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Exploring collective behavior in a swarmalator system

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Swarmalator? $\Longrightarrow$

Oscillators that can sync and swarm

Short for "swarming oscillators"

Ref. K. P. O'Keeffe, H. Hong, and S. H. Strogatz, Nature Commun. 8, 1504 (2017)



S.H. Strogatz and I. Stewart.
Sci. Am. 269 (6), Dec. 68 (1993).

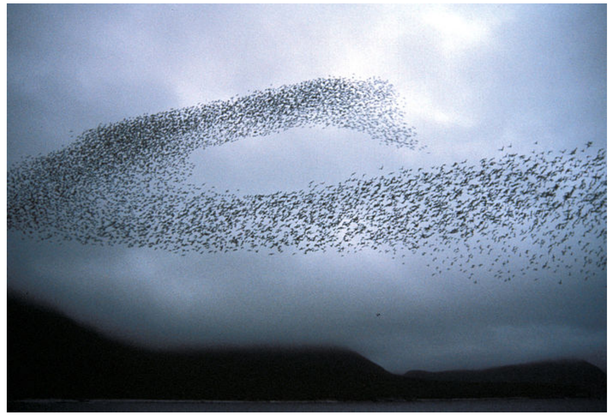
Synchronization

- simultaneous blinking on and off of fireflies
- chorusing crickets
- synchronized clapping of audience
- laser arrays
- heart cells

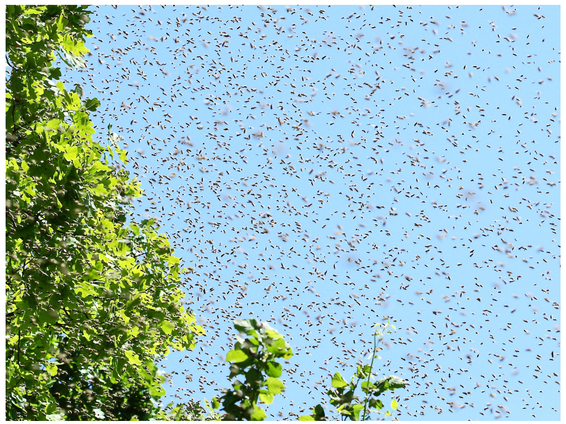
$\Longrightarrow$ Self-organization in "**time**"

Swarming behavior: Self-organization in "space"

A flock of auklets



A swarm of bees





Schooling anchovies



A swarm of jellyfish

- Synchronizing units self-organize **in time**, but not in space.
- Swarming units self-organize **in space**, but not in time.

"Spatiotemporal" self-organization

Photos in https://en.wikipedia.org/wiki/Swarm_behaviour#/

A lot of studies have been conducted so far:

D. Tanaka, Phys. Rev. Lett. **99**, 134103 (2007);

M. Iwasa and D. Tanaka, Phys. Lett. A **381**, 3054 (2017).

→ **K. P. O'Keeffe, H. Hong, and S. H. Strogatz, Nat. Commun. 8, 1504 (2017).**

D. Levis, I. Pagonabarraga, and B. Liebchen, Phys. Rev. Research **1**, 023026 (2019).

B. Zhang, A. Sokolov, and A. Snezhko, Nat. Commun. **11**, 4401 (2020).

A. Barcis and C. Betasteter, IEEE Access **8**, 218752 (2020).

H. Hong, Chaos 28, 103112 (2018).

K. P. O'Keeffe, J. H. M. Evers, and T. Kolokolnikov, Phys. Rev. E **98,** 022203 (2018).

T. A. McLennan-Smith, D. O. Roberts, and H. S. Sidhu, Phys. Rev. E **102**, 032607 (2020).

H. Hong, K. Yeo, and H. K. Lee, PRE 104, 044214 (2021).

H. K. Lee, K. Yeo, and H. Hong, Chaos 31, 033134 (2021).

K. P. O'Keeffe and H. Hong, Phys. Rev. E 105, 064208 (2022).

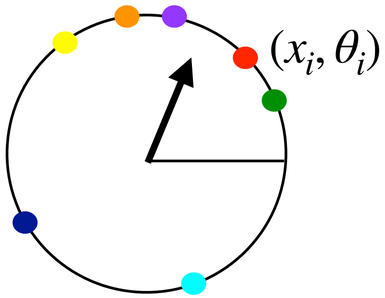
→ **H. Hong, K. P. O'Keeffe, J. S. Lee, and H. Park, Phys. Rev. Research 5, 023105 (2023).**

and so on.

