Unified Hierarchical Relationship Between Thermodynamic Tradeoff Relations

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Basis of thermodynamics

lies in its fundamental inequalities.

$$\Sigma^{\mathrm{tot}} \geq 0$$

Second law

$$\eta \leq \eta_{\mathrm{C}} = 1 - \frac{T_{\mathrm{c}}}{T_{\mathrm{h}}}$$

Carnot bound

$$Q \geq k_{\mathrm{B}} T \ln 2$$

Landauer's principle

Clarifying the relationship is a crucial part of thermodynamics.

Novel Thermodynamic Tradeoff Inequalities

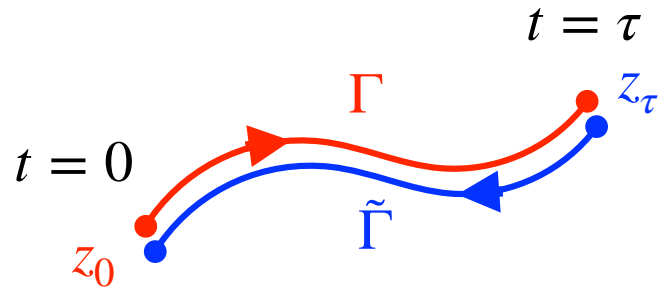
TUR

EB

CSL

PE Tradeoff

Thermodynamic Uncertainty Relation



current-like observable : $J_\tau(\Gamma) = -J_\tau(\tilde{\Gamma})$

$$\underbrace{\frac{\mathrm{Var}[J_\tau]}{\langle J_\tau \rangle^2}}_{\text{relative uncertainty}} \underbrace{\Sigma_\tau}_{\text{cost}} \geq 2 \qquad \text{(steady state)} \quad (k_\mathrm{B} = 1 \text{ unit})$$

$$\frac{\mathrm{Var}[J_\tau]}{\langle J_\tau \rangle^2} \downarrow \quad \Sigma_\tau \uparrow \qquad\qquad \frac{\mathrm{Var}[J_\tau]}{\langle J_\tau \rangle^2} \uparrow \quad \Sigma_\tau \downarrow$$

trade-off relation: we have to pay thermodynamic cost for reducing uncertainty

Novel Thermodynamic Tradeoff Inequalities

TUR

EB

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PE Tradeoff

Thermodynamic Uncertainty Relation

Steady-state TUR

(overdamped Langevin system + Markov jump process)

$$\frac{\mathrm{Var}[J_\tau]}{\langle J_\tau\rangle^2}\Sigma_\tau \geq 2$$

Barato and Seifert, PRL 114, 158101 (2015)
Gingrich et al, PRL 116, 120601 (2016)
Dechant and Sasa, JStatMech 063209 (2018)
Hasegawa and Vu, PRE 99, 062126 (2019)

TUR with an arbitrary state
(overdamped & underdamped Langevin + Markov jump process)

$$\frac{\mathrm{Var}[J_\tau]}{(\hat{O}\langle J_\tau\rangle)^2}\Sigma_\tau \geq 2 \qquad \hat{O} = \tau\partial_\tau - s\partial_s - \omega\partial_\omega$$

Koyuk and Seifert, PRL 125, 260604 (2020)
JSL, Park, Park, PRE 104, L052102 (2021)

thermodynamic second law: $\Sigma_\tau \geq 0$

TUR: stronger bound $\quad \Sigma_\tau \geq 2\frac{(\hat{O}\langle\Theta_\tau\rangle)^2}{\mathrm{Var}[\Theta_\tau]} > 0$

Novel Thermodynamic Tradeoff Inequalities

TUR

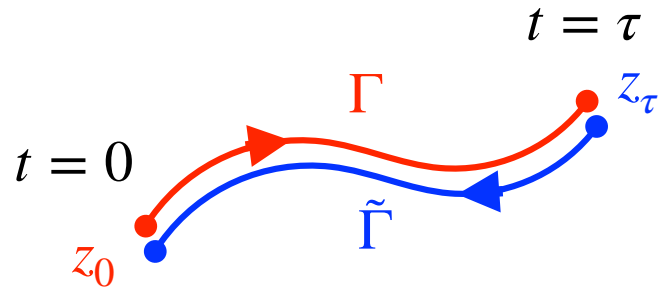
$$\frac{\mathrm{Var}[J_\tau]}{(\hat{O}\langle J_\tau\rangle)^2}\Sigma_\tau \geq 2$$

EB

CSL

PE Tradeoff

Entropic Bound



current-like observable : $J_\tau(\Gamma) = -J_\tau(\tilde{\Gamma})$

$$\langle J_\tau\rangle^2 \leq \chi_\tau \Sigma_\tau$$

Dechant and Sasa, PRE 97, 062101 (2018)
Lee, Kim, Park, JSL PRR 5, 013194 (2023)

$\langle J_\tau\rangle$: irreversible current

χ_τ : (positive) system-dependent constant

EB: stronger bound $\quad \Sigma_\tau \geq \langle J_\tau\rangle^2/\chi_\tau > 0$

Derived only for Langevin dynamics, not for Markov jump process

Novel Thermodynamic Tradeoff Inequalities

TUR

$$\frac{\mathrm{Var}[J_\tau]}{(\hat{O}\langle J_\tau\rangle)^2}\Sigma_\tau \geq 2$$

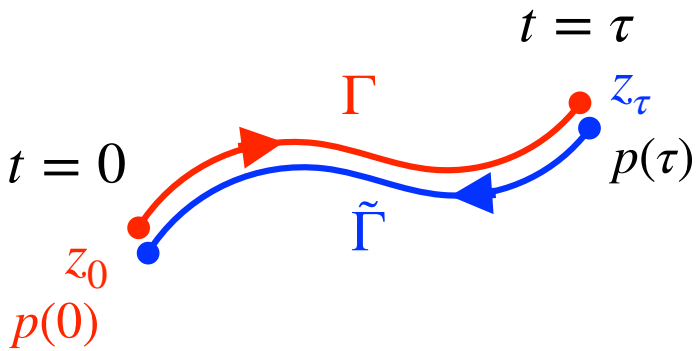
EB

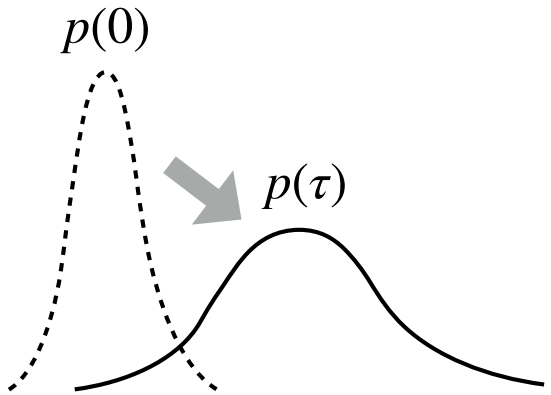
$$\langle J_\tau\rangle^2 \leq \chi_\tau \Sigma_\tau$$

CSL

PE Tradeoff

Classical Speed Limit





$$\tau \geq \frac{d_H^2(p(0),p(\tau))}{\bar{\chi}^H \Sigma_\tau} \quad \Rightarrow \quad \tau\Sigma_\tau \geq \frac{d_H^2(p(0),p(\tau))}{\bar{\chi}^H}$$

Shiraish, Funo, Saito, PRL 121, 070601 (2018)

$d_H(p,q)$: distance between p and q

$\bar{\chi}^H$: mobility (dynamical activity or diffusivity)

$$\tau \downarrow \quad \Sigma_\tau \uparrow \qquad\qquad \tau \uparrow \quad \Sigma_\tau \downarrow$$

trade-off relation: we have to pay more thermodynamic cost for faster transition

Novel Thermodynamic Tradeoff Inequalities

TUR

$$\frac{\mathrm{Var}[J_\tau]}{(\hat{O}\langle J_\tau\rangle)^2}\Sigma_\tau \geq 2$$

EB

$$\langle J_\tau\rangle^2 \leq \chi_\tau \Sigma_\tau$$

CSL

$$\tau \geq \frac{d_H^2(p(0), p(\tau))}{\bar{\chi}^H \Sigma_\tau}$$

PE Tradeoff

Power-Efficiency Tradeoff

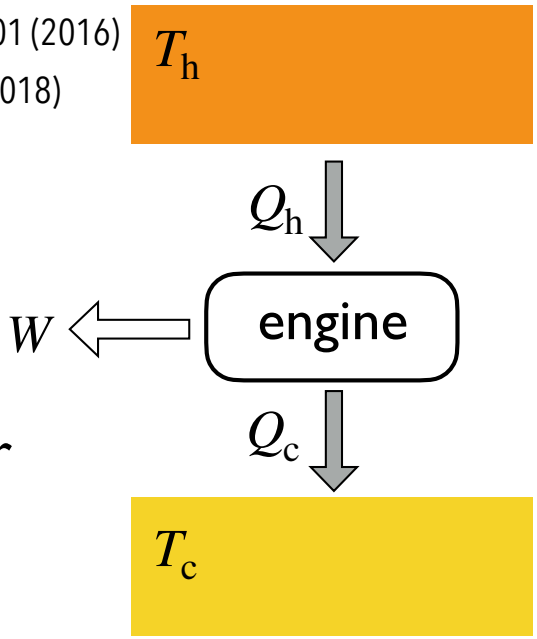
$$P \leq \Xi\eta(\eta_C - \eta)$$

Shiraishi, Saito, Tasaki PRL 117, 190601 (2016)
Dechant and Sasa, PRE 97, 062101 (2018)

$P = W/\tau_c$: engine power

Ξ : system-dependent constant

tradeoff between efficiency and power



Derived for Langevin dynamics (steady-state, cyclic)

Markov jump process (steady-state)

Novel Thermodynamic Tradeoff Inequalities

TUR

$$\frac{\mathrm{Var}[J_\tau]}{(\hat{O}\langle J_\tau\rangle)^2}\Sigma_\tau \geq 2$$

EB

$$\langle J_\tau\rangle^2 \leq \chi_\tau \Sigma_\tau$$

CSL

$$\tau \geq \frac{d_H^2(p(0),p(\tau))}{\bar{\chi}^H \Sigma_\tau}$$

PE Tradeoff

$$P \leq \Xi\eta(\eta_C - \eta)$$

Are these relations independent?

