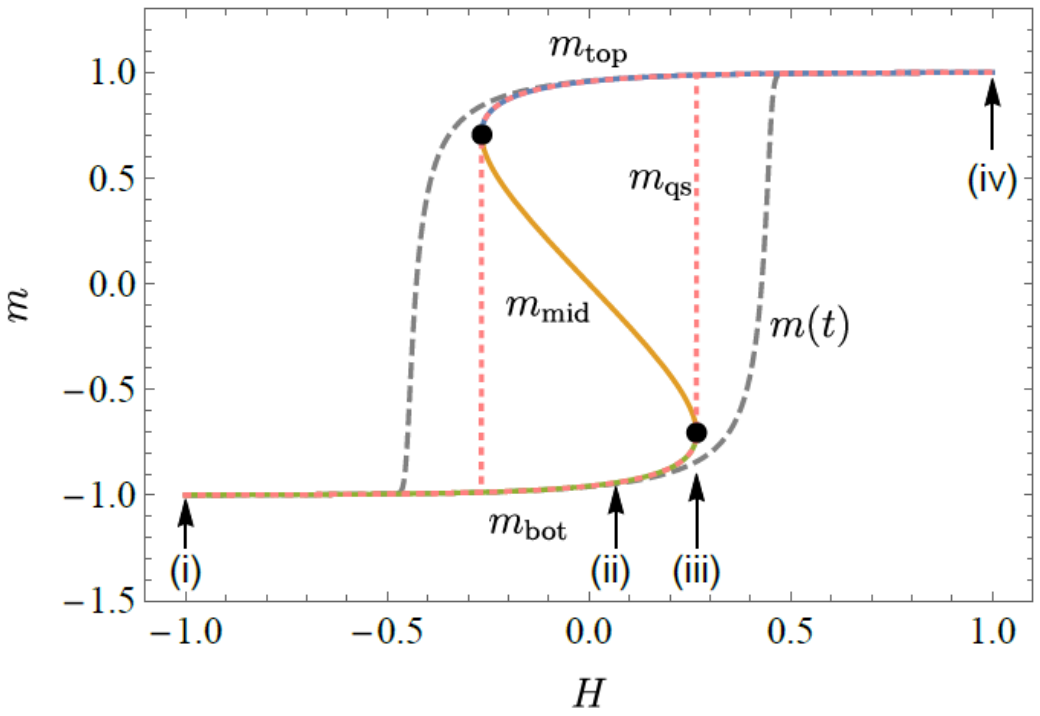
- There can exist one, two or three steady-state solutions depending on the protocol value H.
  - steady state solutions: m_{ss} that let $f(m_{ss}, H) = 0$. Hysteresis implies ergodicity breaking



$\tau_0 = 2, J = 1, \beta = 2$

- For small and large values of H, the system has only one steady state, i.e., the system is monostable.
- For intermediate values of H, there are three steady states: m_{top}, m_{bot}, and m_{mid}, i.e., the system exhibits bistability.
- The two black dots are the turning points $(H_{\pm}, m_{\pm})$, where the metastability appears/disappears.

The mean-field model



Hysteresis

Metastable states

- If H is tuned quasi-statically, the quasi-static state m_{qs} follows the lower branch m_{bot}, even when it is metastable, until the turning point (H_+, m_+). Then the system state switches to the upper branch m_{top}.

- The turning points satisfy

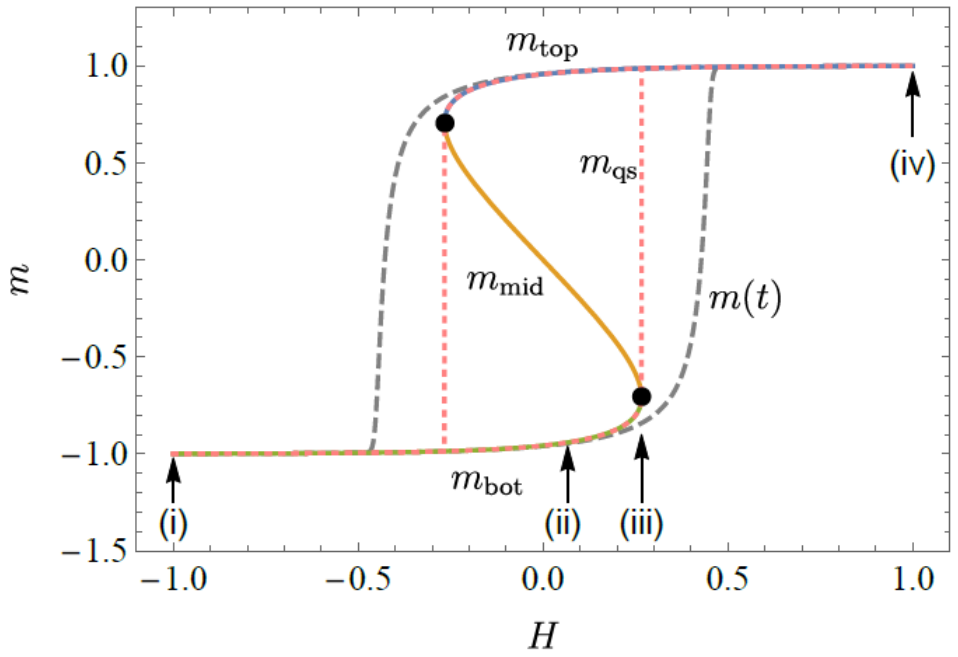
$$\begin{cases} f(m,H)=0, \\ \partial_m f(m,H)=0, \end{cases}$$