Coupled identical swarmalators on a 1D ring

$$\dot{x}_i = \frac{J}{N}\sum_{j=1}^{N}\sin(x_j - x_i)\cos(\theta_j - \theta_i) + \xi_i^x \quad \cdots\cdots \quad (1)$$

$$\dot{\theta}_i = \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)\cos(x_j - x_i) + \xi_i^\theta \quad \cdots\cdots \quad (2)$$



where

$(x_i, \theta_i) \in (S^1, S^1)$: position and phase of the ith swarmalator, periodic with a period 2π

J, K : coupling strengths

ξ_i^x, ξ_i^θ : **thermal noise**

$$\langle \xi_i^x(t) \rangle = 0, \qquad \langle \xi_i^x(t)\xi_j^x(t') \rangle = 2D_x\delta_{ij}\delta(t - t')$$

$$\langle \xi_i^\theta(t) \rangle = 0, \qquad \langle \xi_i^\theta(t)\xi_j^\theta(t') \rangle = 2D_\theta\delta_{ij}\delta(t - t')$$

D_x, D_θ : "**temperature**"

The two noises are independent of each other, i.e., $\langle \xi_i^x(t)\xi_j^\theta(t') \rangle = 0$

What the equation signifies:

$$\dot{x}_i = \frac{J}{N}\sum_{j=1}^{N}\sin(x_j - x_i)\cos(\theta_j - \theta_i) + \xi_i^x \quad \cdots\cdots \quad (1)$$

$$\dot{\theta}_i = \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)\cos(x_j - x_i) + \xi_i^\theta \quad \cdots\cdots \quad (2)$$

The sine terms induce the swarmalators to reduce their phase difference for K>0, while the cosine term implies that the synchronization tendency is stronger for oscillators that are closer in space.

→ “position-dependent synchronization”

“phase-dependent swarming”

How does this model relate to other models?

- **Nonequilibrium generalization of XY model** for a magnetic system, wherein spins possess two degrees of freedom, x and θ, that are coupled.

(i) For $J = K$ and $D_x = D_\theta$ (EQ XY)

$$\mathscr{H} = -\frac{J}{2N}\sum_{i,j=1}^{N}\cos(x_j - x_i)\cos(\theta_j - \theta_i)$$

(ii) For $J \neq K \longrightarrow$ NEQ , where various NEQ (magnetic) phase can emerge.

(iii) For $D_x \neq D_\theta \longrightarrow$ heat engine under the thermal gradient

- Our model can be thought of as **synchronization occurring on a torus.**

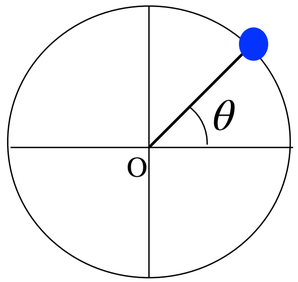
"Generalized version of the **thermal Kuramoto model**"

Kuramoto model (KM)

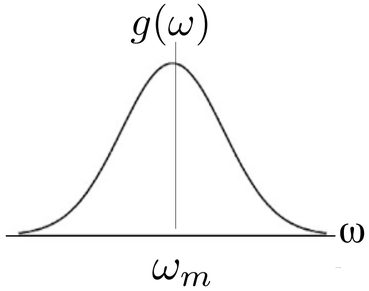
$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)$$

$$i = 1, \cdots, N$$

θ_i : phase (angle) of oscillator i



ω_i: natural frequency of oscillator i

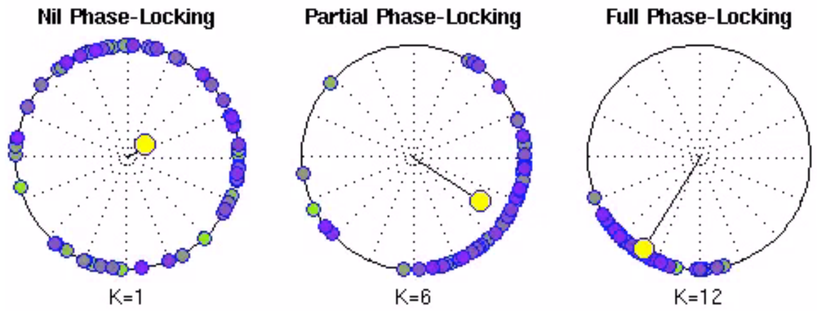


$$\dot{\theta}_i = \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) + \xi_i(t)$$

"Thermal KM"

thermal noise

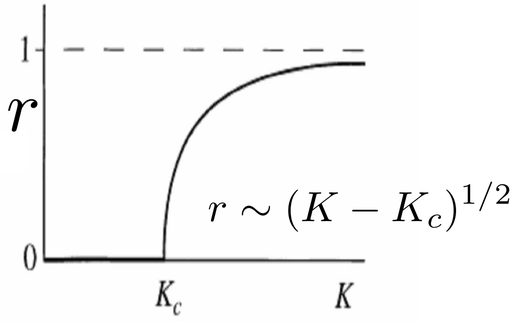
Kuramoto Oscillators



Nil, partial and full phase-locking in an all-to-all network of Kuramoto oscillators. Phase-locking is governed by the coupling strength K and the distribution of intrinisic frequencies ω. Here, the intrinsic frequencies were drawn from a normal distribution (M=0.5Hz, SD=0.5Hz). The yellow disk marks the phase centroid. Its radius is a measure of coherence.

https://en.wikipedia.org/wiki/File:KuramotoModelPhaseLocking.ogv

Order parameter: $re^{i\Theta} \equiv \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$



