Time-cost-error trade-off relation in the stochastic thermodynamics

: Third-law like constraints

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- Separated states
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- Numerical demonstration
- Quantum extension
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1. Background and Motivation

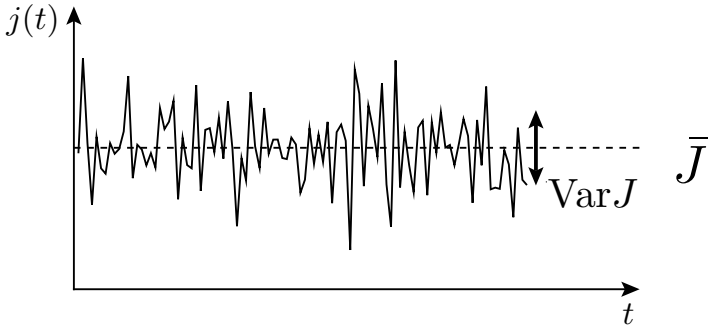
- Two-way trade-off relation in the nonequilibrium thermodynamics

  thermodynamic cost ⟷ precision

  speed

  ⋮

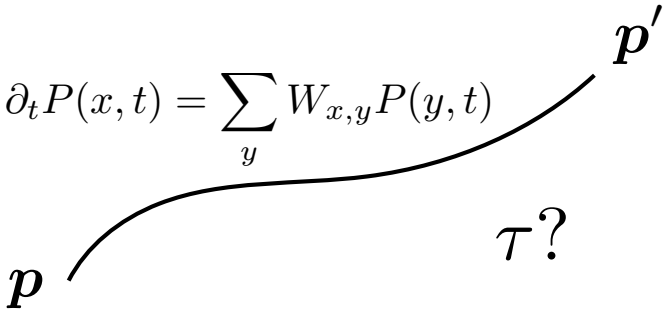
- Thermodynamic uncertainty relation



$$\left(\frac{\mathrm{Var}J}{\bar{J}^2}\right)\sigma \geq 2k_B$$

Barato, Seifert, PRL (2015)
Gingrich et al., PRL (2016)
Dechant, Sasa, JSMech (2018)

- Thermodynamic speed limit



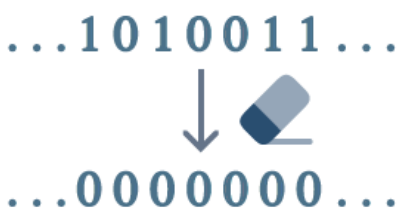
$$\tau \geq \frac{L_1^2(\boldsymbol{p}, \boldsymbol{p}')}{\bar{a}_t \Sigma_\tau}$$

Aurell et. al, JSP (2012)
Shiraishi, Funo, KS, PRL (2018)
Vu and KS, PRX (2023)
Ito, PRL (2018)
Lee, Lee, Kwon, Park, PRL (2022)

Target of thermal operations in this talk

- Examples of important thermal operations

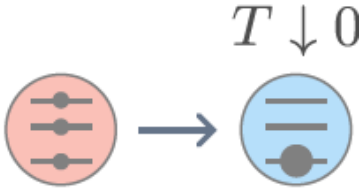
(1) Bit erasure



Landauer principle

Proesmans, Ehrich, Bechhoefer PRL (2020)
Miller, Guarnieri, Mitchison, Goold, PRL (2020)
Rolandi, Perarnau-Llobet, Quantum (2023)
Vu, KS, PRL (2022)

(2) Cooling to the ground state
preparing for an initial state



Third law

Levy, Alicki, Kosloff, PRE (2012)
Massanes, Oppenheim, Nat.Comm(2017)

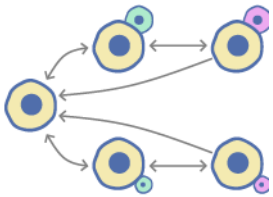
(3) Copying process



Maxwell-demon

Sagawa, Ueda, PRL (2008)