Or can they be understood within a single unified framework?

$$\Sigma^{\mathrm{tot}} \geq 0$$

Second law

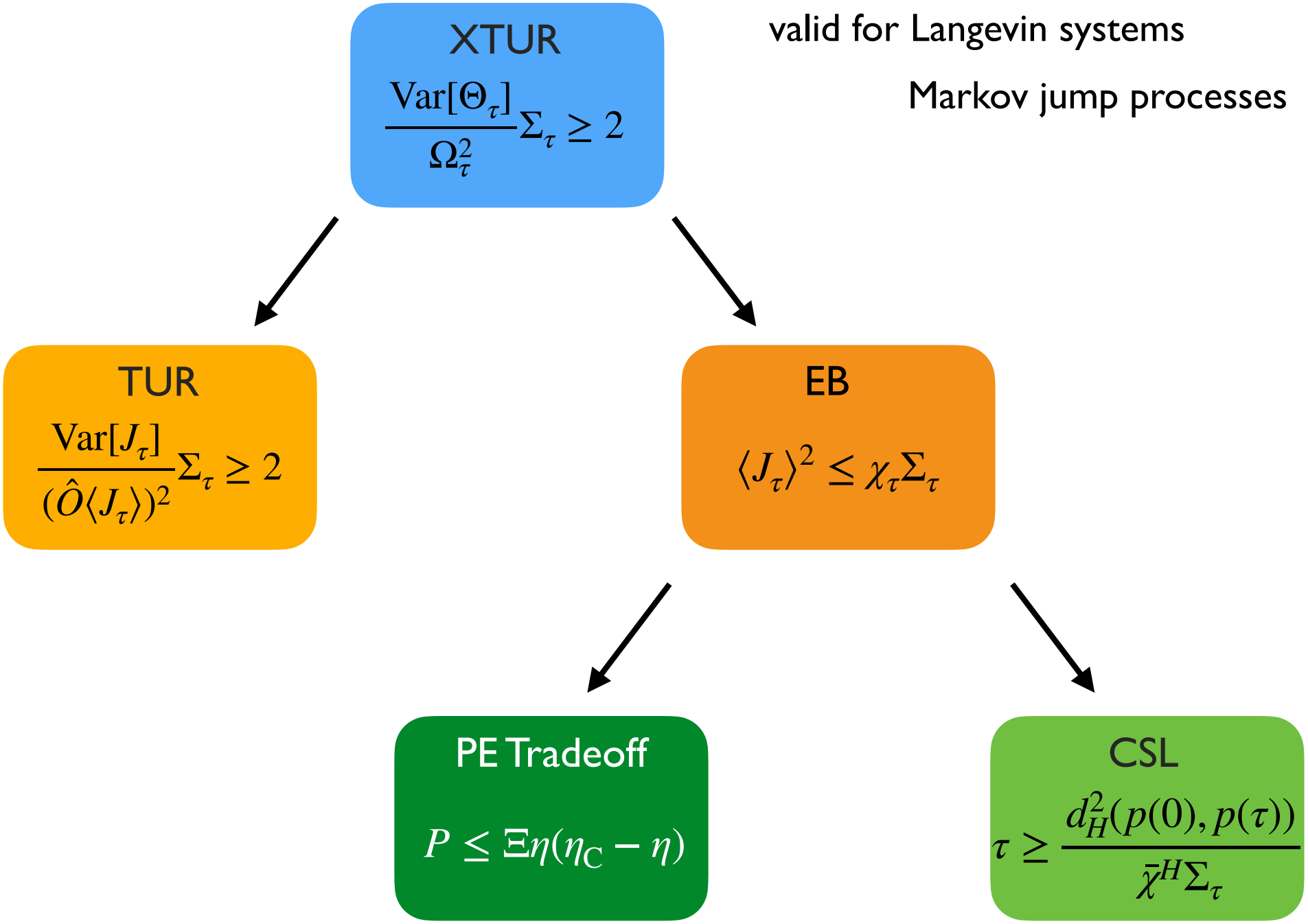
$$\eta \leq \eta_C = 1 - \frac{T_c}{T_h}$$

Carnot bound

$$Q \geq k_B T \ln 2$$

Landauer's principle

Unified Hierarchical Relationship



Setup1: Langevin System

Setup1: *N*-dimensional Langevin systems

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x}, \omega t) + \mathbb{B}(\mathbf{x}, \omega t) \bullet \boldsymbol{\xi}(t, \omega)$$

$\mathbf{x} = (x_1, x_2, \cdots, x_N)^{\mathrm{T}}$: state

$\mathbf{A} = (A_1, A_2, \cdots, A_N)^{\mathrm{T}}$: drift force

ω : parameter controlling speed of protocol

$\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_N)^{\mathrm{T}}$: Gaussian white noise

$\mathbb{B}(\mathbf{x}, \omega t)\mathbb{B}^{\mathrm{T}}(\mathbf{x}, \omega t)/2 = \mathbb{D}(\mathbf{x}, \omega t)$: diffusion matrix

Fokker-Planck equation

$$\partial_t p(\mathbf{x}, t; \omega) = -\nabla^{\mathrm{T}} \mathbf{J}(\mathbf{x}, t; \omega) \qquad \mathbf{J} = \left(\boldsymbol{A}(\mathbf{x}, \omega t) - \left[\nabla^{\mathrm{T}} \mathbb{D}(\mathbf{x}, \omega t) \right]^{\mathrm{T}} \right) p(\mathbf{x}, t; s, \omega)$$

Setup1: Langevin System

Setup1: *N*-dimensional Langevin systems

Entropy production

$$\Sigma_\tau^{\rm tot} = \int_0^\tau dt \int d\mathbf{x}\, \frac{\mathbf{J}^{\rm T}\mathbb{D}^{-1}(\mathbf{x},\omega t)\mathbf{J}}{p(\mathbf{x},t;s,\omega)}$$

$$\langle \mathbf{\Lambda}^{\rm T}\circ\dot{\mathbf{x}}\rangle = \int d\mathbf{x}\,\mathbf{\Lambda}^{\rm T}\mathbf{J}$$

Observable (Γ : trajectory)

current-like : $J_\tau^\Lambda(\Gamma) = \int_0^\tau dt\,\mathbf{\Lambda}^{\rm T}(\mathbf{x}(t),\omega t;s)\circ\dot{\mathbf{x}}(t) \to \langle J_\tau^\Lambda\rangle = \int_0^\tau dt\int d\mathbf{x}\,\mathbf{\Lambda}^{\rm T}\mathbf{J}$

state-dependent : $K_\tau^g(\Gamma) = \int_0^\tau dt\, g(\mathbf{x}(t),\omega t;s) \to \langle K_\tau^g\rangle = \int_0^\tau dt\int d\mathbf{x}\, g(\mathbf{x}(t),\omega t;s)p(\mathbf{x},t;s,\omega)$

composite : $\Theta_\tau^{\Lambda,g}(\Gamma) = J_\tau^\Lambda(\Gamma) + K_\tau^g(\Gamma)$

Setup2: Markov Jump Process

Setup2: Markov jump processes

$$\dot{p}_n(t;\omega) = \sum_{m(\neq n)} j_{nm}(t;\omega) \qquad \text{where } j_{nm}(t;\omega) = R_{nm}(\omega t)p_m(t;\omega) - R_{mn}(\omega t)p_n(t;\omega)$$

Entropy production

$$\Sigma_\tau^{\text{tot}} = \int_0^\tau dt \sum_{n>m} j_{nm} \ln \frac{R_{nm}p_m}{R_{mn}p_n}$$