Let's start exploring our model now:

$$\dot{x}_i = \frac{J}{N}\sum_{j=1}^{N}\sin(x_j - x_i)\cos(\theta_j - \theta_i) + \xi_i^x \quad \cdots\cdots \quad (1)$$

$$\dot{\theta}_i = \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)\cos(x_j - x_i) + \xi_i^\theta \quad \cdots\cdots \quad (2)$$

$$i = 1, \cdots, N$$

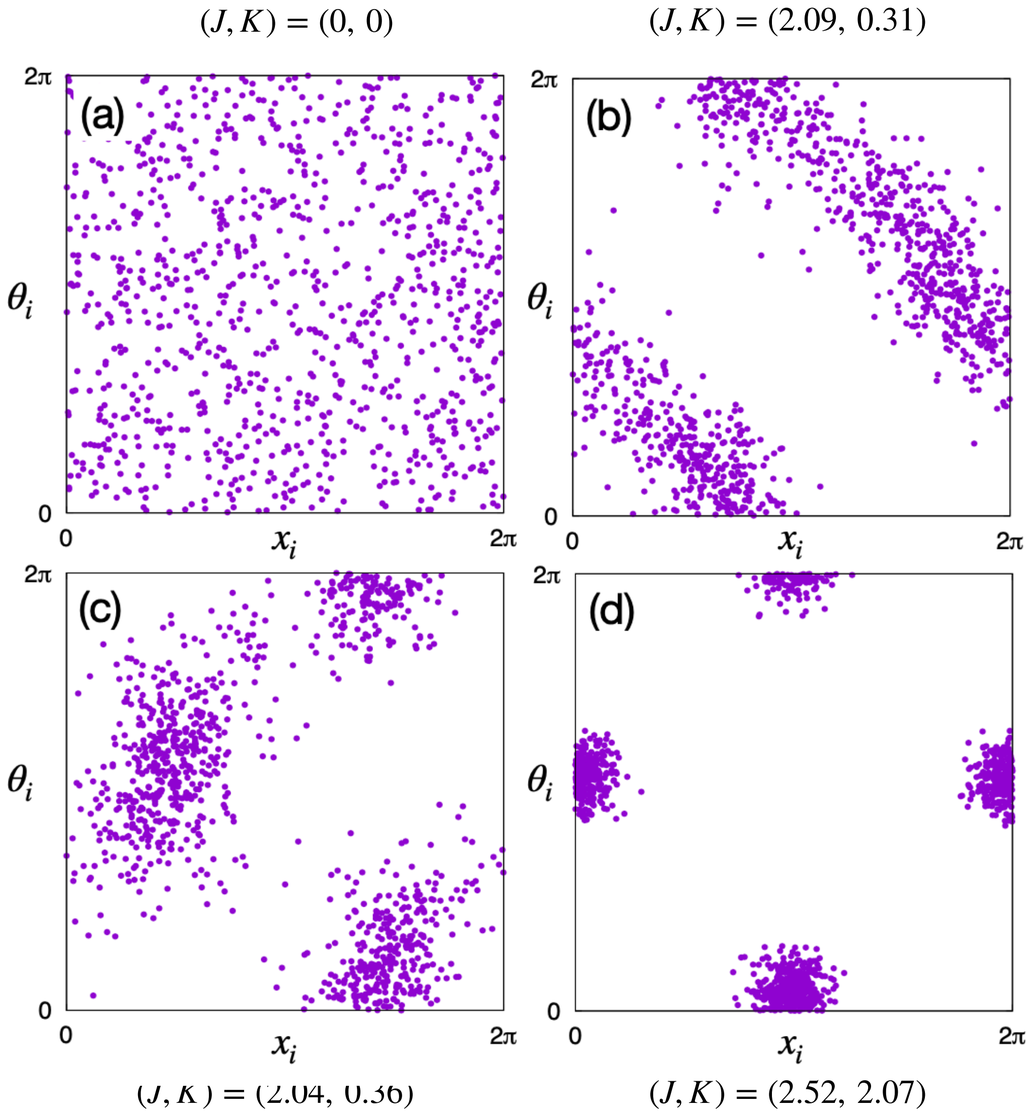
What order parameters would be good?

One may consider the **Kuramoto synchronization order parameters**:

$$Z_x \equiv R_x e^{i\Theta_x} = \frac{1}{N}\sum_{j=1}^{N}\langle e^{ix_j}\rangle,$$

$$Z_\theta \equiv R_\theta e^{i\Theta_\theta} = \frac{1}{N}\sum_{j=1}^{N}\langle e^{i\theta_j}\rangle,$$

To gain insight into the order parameters



(a) uniformly distributed state

(b) **linearly correlated** between x and θ

(c) **mixed** between (b) and (d)

(d) two well-defined isotropic clusters separated by a **distance of π** in both directions.