(4) Proofreading



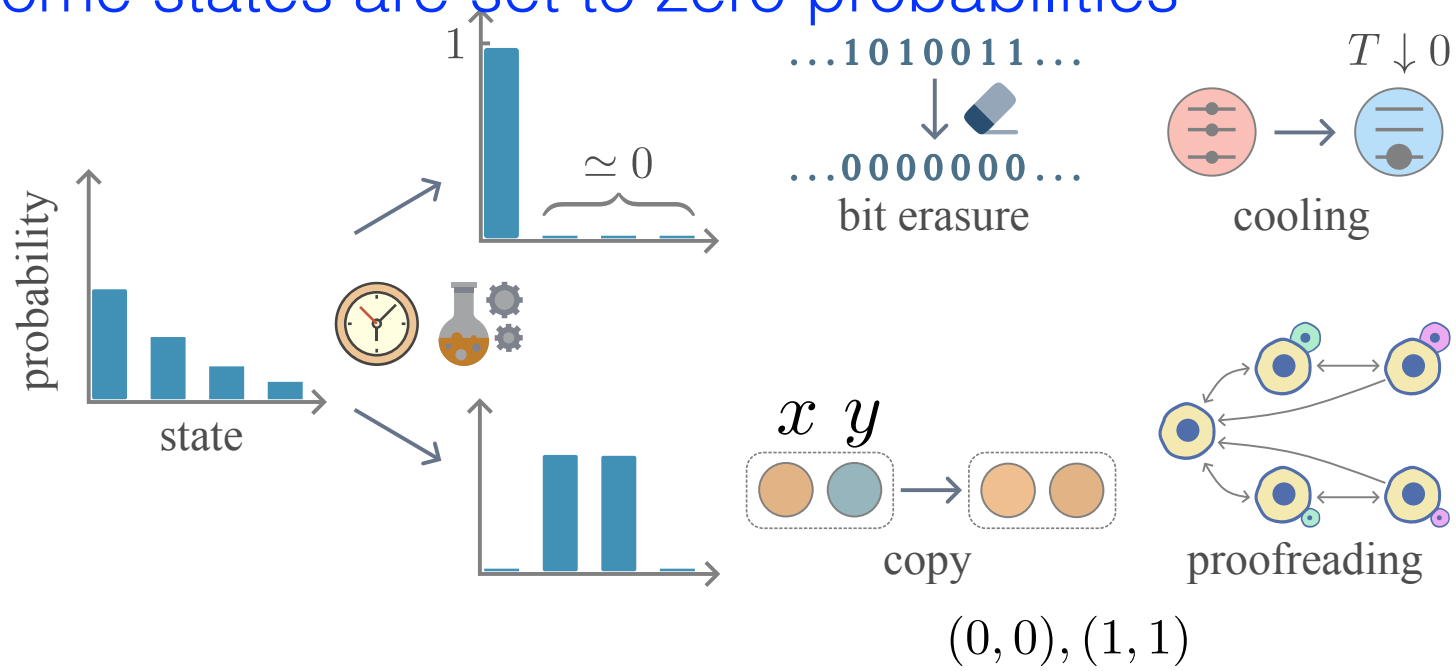
Hopfield、PNAS (1974)

Rao, Peliti, JSMech (2015)
Chiuchiu, Mondal, Pigolotti, NJP (2023)

Common aspects in important thermal operations

- Common aspects

  ``Some states are set to zero probabilities”



- What are required

  (1) Operation must be accurate

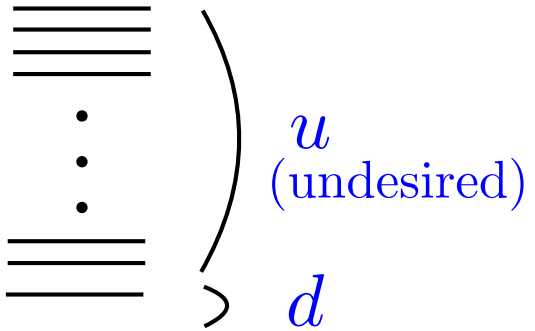
  (2) Operation time must be finite

  (3) But small (thermodynamic) cost is desired

2. Separated states

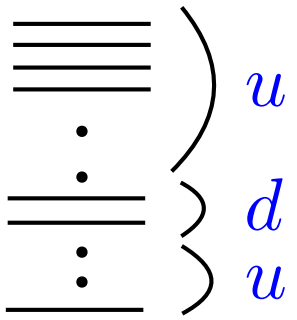
- ``Separated states" $P_u = 0, P_d = 1$

e.g.