which can be solved as $(H_\pm, m_\pm) = (\pm H^*, \pm m^*)$,

where $m^* = -\sqrt{1-\frac{1}{\beta J}}$,

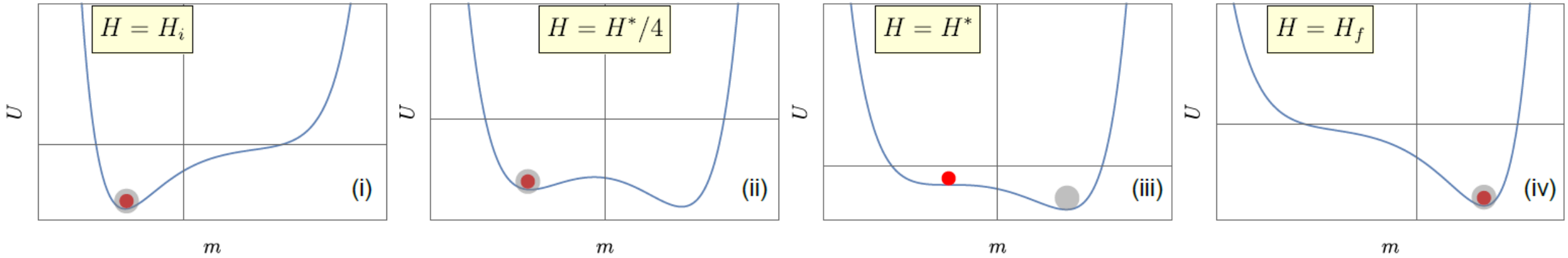
$$H^* = \frac{1}{\beta}\left[\sqrt{\beta J(\beta J-1)} - \operatorname{arctanh}\left(\sqrt{\frac{\beta J-1}{\beta J}}\right)\right].$$

The mean-field model



- When H is tuned at a finite rate, at time 0, the system is prepared in the equilibrium state of H_i, i.e., $m(0) = m_{qs}(H_i)$, but at latter times $m(t) \neq m_{qs}\ (H(t))$. This mismatch becomes evident around the turning point.

- We illustrate the finite-time first-order phase transition with a particle moving in an evolving potential landscape U (where U satisfies $f(m,H) = -\partial_m U/\tau_0$).



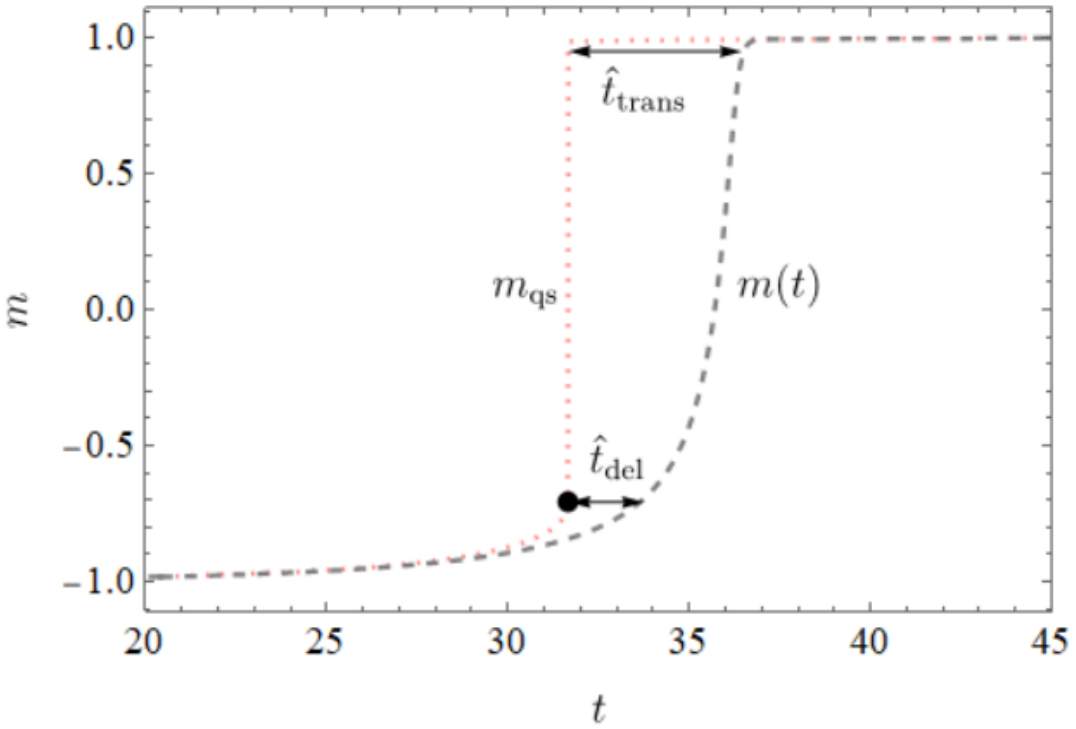
gray circle: quasi-static state m_{qs} red circle: transient state $m(t)$