Observable

$$\langle \dot{N}_{nm}(t) \rangle = R_{nm}(\omega t)p_m(t;\omega)$$

$$\text{current-like} : J_\tau^\Lambda(\Gamma) = \int_0^\tau dt \sum_{n\neq m} \Lambda_{nm}(\omega t)\dot{N}_{nm}(t) \quad \rightarrow \langle J_\tau^\Lambda \rangle = \int_0^\tau dt \sum_{n>m} \Lambda_{nm}(\omega t) j_{nm}(t;\omega)$$

$$\Lambda_{nm} = -\Lambda_{mn}$$

$N_{nm}(t)$: # of jumps from state m to n until time t

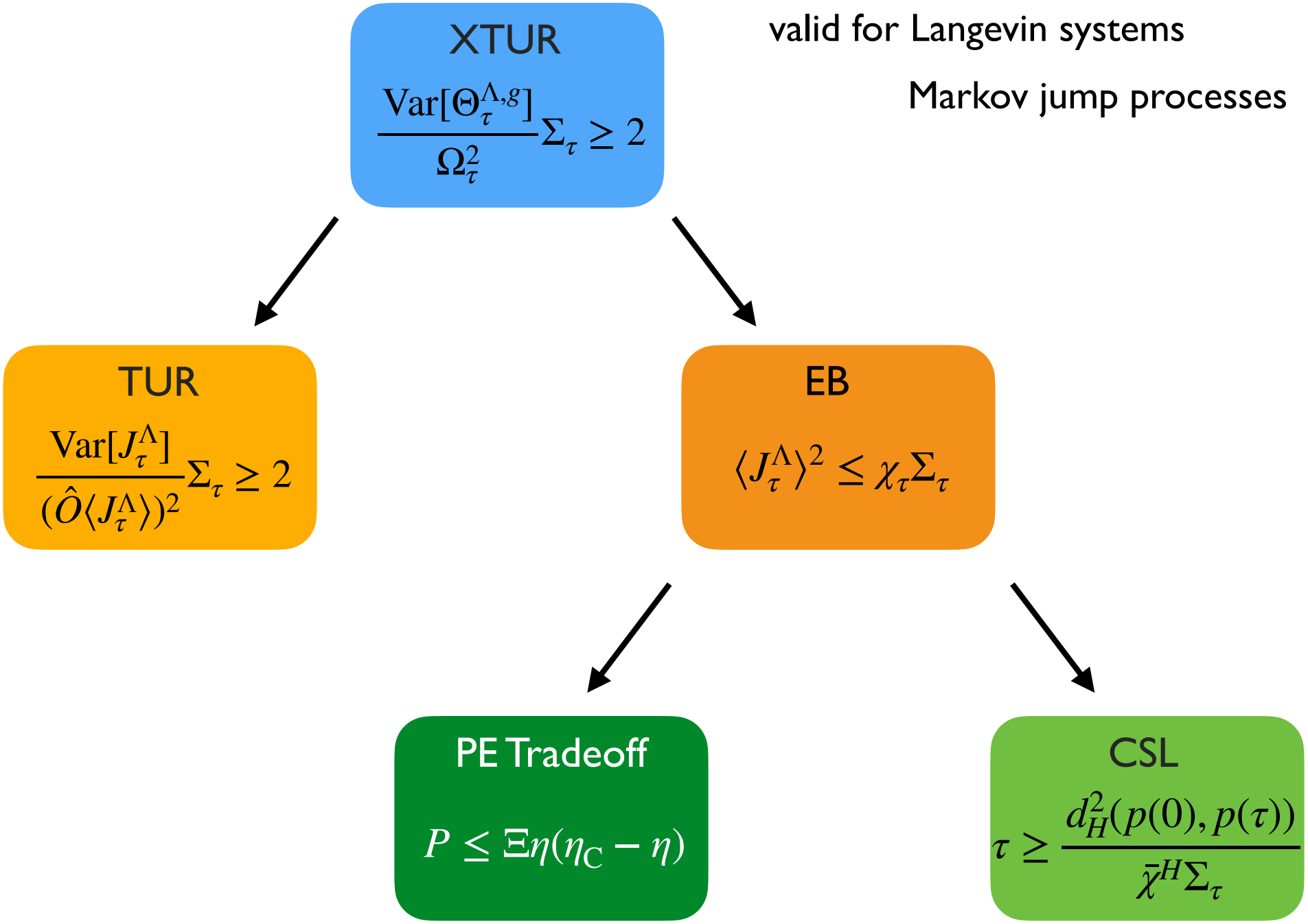
$$\langle \eta_n \rangle = p_n(t;\omega)$$

$$\text{state-dependent} : K_\tau^g(\Gamma) = \int_0^\tau dt \sum_n \eta_n(t) g_n(\omega t) \rightarrow \langle K_\tau^g \rangle = \int_0^\tau dt \sum_n p_n(t) g_n(\omega t)$$

$\eta_n(t) \equiv \delta_{x(t),n}$: state indicator at time t

$$\text{composite} : \Theta_\tau^{\Lambda,g}(\Gamma) = J_\tau^\Lambda(\Gamma) + K_\tau^g(\Gamma)$$

Unified Hierarchical Relationship



XTUR

XTUR

$$\frac{\mathrm{Var}[\Theta_\tau^{\Lambda,g}]}{\Omega_\tau^2}\Sigma_\tau \geq 2$$

derived from Cramér-Rao inequality

composite : $\Theta_\tau^{\Lambda,g}(\Gamma) = J_\tau^\Lambda(\Gamma) + K_\tau^g(\Gamma)$

$\Omega_\tau \equiv \hat{O}_\tau\langle\Theta_\tau^{\Lambda,g}\rangle - \langle K_\tau^g\rangle \quad \leftarrow \hat{O} = \tau\partial_\tau - \omega\partial_\omega$

Σ_τ : EP, or pseudo EP

Observable (Langevin)

$$J_\tau^\Lambda(\Gamma) = \int_0^\tau dt\, \mathbf{\Lambda}^{\mathrm{T}}(\mathbf{x}(t), \omega t) \circ \dot{\mathbf{x}}(t)$$

$$K_\tau^g(\Gamma) = \int_0^\tau dt\, g(\mathbf{x}(t), \omega t)$$

Observable (Markov jump)

$$J_\tau^\Lambda(\Gamma) = \int_0^\tau dt \sum_{n\neq m} \Lambda_{nm}(\omega t)\dot{N}_{nm}(t)$$

$$K_\tau^g(\Gamma) = \int_0^\tau dt \sum_n \eta_n(t) g_n(\omega t)$$

Equality condition (Langevin)

$$\mathbf{\Lambda}^{\mathrm{eq}} = C\frac{\mathbb{D}^{-1}\mathbf{J}}{\tilde{p}}$$

$$g^{\mathrm{eq}} = -C\left[\frac{\mathbf{J}^{\mathrm{T}}\mathbb{D}^{-1}\mathbf{J}}{p^2} + \frac{\nabla^{\mathrm{T}}\mathbf{J}}{p}\right]$$

Equality condition (Markov jump $\Sigma_\tau = \Sigma_\tau^{\mathrm{ps}}$)

$$\Lambda_{nm}^{\mathrm{eq}} = C\frac{j_{nm}}{a_{nm}}$$

$$g_n^{\mathrm{eq}} = -C\sum_{m(\neq n)} R_{mn}\frac{j_{nm}}{a_{nm}}$$

XTUR ⟶ TUR

XTUR

$$\frac{\mathrm{Var}[\Theta_\tau^{\Lambda,g}]}{\Omega_\tau^2}\Sigma_\tau \geq 2$$

composite : $\Theta_\tau^{\Lambda,g}(\Gamma) = J_\tau^\Lambda(\Gamma) + K_\tau^g(\Gamma)$

$$\Omega_\tau \equiv \hat{O}_\tau\langle\Theta_\tau^{\Lambda,g}\rangle - \langle K_\tau^g\rangle \quad \leftarrow \hat{O} = \tau\partial_\tau - \omega\partial_\omega$$

$g = 0$ (overdamped) $\rightarrow \dfrac{\mathrm{Var}[J_\tau^\Lambda]}{[(\tau\partial_\tau - \omega\partial_\omega)J_\tau^\Lambda]^2}\Sigma_\tau^{\mathrm{tot}} \geq 2$

Koyuk and Seifert, PRL 125, 260604 (2020)

Liu, Gong, and Ueda, PRL 125, 140602 (2020)

equality condition cannot be met in TUR since $g^{\mathrm{eq}} \neq 0$

Observable (Langevin)

$$J_\tau^\Lambda(\Gamma) = \int_0^\tau dt\, \mathbf{\Lambda}^{\mathrm{T}}(\mathbf{x}(t), \omega t) \circ \dot{\mathbf{x}}(t)$$

$$K_\tau^g(\Gamma) = \int_0^\tau dt\, g(\mathbf{x}(t), \omega t)$$

Observable (Markov jump)

$$J_\tau^\Lambda(\Gamma) = \int_0^\tau dt \sum_{n\neq m} \Lambda_{nm}(\omega t)\dot{N}_{nm}(t)$$

$$K_\tau^g(\Gamma) = \int_0^\tau dt \sum_n \eta_n(t) g_n(\omega t)$$

Equality condition (Langevin)

$$\mathbf{\Lambda}^{\mathrm{eq}} = C\frac{\mathbb{D}^{-1}\mathbf{J}}{\tilde{p}}$$

$$g^{\mathrm{eq}} = -C\left[\frac{\mathbf{J}^{\mathrm{T}}\mathbb{D}^{-1}\mathbf{J}}{p^2} + \frac{\nabla^{\mathrm{T}}\mathbf{J}}{p}\right]$$