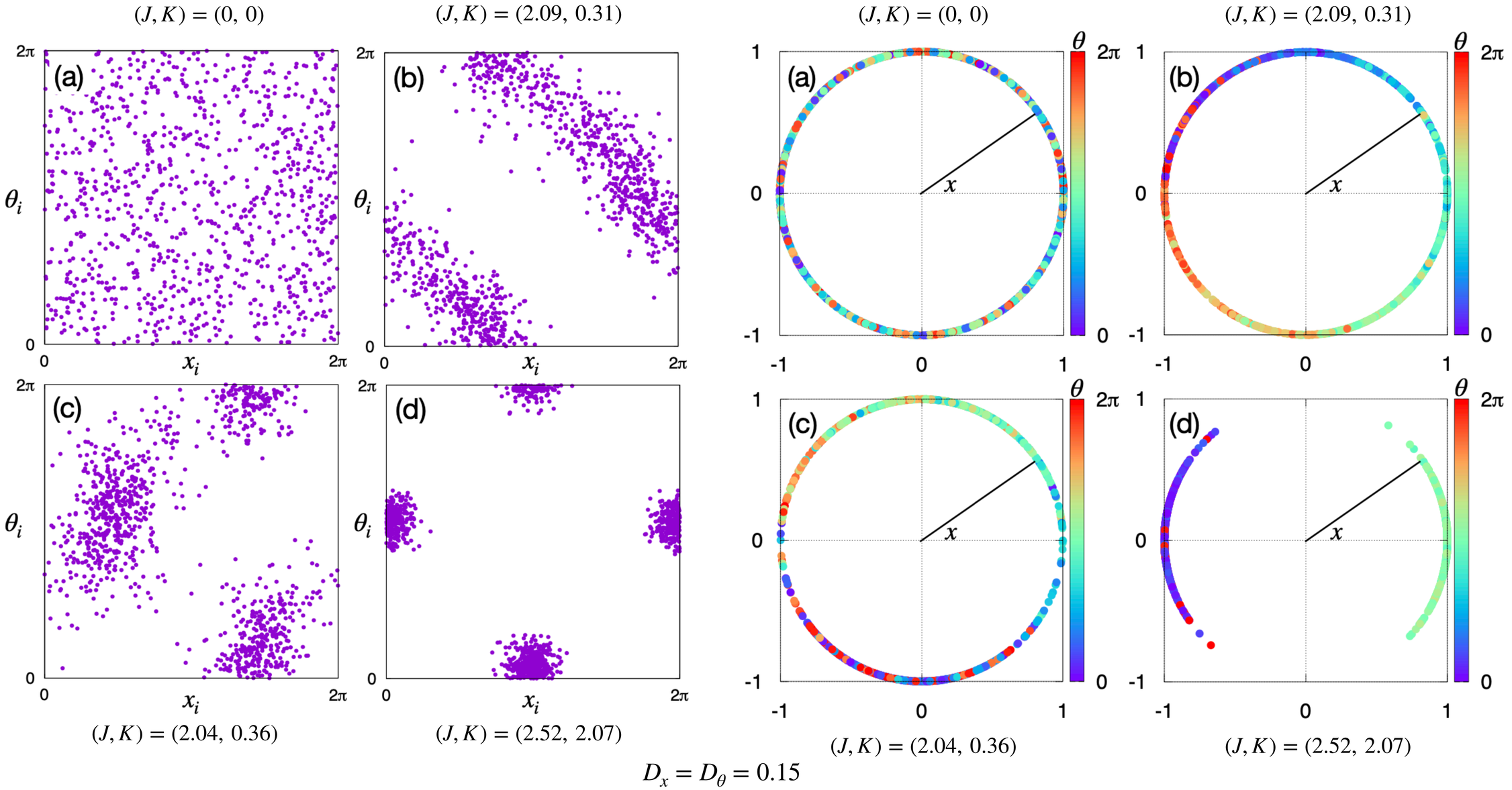
π-symmetry in Eq. (1) and (2):
invariant under

$$x_j \to x_j + \pi$$
$$\theta_j \to \theta_j + \pi$$

The dynamics of any (x, θ) state is equivalent to that of the $(x + \pi,\ \theta + \pi)$ state.
The Kuramoto order parameters are zero!!! :

→ $Z_x = Z_\theta = 0$

Long-term states



Alternative order parameters:

$$W_+ \equiv S_+ e^{i\Phi_+} = \frac{1}{N}\sum_{j=1}^{N}\langle e^{i(x_j+\theta_j)}\rangle,$$

$$W_- \equiv S_- e^{i\Phi_-} = \frac{1}{N}\sum_{j=1}^{N}\langle e^{i(x_j-\theta_j)}\rangle,$$

"**correlation measure**" between x and θ.

This particular form of the correlation measure can be inferred from the linearity of correlations between x and θ with slopes of ± 1.

With these order parameters,

(a) **Disordered (incoherent) state** with $(S_+,\ S_-) = (0,\ 0)$

(b) **Phase wave state** with $(S_+,\ S_-) = (S,\ 0)$ or $(0,\ S)$

(c) **Mixed state** with $(S_+,\ S_-) = (S_1,\ S_2)$ or $(S_2,\ S_1)$, where $S_1 \neq S_2$

(d) **Synchronized (ordered) state** with $(S_+,\ S_-) = (S,\ S)$

Coordinate transformation

$$\dot{x}_i = \frac{J}{N}\sum_{j=1}^{N}\sin(x_j - x_i)\cos(\theta_j - \theta_i) + \xi_i^x \quad \cdots\cdots \quad (1)$$

$$\dot{\theta}_i = \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)\cos(x_j - x_i) + \xi_i^\theta \quad \cdots\cdots \quad (2)$$