Scaling relation in large systems -- the delay time

- Consider the case where $H(t)$ is tuned linearly

$$H(t) = H_i + (H_f - H_i)\frac{t}{t_f},$$

with the quench rate $v = (H_f - H_i)/t_f$ and the time duration t_f .



- While the potential landscape has drastically changed from bistable to monostable at the turning time t^*, the transition of the system occurs later by a delay time $\hat{t}_{del}$, defined through $m(t^* + \hat{t}_{del}) = m^*$.

- We shift the variables: $\hat{H} = H - H^*, \hat{m} = m - m^*, \hat{t} = t - t^*$. The evolution equation is turned into

$$\frac{d\hat{m}}{d\hat{t}} = f(m^* + \hat{m}, H^* + v\hat{t})$$

Scaling relation in large systems -- the delay time

- The evolution equation expanded around the turning point for slow quench approximation reads

$$\frac{d\hat{m}}{d\hat{t}} = f(m^* + \hat{m}, H^* + v\hat{t}) \approx \frac{2}{\tau_0}\left[\beta J\sqrt{\beta J - 1}\hat{m}^2 + \sqrt{\frac{\beta}{J}}v\hat{t}\right].$$

- It can be adimensionalized into

$$\frac{du}{ds} = u^2 + s,$$

by rescaling the variables $u = \alpha\hat{m}, \quad s = \gamma\hat{t},$ where

$$\alpha = \left(\beta J\sqrt{\beta J - 1}\right)^{2/3}\left(\frac{\tau_0}{2}\sqrt{\frac{\beta}{J}}v\right)^{-1/3},$$

$$\gamma = \left(\beta J\sqrt{\beta J - 1}\right)^{1/3}\left(\frac{4}{\tau_0^2}\sqrt{\frac{\beta}{J}}v\right)^{1/3}.$$

- Under the asymptotic boundary condition $s \to -\infty, u \to \sqrt{-s}$, it is solved in terms of the Airy function

$$u(s) = \frac{\mathrm{Ai}'(-s)}{\mathrm{Ai}(-s)}.$$

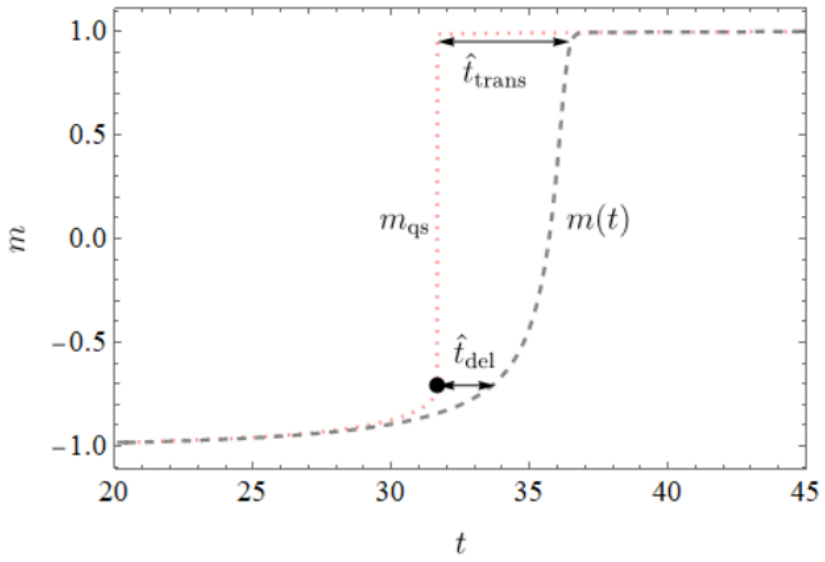
- In this way, the delay time can be computed as