Equality condition (Markov jump $\Sigma_\tau = \Sigma_\tau^{\mathrm{ps}}$)

$$\Lambda_{nm}^{\mathrm{eq}} = C\frac{j_{nm}}{a_{nm}}$$

$$g_n^{\mathrm{eq}} = -C\sum_{m(\neq n)} R_{mn}\frac{j_{nm}}{a_{nm}}$$

XTUR ⟶ TUR

XTUR

$$\frac{\mathrm{Var}[\Theta_\tau^{\Lambda,g}]}{\Omega_\tau^2}\Sigma_\tau \geq 2$$

composite : $\Theta_\tau^{\Lambda,g}(\Gamma) = J_\tau^\Lambda(\Gamma) + K_\tau^g(\Gamma)$

$$\Omega_\tau \equiv \hat{O}_\tau\langle\Theta_\tau^{\Lambda,g}\rangle - \langle K_\tau^g\rangle \quad \leftarrow \hat{O} = \tau\partial_\tau - \omega\partial_\omega$$

$g = -\mathbf{\Lambda}^{\mathrm{T}}\mathbf{J}/p$ (overdamped, steady) $\rightarrow$ $\dfrac{\mathrm{Var}_{\mathrm{ss}}(J_\tau^\Lambda + K_\tau^g)}{\langle J_\tau^\Lambda\rangle_{\mathrm{ss}}^2}\Sigma_\tau^{\mathrm{tot}} \geq 2$

relative TUR (RTUR)

Dechant and Sasa, PRR 3, L042012 (2021)

$$\langle K_\tau^g\rangle = -\int_0^\tau dt\,\mathbf{\Lambda}^{\mathrm{T}}\mathbf{J}$$

$$\langle J_\tau^\Lambda\rangle = \int_0^\tau dt\,\mathbf{\Lambda}^{\mathrm{T}}\mathbf{J}$$

$\rightarrow \langle\Theta_\tau^{\Lambda,g}\rangle = 0$

Observable (Langevin)

$$J_\tau^\Lambda(\Gamma) = \int_0^\tau dt\,\mathbf{\Lambda}^{\mathrm{T}}(\mathbf{x}(t),\omega t)\circ\dot{\mathbf{x}}(t)$$

$$K_\tau^g(\Gamma) = \int_0^\tau dt\,g(\mathbf{x}(t),\omega t)$$

Observable (Markov jump)

$$J_\tau^\Lambda(\Gamma) = \int_0^\tau dt\sum_{n\neq m}\Lambda_{nm}(\omega t)\dot{N}_{nm}(t)$$

$$K_\tau^g(\Gamma) = \int_0^\tau dt\sum_n \eta_n(t)g_n(\omega t)$$

Equality condition (Langevin)

$$\mathbf{\Lambda}^{\mathrm{eq}} = C\frac{\mathbb{D}^{-1}\mathbf{J}}{\tilde{p}}$$

$$g^{\mathrm{eq}} = -C\left[\frac{\mathbf{J}^{\mathrm{T}}\mathbb{D}^{-1}\mathbf{J}}{p^2} + \frac{\nabla^{\mathrm{T}}\mathbf{J}}{p}\right]$$

Equality condition (Markov jump $\Sigma_\tau = \Sigma_\tau^{\mathrm{ps}}$)

$$\Lambda_{nm}^{\mathrm{eq}} = C\frac{j_{nm}}{a_{nm}}$$

$$g_n^{\mathrm{eq}} = -C\sum_{m(\neq n)} R_{mn}\frac{j_{nm}}{a_{nm}}$$

Numerical Confirmation for XTUR

Brownian particle pulled by optical tweezers

Equation of motion: $\gamma\dot{x} = -k(x - \omega v t) + \sqrt{2\gamma T}\xi(t)$

ω: protocol-speed controlling parameter

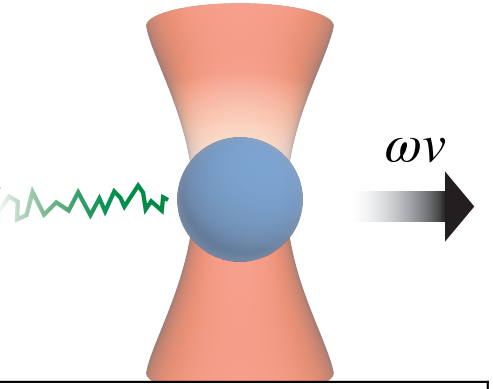
XTUR

$$\frac{\mathrm{Var}[\Theta_\tau^{\Lambda,g}]}{\Omega_\tau^2}\Sigma_\tau \geq 2$$

Observables: $J_\tau^\Lambda(\Gamma) = \int_0^\tau dt\,\dot{x} = x_\tau - x_0 \quad \rightarrow \Theta_\tau^{\Lambda,g} = J_\tau^\Lambda + K_\tau^g$

$$K_\tau^g(\Gamma) = \frac{\alpha k}{\gamma}\int_0^\tau dt\,(x_t - \omega v t) \quad \rightarrow \alpha = 0 : \text{XTUR} \Rightarrow \text{TUR}$$

α: intensity of state-dependent observable

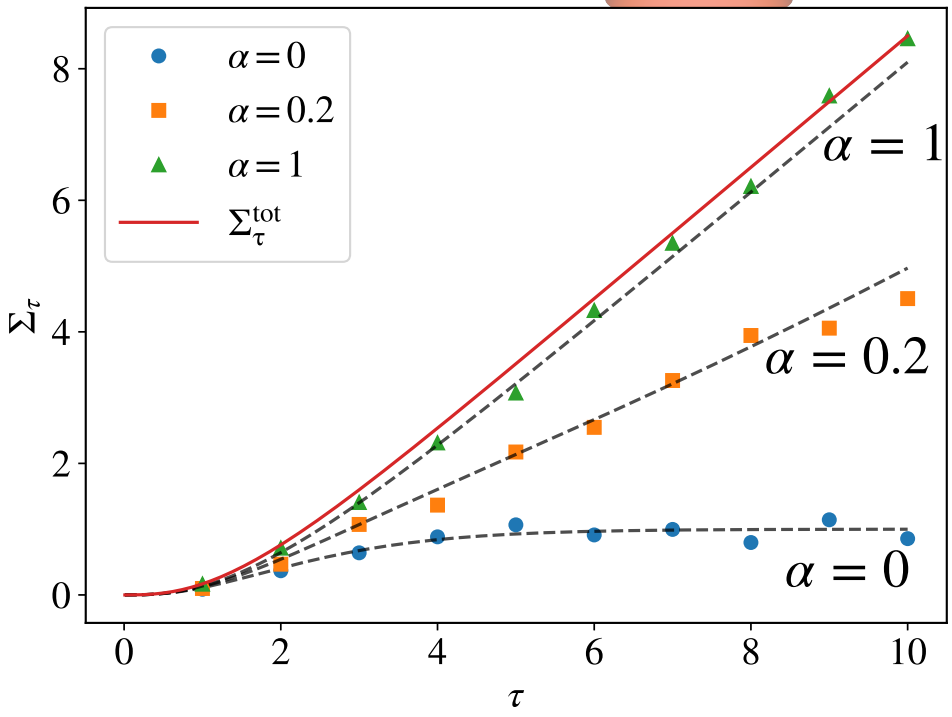


XTUR:

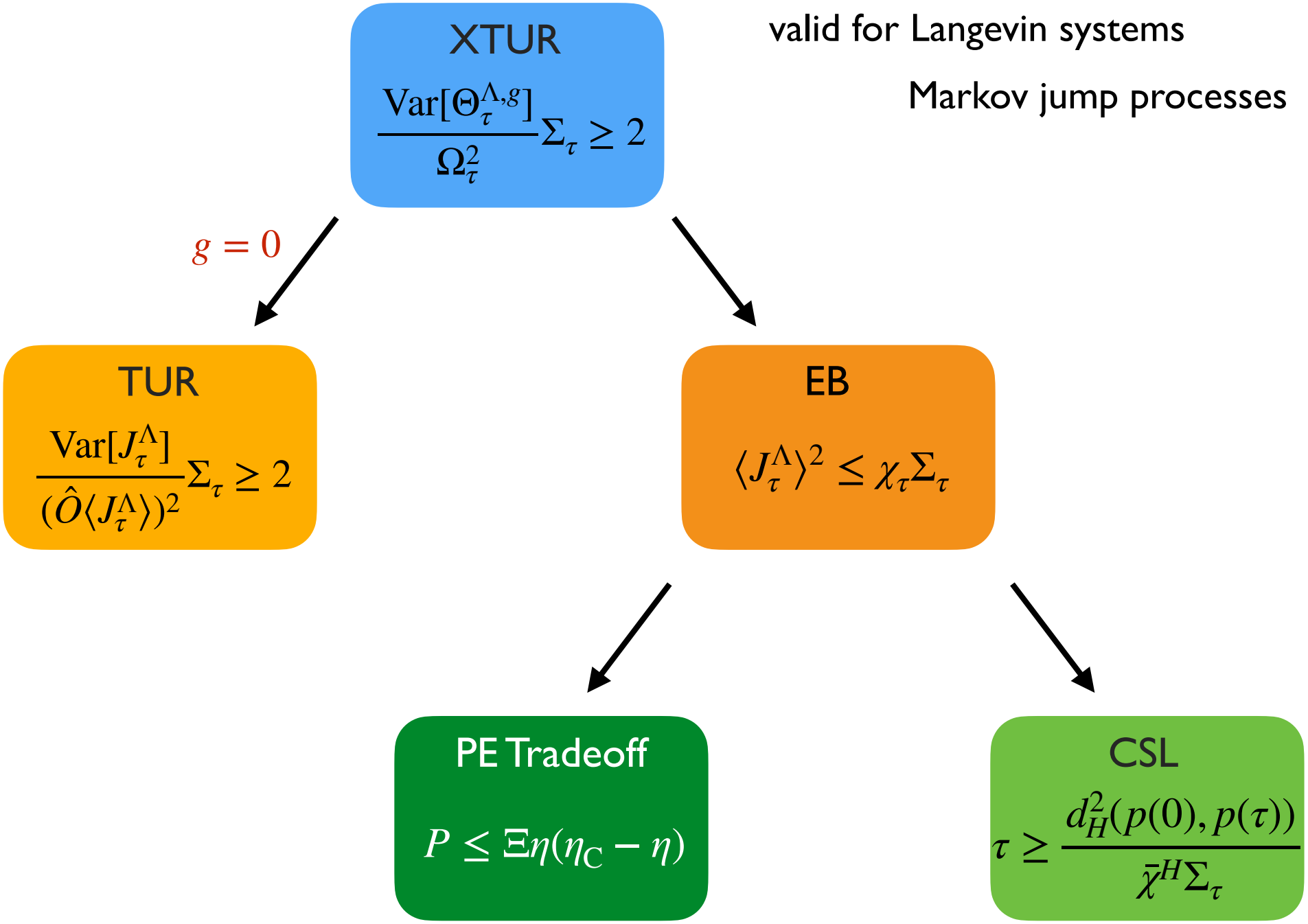
$$\Sigma_\tau^{\mathrm{b}} \equiv \frac{2[(\tau\partial_\tau - \omega\partial_\omega)\langle\Theta_\tau^{\Lambda,g}\rangle - \langle K_\tau^g\rangle]^2}{\mathrm{Var}[\Theta_\tau^{\Lambda,g}]} \leq \Sigma_\tau^{\mathrm{tot}}$$

$$\partial_\tau\langle\Theta_\tau^{\Lambda,g}\rangle(\omega) \approx \frac{\langle\Theta_{\tau+\Delta\tau}^{\Lambda,g}\rangle(\omega) - \langle\Theta_\tau^{\Lambda,g}\rangle(\omega)}{\Delta\tau}$$

$$\partial_\omega\langle\Theta_\tau^{\Lambda,g}\rangle(\omega) \approx \frac{\langle\Theta_\tau^{\Lambda,g}\rangle(\omega+\Delta\omega) - \langle\Theta_\tau^{\Lambda,g}\rangle(\omega)}{\Delta\omega}$$



Unified Hierarchical Relationship



XTUR ⟶ EB Langevin dynamics

XTUR

$$\frac{\mathrm{Var}[\Theta_\tau^{\Lambda,g}]}{\Omega_\tau^2}\Sigma_\tau \geq 2$$

EB

$$\langle J_\tau^\Lambda \rangle^2 \leq \chi_\tau \Sigma_\tau$$

Observable

$$J_\tau^\Lambda(\Gamma) = \int_0^\tau dt \boldsymbol{\Lambda}^{\mathrm{T}} \circ \dot{\mathbf{x}} = \int_0^\tau dt \boldsymbol{\Lambda}^{\mathrm{T}} \bullet \dot{\mathbf{x}} + \int_0^\tau dt\, \mathrm{tr}[\mathbb{D}\nabla\boldsymbol{\Lambda}^{\mathrm{T}}]$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x}, \omega t) + \mathbb{B}(\mathbf{x}, \omega t) \bullet \boldsymbol{\xi}(t)$$

$$K_\tau^g(\Gamma) = \int_0^\tau dt\, g \quad \leftarrow g^{\mathrm{EB}} = -\boldsymbol{\Lambda}^{\mathrm{T}}\mathbf{A}(\mathbf{x}, \omega t) - \mathrm{tr}[\mathbb{D}\nabla\boldsymbol{\Lambda}^{\mathrm{T}}]$$

$$\Theta_\tau^{\Lambda,g^{\mathrm{EB}}}(\Gamma) = J_\tau^\Lambda(\Gamma) + K_\tau^{g^{\mathrm{EB}}}(\Gamma) = \int_0^\tau dt \boldsymbol{\Lambda}^{\mathrm{T}}\mathbb{B}(\mathbf{x}, \omega t) \bullet \boldsymbol{\xi}(t)$$

$$\langle \Theta_\tau^{\Lambda,g^{\mathrm{EB}}} \rangle = \int_0^\tau dt \langle \boldsymbol{\Lambda}^{\mathrm{T}}\mathbb{B}(\mathbf{x}, \omega t)\rangle\langle \boldsymbol{\xi}(t)\rangle = 0$$

$$\mathrm{Var}(\Theta_\tau^{\Lambda,g^{\mathrm{EB}}}) = \left\langle \left(\Theta_\tau^{\Lambda,g^{\mathrm{EB}}}\right)^2 \right\rangle = 2\int_0^\tau dt \langle \boldsymbol{\Lambda}^{\mathrm{T}}\mathbb{D}\boldsymbol{\Lambda}\rangle$$

$$\Omega_\tau \equiv \hat{O}_\tau \langle \Theta_\tau^{\Lambda,g^{\mathrm{EB}}} \rangle - \langle K_\tau^{g^{\mathrm{EB}}} \rangle$$

$$= \int_0^\tau dt \int d\mathbf{x}\, \boldsymbol{\Lambda}^{\mathrm{T}}\mathbf{J} = \langle J_\tau^\Lambda \rangle$$

$$\langle J_\tau^\Lambda \rangle^2 \leq \chi_\tau \Sigma_\tau$$

$$\chi_\tau = \int_0^\tau dt \langle \boldsymbol{\Lambda}^{\mathrm{T}}\mathbb{D}\boldsymbol{\Lambda}\rangle$$

XTUR ⟶ EB Markov process

XTUR

$$\frac{\mathrm{Var}[\Theta_\tau^{\Lambda,g}]}{\Omega_\tau^2}\Sigma_\tau \geq 2$$

EB

$$\langle J_\tau^\Lambda \rangle^2 \leq \chi_\tau \Sigma_\tau$$

Observable

$$\langle \dot{N}_{nm}(t) \rangle = R_{nm} p_m$$

$$J_\tau^\Lambda(\Gamma) = \int_0^\tau dt \sum_{n\neq m} \Lambda_{nm}(\omega t)\dot{N}_{nm}(t) \quad \to \langle J_\tau^\Lambda \rangle = \int_0^\tau dt \sum_{n\neq m} \Lambda_{nm} R_{nm} p_n$$

$$\langle \eta_n \rangle = p_n$$

$$K_\tau^g(\Gamma) = \int_0^\tau dt \sum_m \eta_m(t) g_m(\omega t) \quad \to \langle K_\tau^{g^{\mathrm{EB}}} \rangle = -\int_0^\tau dt \sum_{n\neq m} \Lambda_{nm} R_{nm} p_m$$