$(x_i,\ \theta_i) \longrightarrow (X_i,\ Y_i)$

$X_i \equiv x_i + \theta_i$ and $Y_i \equiv x_i - \theta_i$

Eq. (1) and (2) $\longrightarrow$

$$\dot{X}_i = J_+ S_+ \sin(\Phi_+ - X_i) + J_S_\sin(\Phi_ - Y_i) + \xi_i^X \quad \cdots\cdots \quad (3)$$

$$\dot{Y}_i = J_S_+ \sin(\Phi_+ - X_i) + J_+ S_\sin(\Phi_ - Y_i) + \xi_i^Y \quad \cdots\cdots \quad (4)$$

where

$$J_\pm = \frac{J \pm K}{2}$$

$$S_+ e^{i\Phi_+} = \frac{1}{N}\sum_{j=1}^{N}\langle e^{iX_j}\rangle\ , \qquad S_e^{i\Phi_} = \frac{1}{N}\sum_{j=1}^{N}\langle e^{iY_j}\rangle$$

← Order parameters

$$\xi_i^X = \xi_i^x + \xi_i^\theta$$

$$\xi_i^Y = \xi_i^x - \xi_i^\theta$$

not independent of each other in general

$$\langle \xi_i^X \rangle = \langle \xi_i^Y \rangle = 0$$

$$\langle \xi_i^X(t)\xi_j^X(t')\rangle = 2D_X\delta_{ij}\delta(t-t')$$

$$\langle \xi_i^Y(t)\xi_j^Y(t')\rangle = 2D_Y\delta_{ij}\delta(t-t')$$

$$\langle \xi_i^X(t)\xi_j^Y(t')\rangle = 2D_{XY}\delta_{ij}\delta(t-t')$$

$D_{XY} = 0$ for $D_x = D_\theta$.

$$D_X = D_Y = D_x + D_\theta \equiv D$$

$$D_{XY} = D_x - D_\theta \equiv d$$

When $J = K$ $(J_- = 0)$ and $D_x = D_\theta$ $(d = 0)$,

$$\dot{X}_i = J_+ S_+ \sin(\Phi_+ - X_i) + J_- S_- \sin(\Phi_- - Y_i) + \xi_i^X \quad \cdots\cdots \quad (3)$$

$$\dot{Y}_i = J_- S_+ \sin(\Phi_+ - X_i) + J_+ S_- \sin(\Phi_- - Y_i) + \xi_i^Y \quad \cdots\cdots \quad (4)$$

$\Longrightarrow$ **EQ XY** model for X and Y, which can be solved exactly.

EQ magnetic transition occurs at $J_c = 2D_X = 4D_x$ from $(0,0)$ to (S, S) without an intermediate phase wave or mixed state.

It may emerge when the system is driven out of EQ.,
either by an external **nonconservative force** $(J \neq K)$
or a thermal gradient caused by **multiple reservoirs** $(D_x \neq D_\theta)$.

Fourier mode analysis

Probability density function (PDF): $\rho(X, Y, t)$

X, Y : periodic with 2π $\Longrightarrow$ normalization condition: $\int_0^{2\pi}\int_0^{2\pi} dXdY\rho(X, Y, t) = 1$

Fokker-Planck equation $$\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial X}\mathcal{J}_X - \frac{\partial}{\partial Y}\mathcal{J}_Y \quad \cdots\cdots (5)$$

where $$\mathcal{J}_X = (J_+F_+ + J_-F_-)\rho - D\frac{\partial\rho}{\partial X} - d\frac{\partial\rho}{\partial Y}$$

$$\mathcal{J}_Y = (J_-F_+ + J_+F_-)\rho - D\frac{\partial\rho}{\partial Y} - d\frac{\partial\rho}{\partial X}$$

$$D = D_X = D_Y = D_x + D_\theta$$

$$d = D_{XY} = D_x - D_\theta$$

$$J_\pm = \frac{J \pm K}{2}$$

with $F_+ = S_+\mathrm{Im}\, e^{i(\Phi_+ - X)}$ and $F_- = S_-\mathrm{Im}\, e^{i(\Phi_- - Y)}$

$\Longrightarrow$ $$\rho(X, Y, t) = \frac{1}{(2\pi)^2}\sum_{n,m=-\infty}^{\infty}\alpha_{n,m}(t)e^{in(X-\Phi_+)}e^{im(Y-\Phi_-)} \quad \cdots\cdots (6)$$

where $\alpha_{00} = 1$ and $\alpha^*_{nm} = \alpha_{-n,-m}$

$$\rho(X,Y,t) = \frac{1}{(2\pi)^2} \sum_{n,m=-\infty}^{\infty} \alpha_{n,m}(t) e^{in(X-\Phi_+)} e^{im(Y-\Phi_-)}$$

The order parameters can be expressed in terms of Fourier coefficients as

$$S_+ = \iint dX dY e^{i(X-\Phi_+)} \rho(X,Y,t) = \alpha_{-1,0} = \alpha^*_{1,0} = \alpha_{1,0}$$

$$S_- = \iint dX dY e^{i(Y-\Phi_-)} \rho(X,Y,t) = \alpha_{0,-1} = \alpha^*_{0,1} = \alpha_{0,1}$$