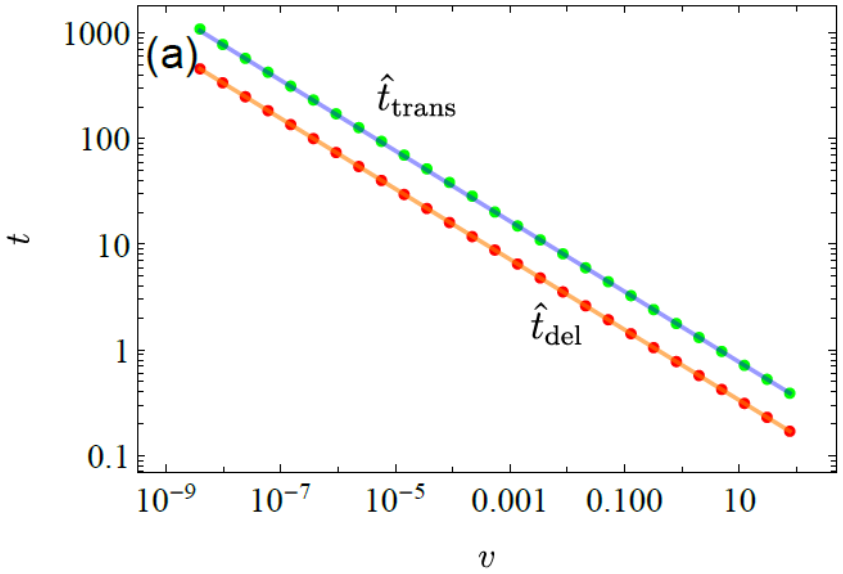
$$\hat{t}_{\mathrm{del}} = -A_1'\left[4\sqrt{\beta J(\beta J - 1)}\frac{\beta}{\tau_0^2}v\right]^{-1/3},$$

where $A_1' \approx -1.019$ represents the first zero point of the Airy Prime function.

P. Jung, G. Gray, R. Roy, and P. Mandel, Phys. Rev. Lett. 65, 1873 (1990).

Scaling relation in large systems -- the delay time





- The transition time $\hat{t}_{trans}$ to reach the monostable state is defined through

$$m(t^* + \hat{t}_{\rm trans}) = m_{\rm nd}$$

Here m_{nd} stands for the non-degenerate root of $f(m, H_+) = 0$. For $\beta J \gg 1$, m_{nd} can be approximated by 1.

- The transition time can be calculated as

$$\hat{t}_{\rm trans} = -A_1 \left[4\sqrt{\beta J(\beta J - 1)} \frac{\beta}{\tau_0^2} v \right]^{-1/3},$$

where $A_1 \approx -2.338$ stands for the first zero point of the Airy function.

J. H. Li et al, Phys. D: Nonlinear Phenom. 395, 7 (2019).

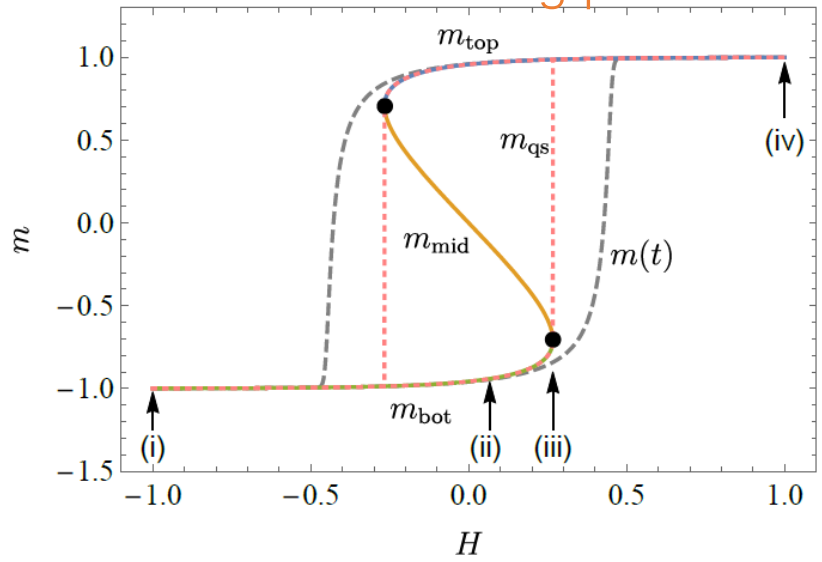
Scaling relation in large systems -- the average work (mean field result)

- The non-catching-up between the transient state and the quasi-static state results in the excess work, which characterizes the irreversibility of the process. It becomes evident around the turning point.

- The work (per site) performed on the system is

$$w = \int_0^{t_f} dt \frac{\partial \mathcal{H}}{\partial H} \dot{H} = -v \int_0^{t_f} dt m(t).$$