$$g_m^{\mathrm{EB}} = -\sum_{n(\neq m)} \Lambda_{nm} R_{nm}$$

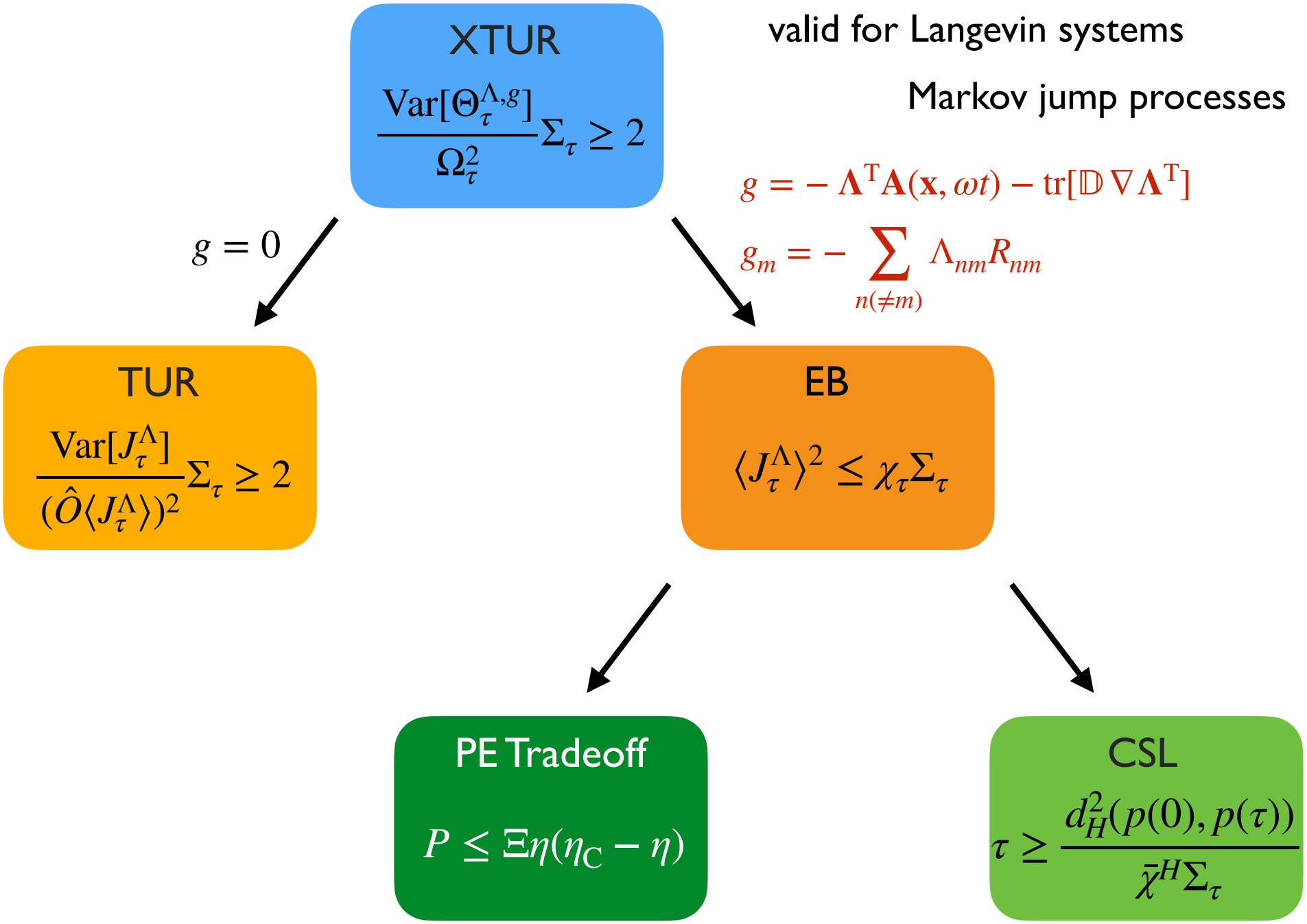
$$\Theta_\tau^{\Lambda,g^{\mathrm{EB}}}(\Gamma) = J_\tau^\Lambda(\Gamma) + K_\tau^{g^{\mathrm{EB}}}(\Gamma)$$

$$\langle \Theta_\tau^{\Lambda,g^{\mathrm{EB}}} \rangle = 0$$

$$\mathrm{Var}(\Theta_\tau^{\Lambda,g^{\mathrm{EB}}}) = \int_0^\tau dt \sum_{n\neq m} \Lambda_{nm}^2 R_{nm} p_m \equiv 2\chi_\tau \quad \Rightarrow \quad \langle J_\tau^\Lambda \rangle^2 \leq \chi_\tau \Sigma_\tau$$

$$\Omega_\tau \equiv \hat{O}_\tau \langle \Theta_\tau^{\Lambda,g^{\mathrm{EB}}} \rangle - \langle K_\tau^{g^{\mathrm{EB}}} \rangle = \langle J_\tau^\Lambda \rangle$$

Unified Hierarchical Relationship



EB ⟶ PE tradeoff

EB

$$\langle J_\tau^\Lambda \rangle^2 \le \chi_\tau \Sigma_\tau$$

PE Tradeoff

$$P \le \Xi \eta (\eta_C - \eta)$$

Observable

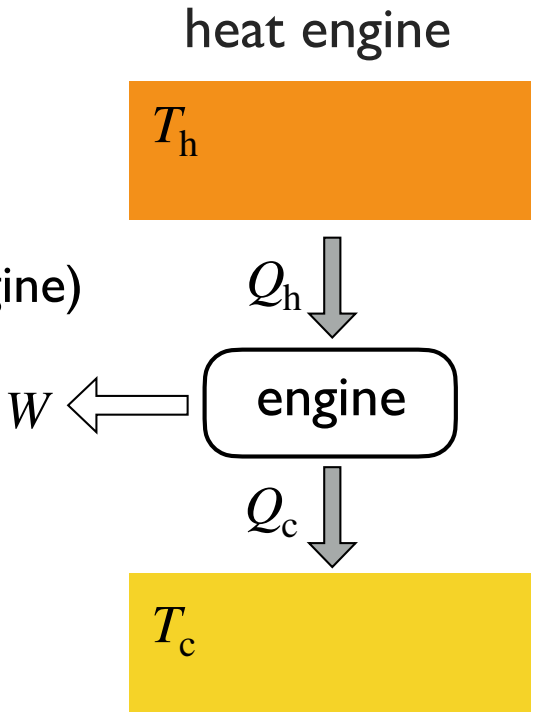
$$\langle J_\tau^\Lambda \rangle = \langle Q_h \rangle$$

$$\Sigma_\tau = -\frac{\langle Q_h \rangle}{T_h} + \frac{\langle Q_c \rangle}{T_c} \quad \text{(steady state, cyclic engine)}$$

$$= \frac{\langle Q_h \rangle}{T_c}(\eta_C - \eta) \quad \leftarrow \eta = \frac{\langle W \rangle}{\langle Q_h \rangle}$$

$$\eta_C = 1 - \frac{T_c}{T_h}$$

$$\langle W \rangle = \langle Q_h \rangle - \langle Q_c \rangle$$



$$\bar{\chi} \equiv \frac{\chi_\tau}{\tau}$$

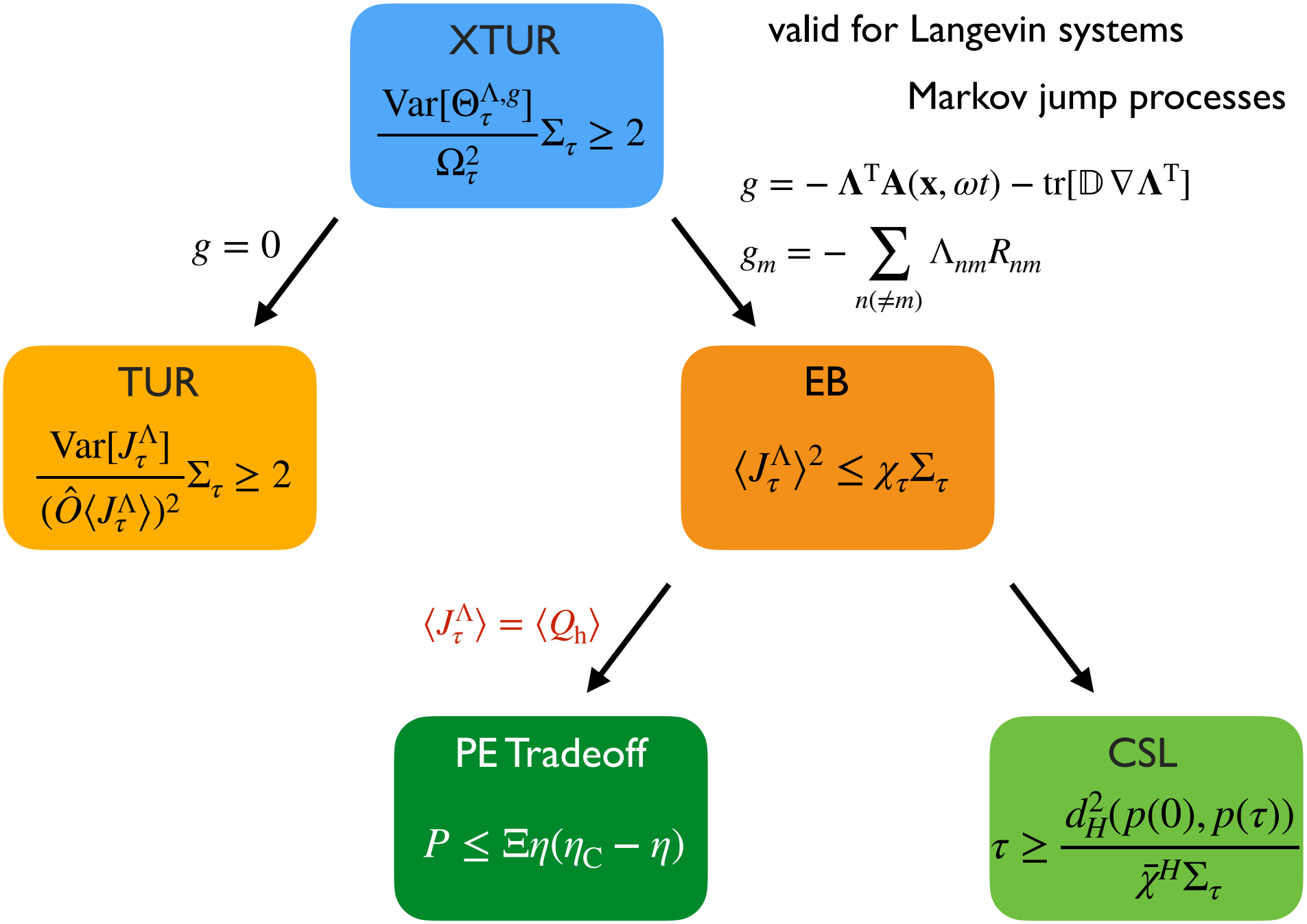
$$\chi_\tau = \frac{1}{2}\int_0^\tau dt \sum_{n \ne m} \Lambda_{nm}^2 R_{nm} p_m \ : \text{Markov jump}$$

$$\chi_\tau = \int_0^\tau dt \langle \mathbf{\Lambda}^{\mathrm{T}} \mathbb{D} \mathbf{\Lambda} \rangle \ : \text{Langevin}$$

$$\text{EB} \rightarrow \langle Q_h \rangle^2 \le \bar{\chi}\tau \frac{\langle Q_h \rangle}{T_c}(\eta_C - \eta) \qquad \therefore P = \frac{\langle W \rangle}{\tau} \le \frac{\bar{\chi}}{T_c}\eta(\eta_C - \eta)$$

$$\langle Q_h \rangle^2 \times \frac{\langle W \rangle}{\tau \langle Q_h \rangle^2} \le \bar{\chi}\tau \frac{\langle Q_h \rangle}{T_c} \times \frac{\langle W \rangle}{\tau \langle Q_h \rangle^2}(\eta_C - \eta)$$