$S_\pm$: real $\rightarrow$ $\alpha_{1,0}$ and $\alpha_{0,1}$: real

Mode coupled equations:

$$\begin{aligned}
\dot{\alpha}_{n,m} - i(n\dot{\Phi}_+ + m\dot{\Phi}_-)\alpha_{n,m} &= -[D(n^2+m^2) + 2dnm]\alpha_{n,m} \\
&\quad + \frac{1}{2}(nJ_+ + mJ_-)S_+(\alpha_{n-1,m} - \alpha_{n+1,m}) \\
&\quad + \frac{1}{2}(nJ_- + mJ_+)S_-(\alpha_{n,m-1} - \alpha_{n,m+1})
\end{aligned} \qquad \cdots\cdots (7)$$

Steady-state solutions

The steady-state solutions can be obtained by setting $\dot{\alpha}_{n,m} = \dot{\Phi}_{\pm} = 0.$

(1) **Incoherent state** $(0,\ 0)$

$\alpha_{n,m} = 0$ except for $\alpha_{0,0} = 1$, resulting in $S_+ = S_- = 0$ and $\rho = 1/(2\pi)^2$.

(2) **Phase wave state** $(S,\ 0)$ or $(0,\ S)$

With $S_- = 0$, we find the steady-state equations for $\alpha_{n,0}$ as

$$0 = -Dn\alpha_{n,0} + \frac{1}{2}J_+S_+(\alpha_{n-1,0} - \alpha_{n+1,0}).$$

Using the recurrence relation of the modified Bessel function, we find the solution is given by

$$\alpha_{n,0} = \frac{I_{|n|}\left(\frac{J_+S_+}{D}\right)}{I_0\left(\frac{J_+S_+}{D}\right)}$$

$\alpha_{1,0} = S_+$

For small S_+, we obtain $S_+ \approx \sqrt{2(J_+/2D - 1)}$ for $J_+/D \geq 2.$

$$\rho = \frac{1}{(2\pi)^2}\frac{e^{(J_+S_+/D)\cos(X-\Phi_+)}}{I_0(J_+S_+/D)}$$

(3) Mixed state $(S_1,\ S_2)$ or $(S_2,\ S_1)$

(4) Ordered state $(S,\ S)$

→ These states can not be derived analytically.

Instead, we **truncate higher-order Fourier modes** in the dynamic equation by keeping the mode of $(n,\ m)$ up to order $\ell (= |n| + |m|)$.

Then, steady-state solutions are **calculated by numerical iterations** with $\dot{\Phi}_{\pm} = 0$

The solutions **well converge** with increasing ℓ and **saturate around** $\ell = 10$.

This fast convergence is due to the **exponential decay of** $\alpha_{n,m}$ with ℓ in the steady state.

Comparison of mode analysis and numerical simulations

$$S_{+}e^{i\Phi_{+}} = \frac{1}{N}\sum_{j=1}^{N}\langle e^{iX_j}\rangle$$

$$S_{-}e^{i\Phi_{-}} = \frac{1}{N}\sum_{j=1}^{N}\langle e^{iY_j}\rangle$$

Four states:

(S_+, S_-)