with the Hamiltonian density $\mathcal{H}(m) = -\frac{J}{2} m^2 - Hm.$



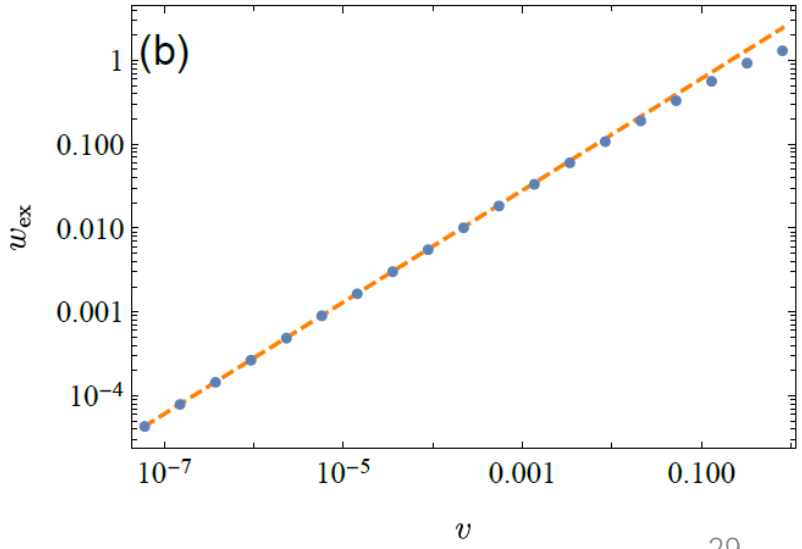
- When the protocol is tuned quasi-statically, the work is w_{qs}.

- The excess work, defined as $w_{ex} = w - w_{qs}$, describes the extra work when H is tuned at a finite rate rather than quasi-statically

$$w_{\text{ex}} \approx v\hat{t}_{\text{trans}}(m_{\text{nd}} - m^*)$$

Mean field result:

$$\approx \frac{-A_1(1+\sqrt{1-\frac{1}{\beta J}})}{\left[4\sqrt{\beta J(\beta J-1)}\frac{\beta}{\tau_0^2}\right]^{1/3}} v^{2/3},$$



F. Zhong et al, J. Phys. Condens. Matter 6, 7785 (1994).

C. N. Luse et al, Phys. Rev. E 50, 224 (1994).

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Scaling relations for small systems

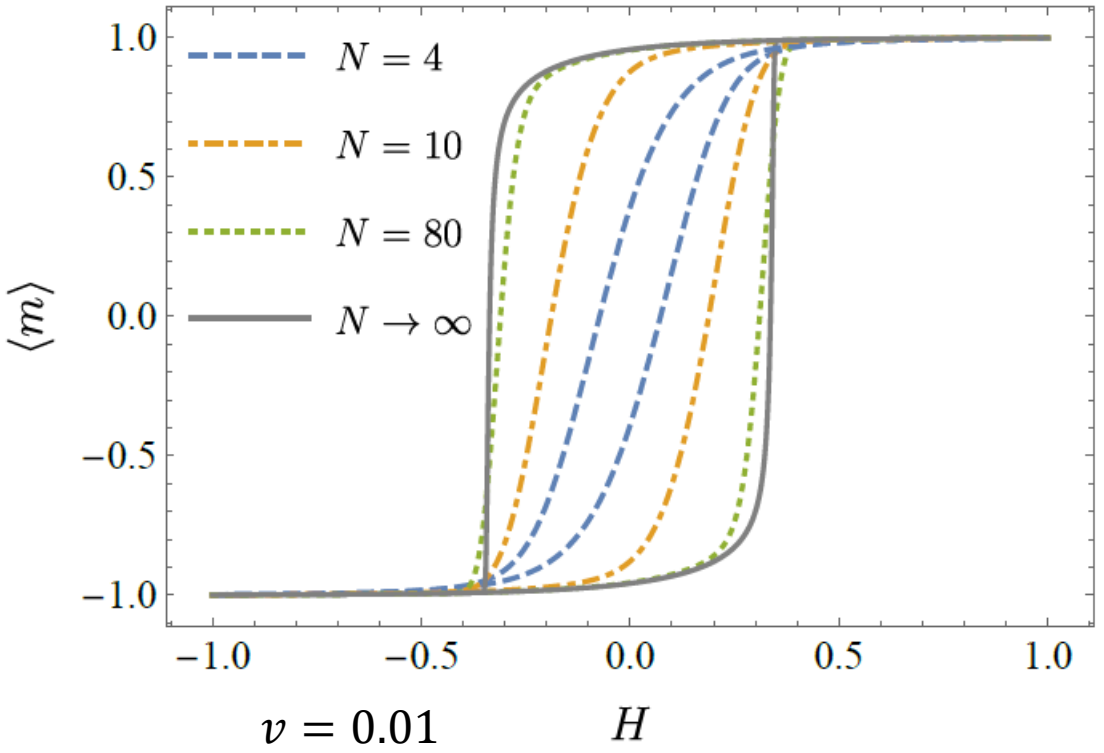
- For small systems, fluctuations become non-negligible.