Unified Hierarchical Relationship



EB ⟶ CSL

EB

$$\langle J_\tau^\Lambda \rangle^2 \le \chi_\tau \Sigma_\tau$$

CSL

$$\tau \ge \frac{d_H^2(p(0), p(\tau))}{\bar{\chi}^H \Sigma_\tau}$$

$h \in H$
: continuous
almost everywhere differentiable

(Langevin)

$$\langle J_\tau^\Lambda \rangle = \int_0^\tau dt \int d\mathbf{x}\, \mathbf{\Lambda}^{\mathrm{T}} \mathbf{J} \qquad \rightarrow \mathbf{\Lambda} = \nabla h(\mathbf{x})$$

(Markov jump)

$$\langle J_\tau^\Lambda \rangle = \int_0^\tau dt \sum_{n>m} \Lambda_{nm}(\omega t) j_{nm}(t;\omega) \rightarrow \Lambda_{nm} = h_n - h_m \equiv \nabla h_{nm}$$

$$\rightarrow \langle J_\tau^\Lambda \rangle = \langle h \rangle_{p(\tau)} - \langle h \rangle_{p(0)}$$

EB : $$\left(\langle h \rangle_{p(\tau)} - \langle h \rangle_{p(0)} \right)^2 \le \chi_\tau \Sigma_\tau$$

taking supremum over $h \in H$: $d_H^2(p(\tau), p(0)) \le \chi_\tau^H \Sigma_\tau$

$$d_H(p,q) = \sup_{h \in H} \left| \langle h \rangle_p - \langle h \rangle_q \right| \text{ : Integral Probability Metric (IPM)}$$

$$\chi_\tau^H \equiv \sup_{h \in H} \int_0^\tau dt \int_{\mathbb{R}^N \setminus V_h} d\mathbf{x}\, p \nabla h^T \mathbb{D} \nabla h \quad \text{(Langevin)}$$

$$\chi_\tau^H = \frac{1}{2} \sup_{h \in H} \int_0^\tau dt \sum_{n<m} (h_n - h_m)^2 a_{nm} \quad \text{(Markov jump)}$$

EB ⟶ CSL

EB

$$\langle J_\tau^\Lambda \rangle^2 \leq \chi_\tau \Sigma_\tau$$

↓

CSL

$$\tau \geq \frac{d_H^2(p(0), p(\tau))}{\bar{\chi}^H \Sigma_\tau}$$

taking supremum over $h \in H$: $d_H^2(p(\tau), p(0)) \leq \chi_\tau^H \Sigma_\tau$

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$$\Rightarrow \bar{\chi}^H \equiv \frac{\chi_\tau^H}{\tau}$$

$$\therefore \tau \geq \frac{d_H^2(p(0), p(\tau))}{\bar{\chi}^H \Sigma_\tau}$$: pristine form of speed limit

→ previously reported speed limits

EB ⟶ CSL

EB

$$\langle J_\tau^\Lambda \rangle^2 \le \chi_\tau \Sigma_\tau$$

CSL

$$\tau \ge \frac{d_H^2(p(0), p(\tau))}{\bar{\chi}^H \Sigma_\tau}$$

$$d_H(p,q) = \sup_{h \in H} \left| \langle h \rangle_p - \langle h \rangle_q \right| \text{ : Integral Probability Metric (IPM)}$$

Choose

$H = H_{\mathrm{Lip}_1}$: Lipschitz-1 $\qquad \mathrm{Lip}_1 \equiv \{h \,|\, d(x,y) \ge |h(x) - h(y)|, \forall x, y\}$

$\rightarrow d_{\mathrm{Lip}_1}(p,q) = W_1(p,q)$: Wasserstein-1 distance

$$\Rightarrow \tau \ge \frac{W_1^2(p(0), p(\tau))}{\bar{\chi}^H \Sigma_\tau}$$

$H = H_1$: set of functions bounded within the range $[-1,1]$

$\rightarrow d_{H_1}(p,q) = 2 d_{\mathrm{TV}}(p,q)$: total variation distance

$$d_{\mathrm{TV}}(p,q) \equiv \frac{1}{2} \sum_{x \in \Omega} |p(x) - q(x)| = \frac{1}{2} \sup_{h \in [-1,1]} |\langle h \rangle_p - \langle h \rangle_q|$$

$$\Rightarrow \tau \ge \frac{4 d_{\mathrm{TV}}^2(p(0), p(\tau))}{\bar{\chi}^H \Sigma_\tau}$$

$$\therefore \tau \ge \frac{d_H^2(p(0), p(\tau))}{\bar{\chi}^H \Sigma_\tau}$$: pristine form of speed limit

$\rightarrow$ previously reported speed limits

EB ⟶ CSL

EB

$$\langle J_\tau^\Lambda \rangle^2 \le \chi_\tau \Sigma_\tau$$

CSL

$$\tau \ge \frac{d_H^2(p(0),p(\tau))}{\bar{\chi}^H \Sigma_\tau}$$

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$H = H_{\mathrm{Lip}_1}$: Lipschitz-1 $\quad \mathrm{Lip}_1 \equiv \{h \,|\, d(x,y) \ge |h(x)-h(y)|, \forall x,y\}$

$$\Rightarrow \tau \ge \frac{W_1^2(p(0),p(\tau))}{\bar{\chi}^H \Sigma_\tau}$$

(Langevin)

$$\Rightarrow \tau \ge \frac{W_1^2(p(0),p(\tau))}{\Sigma_\tau^{\mathrm{tot}} \overline{\langle \rho(\mathbb{D}) \rangle}}$$

$\rho(\mathbb{D})$: largest eigenvalue of $\mathbb{D}$

valid even when $\mathbb{D}$ depends on time and space

cf. $\tau \ge \dfrac{W_2^2(p(0),p(\tau))}{\Sigma_\tau^{\mathrm{tot}} D}$

Dechant and Sakurai, arXiv:1912.08405
Vu and Saito, PRX 13, 011013 (2023)
Nakazato and Ito, PRR 3, 043093 (2021)

valid for only constant $\mathbb{D}$

(Markov jump)

$$\Rightarrow \tau \ge \frac{W_1^2(p(0),p(\tau))}{\Sigma_\tau \bar{A}/2}$$

Shiraish, Funo, Saito, PRL 121, 070601 (2018)

$$\Rightarrow \tau \ge \frac{W_1}{\bar{A} \tanh\left(\frac{\Sigma_\tau^{\mathrm{tot}}}{2W_1}\right)}$$

: tighter bound

Vu and Saito, PRX 13, 011013 (2023)
Dechant, JPhysA: Math. Their 55, 094001 (2022)

$H = H_1$: set of functions bounded within the range $[-1,1]$

$$\Rightarrow \tau \ge \frac{4 d_{\mathrm{TV}}^2(p(0),p(\tau))}{\bar{\chi}^H \Sigma_\tau}$$

EB ⟶ CSL

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$$\langle J_\tau^\Lambda \rangle^2 \le \chi_\tau \Sigma_\tau$$