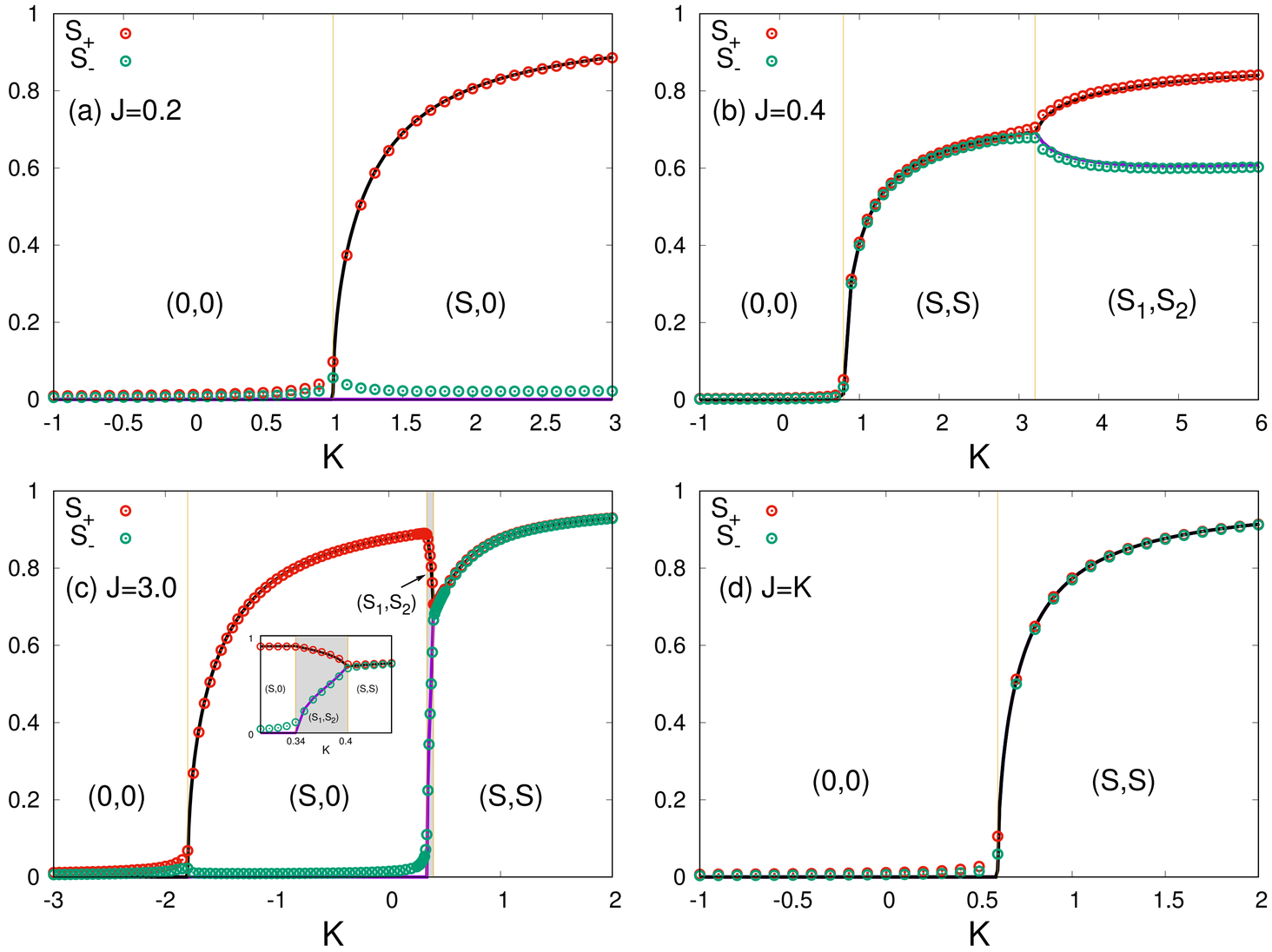
$(0,0)$

$(S,0)/(0,S)$

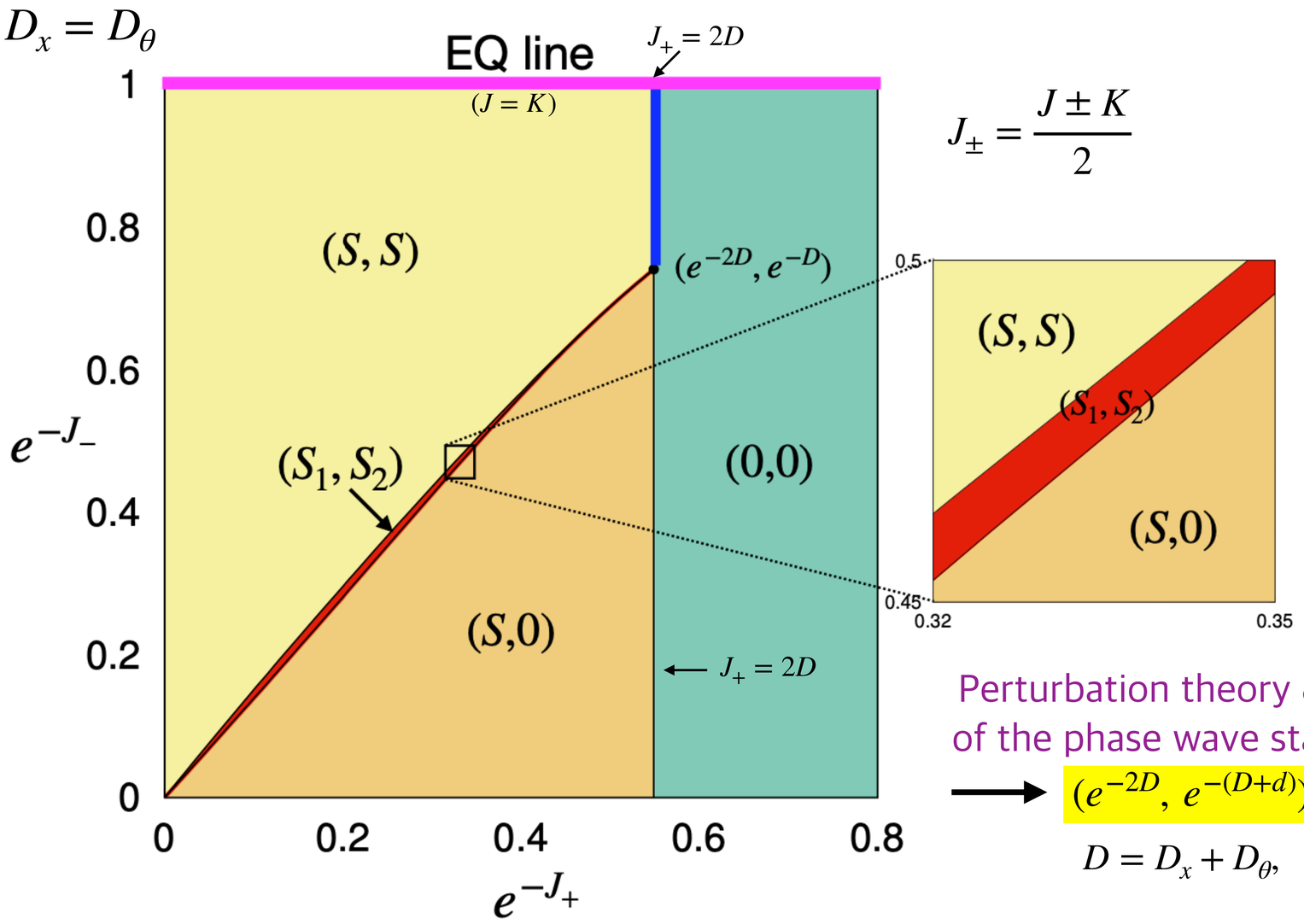
(S, S)