- The probability distribution of the system state $P(M,t)$ evolves under the master equation

$$\frac{\partial P(M,t)}{\partial t} = \sum_{\eta=\pm}\Big[W_\eta(M_{-\eta},H(t))P(M_{-\eta},t) - W_\eta(M,H(t))P(M,t)\Big],$$

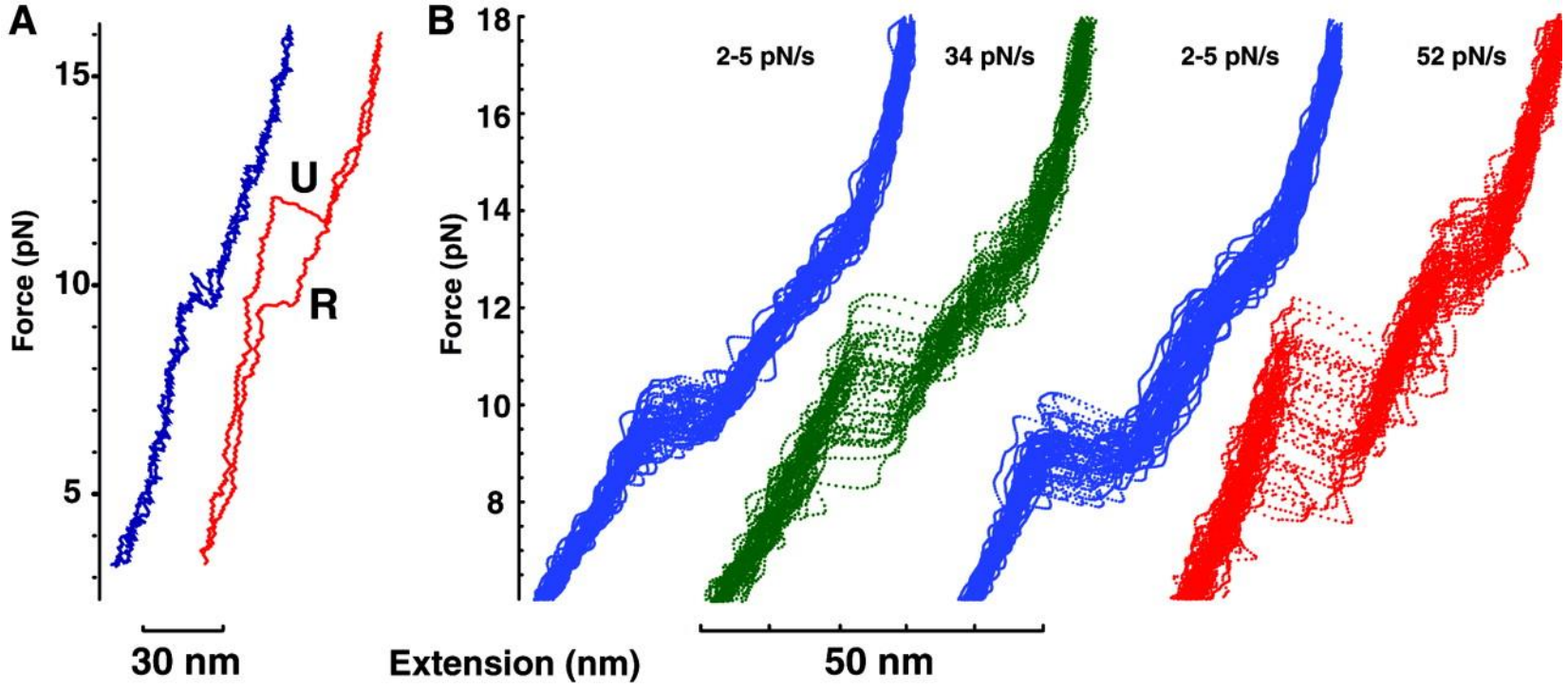
- The site-averaged mean magnetization can be calculated from the solution of the master equatic

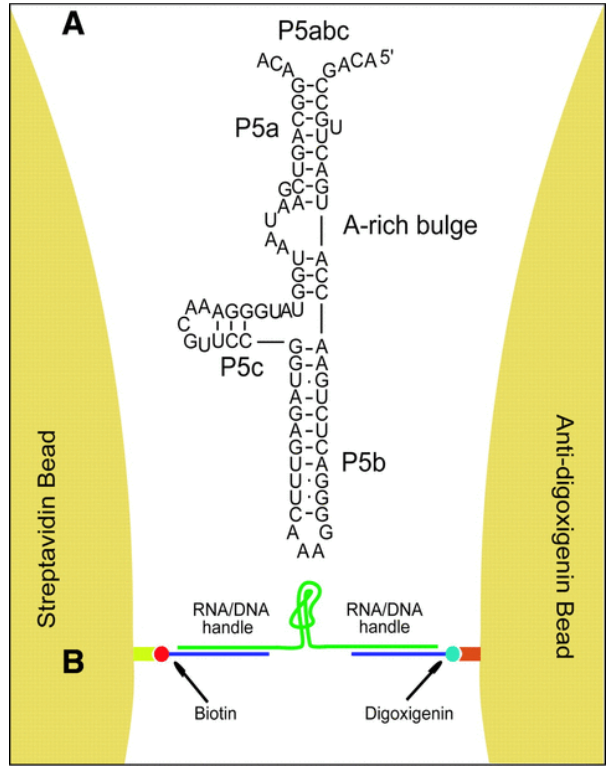
$$\langle m\rangle = \frac{1}{N}\sum_M MP(M,t).$$