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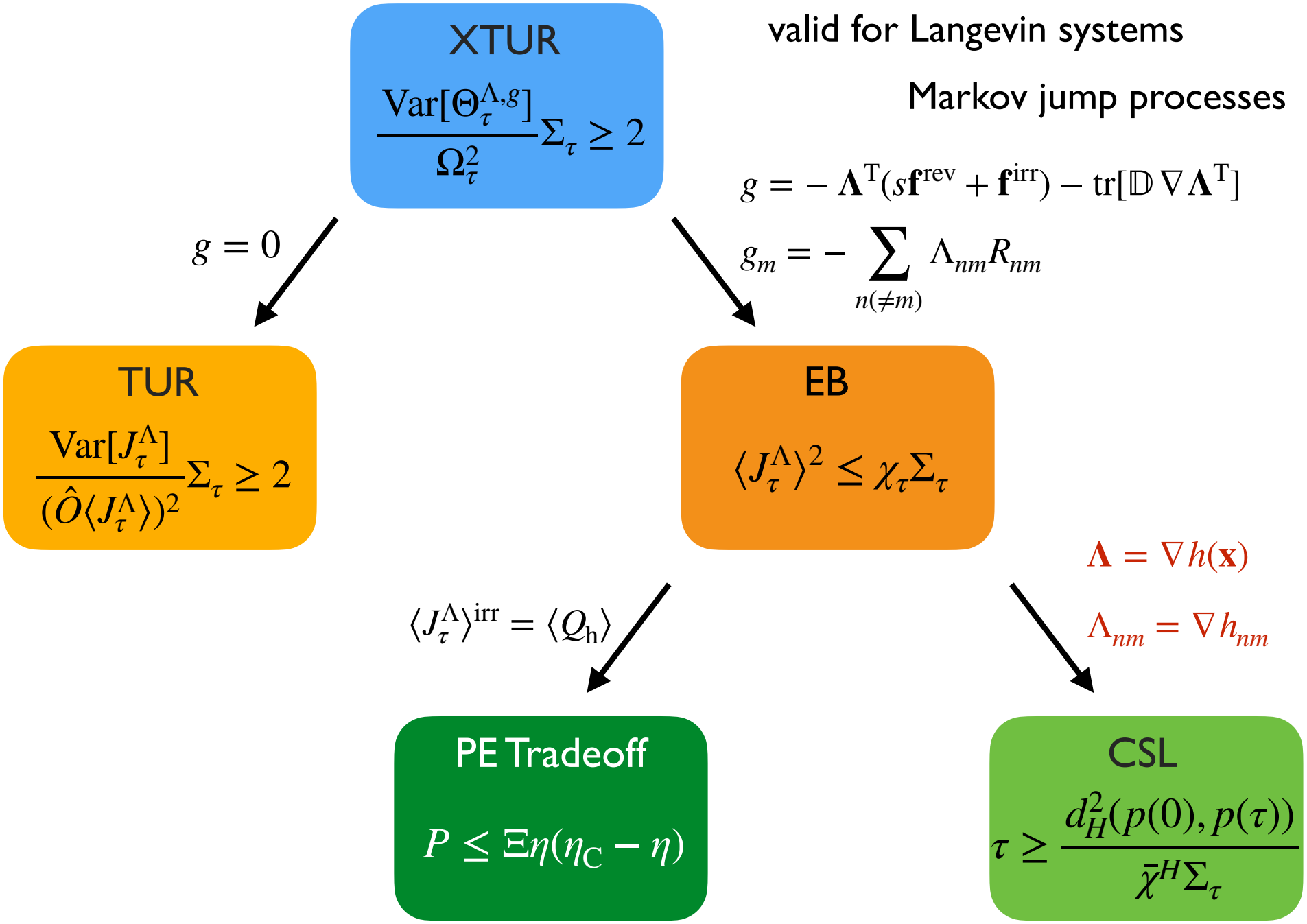
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$$\Rightarrow \tau \ge \frac{d_{\mathrm{TV}}}{\bar{A} \tanh\left(\frac{\Sigma_\tau^{\mathrm{tot}}}{2 d_{\mathrm{TV}}}\right)}$$: tighter bound

JSL, Lee, Kwon, Park, PRL 129, 120603 (2022)

Unified Hierarchical Relationship



Numerical Confirmation for New Speed Limits

Example: inhomogeneous temperature field

Equation of motion: $\gamma\dot{x} = -\dfrac{\partial V(x,t)}{\partial x} + \sqrt{2\gamma T(x,t)} \bullet \xi(t) \qquad (\gamma = 1)$

Space-dependent temperature: $T(x) = \begin{cases} T_1 & \text{for } x < 0 \\ T_2 & \text{for } x \geq 0 \end{cases}$

When $t = 0$: equilibrium at $T_1 = T_2 = 1$ with $V(x) = \dfrac{1}{2}k(x - x_0)^2$

When $t > 0$: $T_1 = 1,\ T_2 = 4$ with $V(x,t)$

CSL:

$$\tau \geq \frac{W_1^2(p(0), p(\tau))}{\Sigma_\tau^{\mathrm{tot}} \overline{\langle \rho(\mathbb{D}) \rangle}} \qquad \rho(\mathbb{D}) = T(x,t) \Rightarrow \sqrt{\Sigma_\tau^{\mathrm{tot}} \tau \overline{\langle T \rangle}} \geq W_1(p(0), p(\tau))$$

valid even when $\mathbb{D}$ depends on x, t

$$\tau \geq \frac{W_2^2(p(0), p(\tau))}{\Sigma_\tau^{\mathrm{tot}} D} \Rightarrow \sqrt{\Sigma_\tau^{\mathrm{tot}} \tau T} \geq W_2(p(0), p(\tau))$$

derived for constant $\mathbb{D}$

Dechant and Sakurai, arXiv:1912.08405
Vu and Saito, PRX 13, 011013 (2023)
Nakazato and Ito, PRR 3, 043093 (2021)

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CSL:

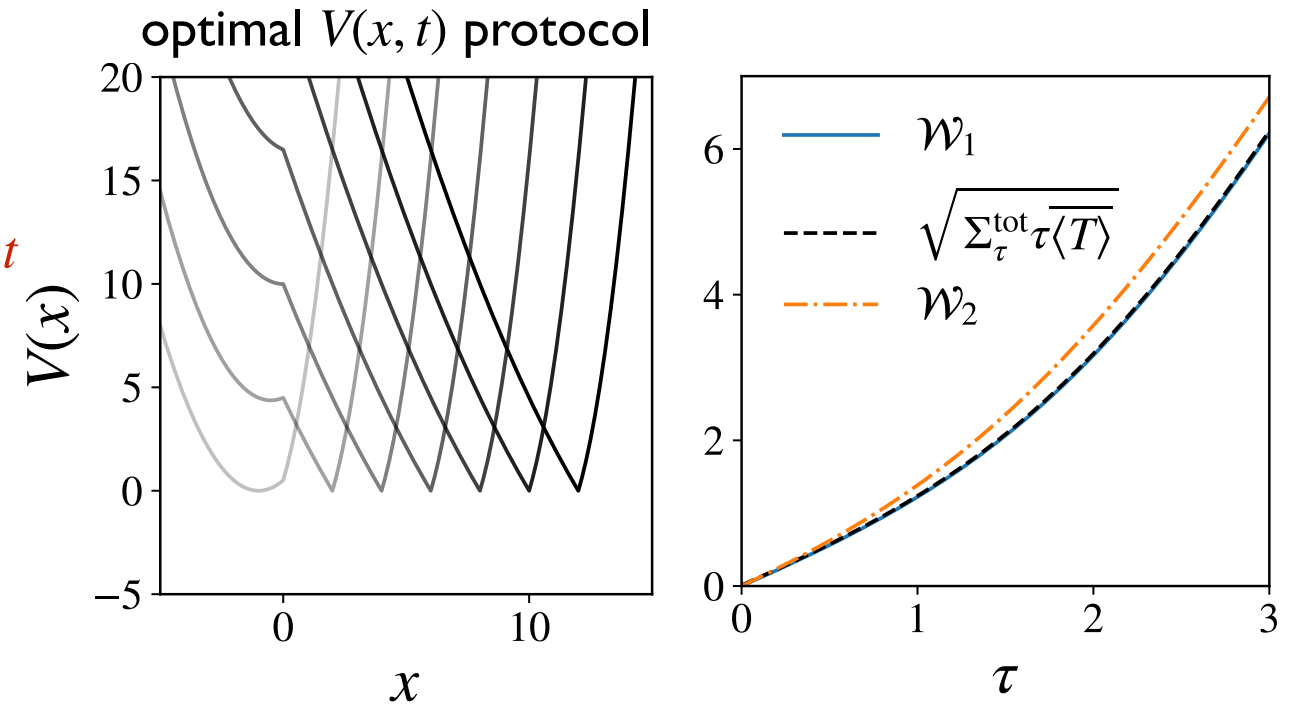
$$\sqrt{\Sigma_\tau^{\mathrm{tot}} \tau \overline{\langle T \rangle}} \geq W_1(p(0), p(\tau))$$

valid even when $\mathbb{D}$ depends on x, t

$$\sqrt{\Sigma_\tau^{\mathrm{tot}} \tau \overline{\langle T \rangle}} \geq W_2(p(0), p(\tau))$$

valid for inhomogeneous $\mathbb{D}$?

Dechant and Sakurai, arXiv:1912.08405
Vu and Saito, PRX 13, 011013 (2023)
Nakazato and Ito, PRR 3, 043093 (2021)



Summary

1. We unveiled the unified hierarchical relationship between tradeoff relations.
   - TUR and EB are derived from XTUR.
   - PE tradeoff and CSL are derived from EB.

2. We derived new tradeoff relations
   - XTUR
   - EB for Markov jump process
   - PE tradeoff for Markov jump process
   - CSL for time- and space-varying temperature

3. TUR, EB, PE tradeoff, CSL can be categorized into a single class, "XTUR class".
   - Carnot bound, Landauer's principle: second-law class
   - if new bound (not XTUR class) found, it is really NEW class bound.

4. Preprint arXiv:2311.01098

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