(S_1, S_2)

$D_x = D_\theta = 0.15$



Phase diagram



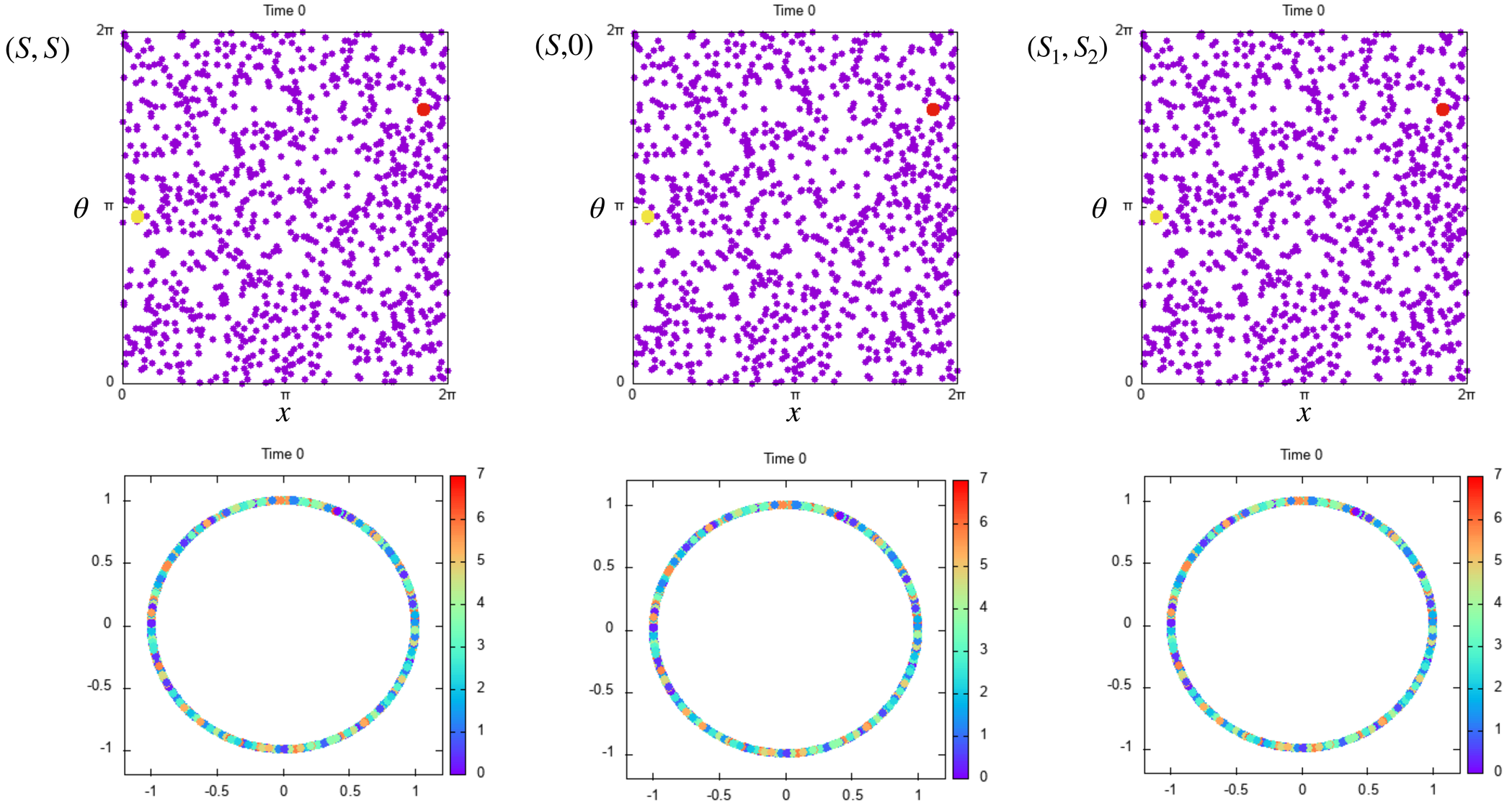
$$J_\pm = \frac{J \pm K}{2}$$

Perturbation theory & linear stability analysis of the phase wave state

→ $(e^{-2D},\ e^{-(D+d)})$

$D = D_x + D_\theta, \quad d = D_x - D_\theta$

Long-term states



Summary

- We explored the **joint action of swarming and synchronization** by studying a simple 1D model of swarmalators.
- We found four collective states, illustrated their transitions with order parameter curves $S_{\pm}(J, K)$, and encapsulated their stabilities in a phase diagram.
- We showed that the **four states can be induced by means of only the thermal noise and the interactions between the units in the system.**
- We have conducted a stability analysis of the phase wave state using the perturbation theory, and determined the multi critical point where the four states converge.

Ref: H. Hong, K. P. O'Keeffe, J. S. Lee, and H. Park, Phys. Rev. Research 5, 023105 (2023).

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