Y. X. Wu, J. F Chen, **H. T. Quan**, arXiv preprint arXiv:2401.15592

“Hysteresis” in nonequilibrium processes





J. Liphardt S. Dumont, S. B. Smith, I. Tinoco, JR., and C. Bustamenta, Science 296, 1832 (2002)

Scaling relations for small systems

- The average work done on a system during the manipulation can be computed from

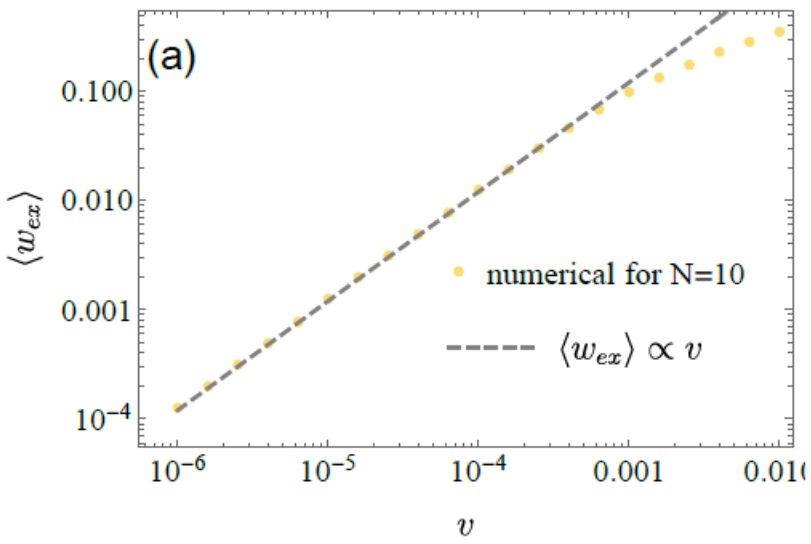
$$\langle w \rangle = -v \int_0^{t_f} dt \langle m(t) \rangle.$$

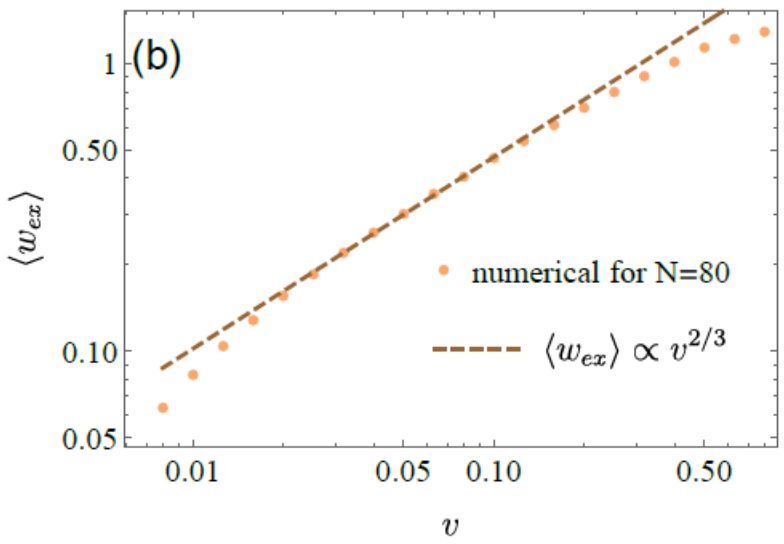
- The average excess work is $\langle w_{ex} \rangle = \langle w \rangle - w_{qs}.$
- For small systems, in the slow driving regime, the dynamics falls within the near-equilibrium regime

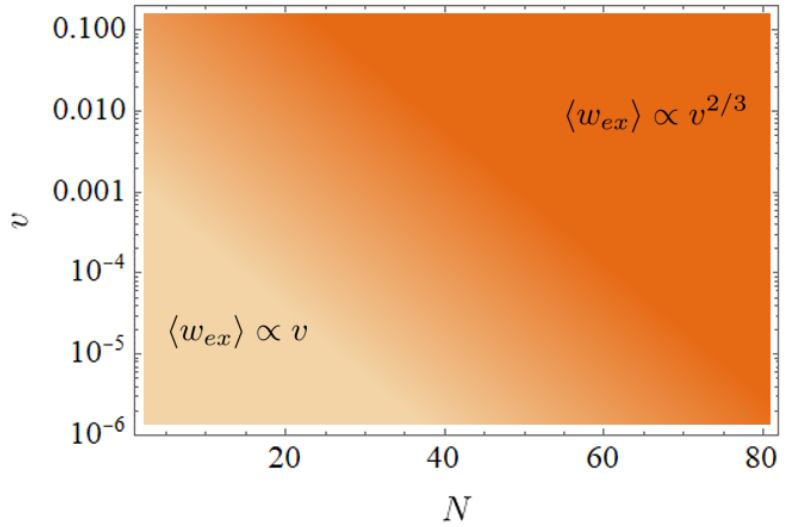
$$\langle w \rangle = \langle w_{ex} \rangle \sim v$$

- For large systems, when the quench rate is slow but not too slow, quasi-static work can be approximated with that in the deterministic limit

$$\langle w_{ex} \rangle \propto v^{2/3}$$







Y. X. Wu, J. F Chen, **H. T. Quan**, arXiv preprint arXiv:2401.15592

Work distribution for small systems

- Stochastic thermodynamics brings us more than the average value of work. We can calculate the probability distribution of work, which contains all the statistics of work fluctuation, and verify the Jarzynski's equality.

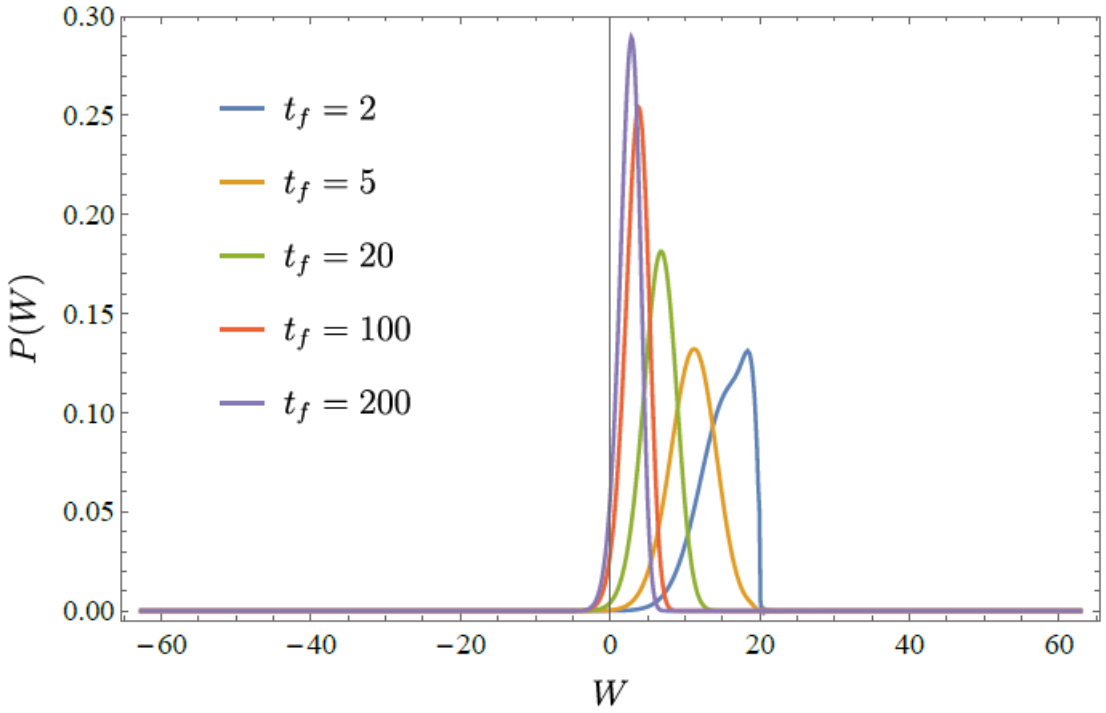


Fig. Work distribution for $N = 10$

Jarzynski's equality

$$\langle e^{-\beta W} \rangle = \frac{Z_{H_f}}{Z_{H_i}}.$$

Y. X. Wu, J. F Chen, **H. T. Quan**, arXiv preprint arXiv:2401.15592

K. Matsuo, J. Stat. Phys. 16, 169 (1977).

A. Imparato et al, Phys. Rev. E 72, 046114 (2005).

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Summary

- We study the scaling relation between the excess work and the quench rate in the finite-time second-order and the first-order phase transitions.

- For the second-order phase transition, the scaling exponents of the excess work depends on the critical exponents.

- For the first-order phase transition, under mean-field approximation, the scaling is found to be $\langle w_{ex} \rangle \sim v^{2/3}$. While in small systems, the excess work is linear with the quench rate $\langle w_{ex} \rangle \sim v$.

- The crossover in the scaling has its origin in the time-scale separation and ergodicity breaking. In small systems, no phase transition occurs. The linear scaling relation can be explained with the linear response theory.

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Now at ZJSTU



Fan Zhang



Yuxin Wu

[1] Z. Fei, N. Freitas, V. Cavina, H. T. Quan, M. Esposito, Physical Review Letters 124 (17), 170603 (2020).
[2] F. Zhang, H. T. Quan, Physical Review E 105 (2), 024101 (2022).
[3] Y. X. Wu, J. F Chen, H. T. Quan, arXiv preprint arXiv:2401.15592

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