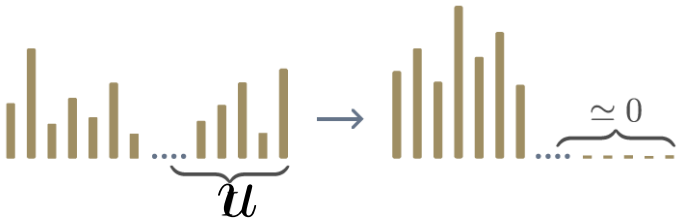


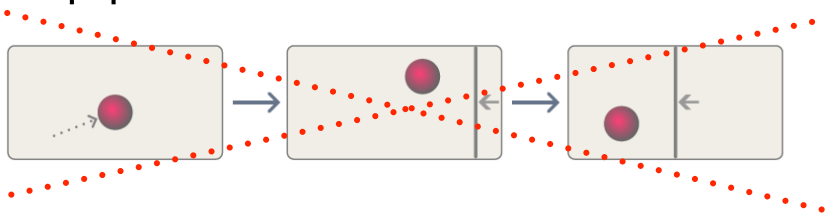
$$P_d := \sum_{i \in d} P(i)$$

$$P_u := \sum_{i \in u} P(i)$$

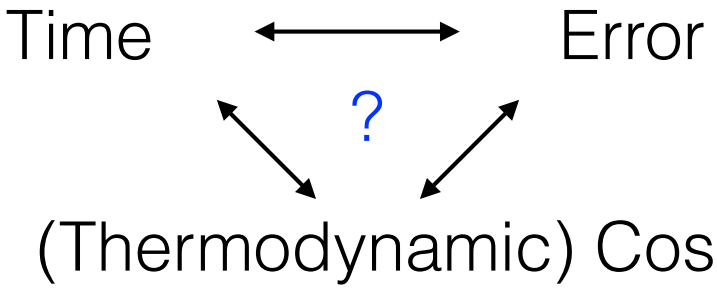


- One remark on the definition: supports are never reduced





- Three-way trade-off relations for making the separate states?



Stochastic thermodynamics

- Classical probabilistic process

$$\frac{\partial}{\partial t}P_n(t) = \sum_{m(\neq n)} W_{n,m}(t)P_m(t) - W_{m,n}(t)P_n(t)$$

- Crucial quantities

  1. Entropy production rate: irreversibility

$$\sigma_t = \frac{1}{2}\sum_{m,n}(W_{n,m}P_m - W_{m,n}P_n)\ln\left[\frac{W_{n,m}P_m}{W_{m,n}P_n}\right] \geq 0$$

  2. Activity: number of transitions

$$a_t = \sum_{m\neq n} W_{m,n}P_n + W_{n,m}P_m \geq 0$$

- Quantum dynamics: the GKSL equation, System+Bath

3. Main results: three-way trade-off relation for separated states

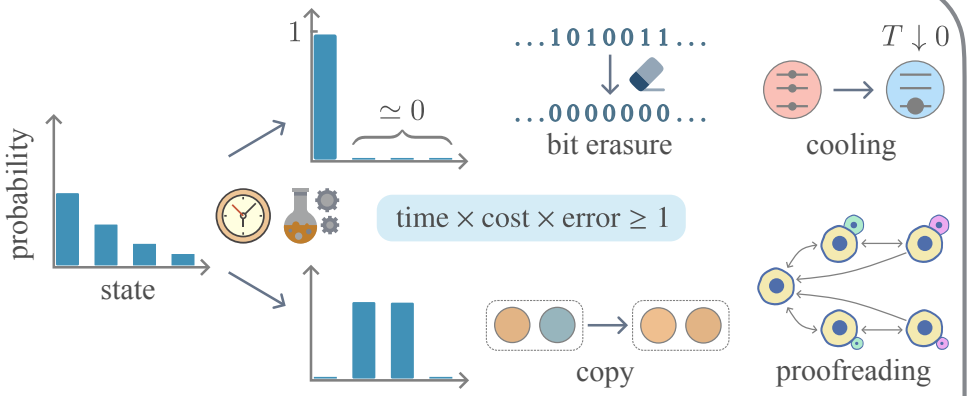
- Time-cost-error trade-off relation

$$\tau \cdot C \cdot \epsilon_\tau \geq \eta_\tau$$

Time Cost Error

Relative error ratio

$$\eta_\tau := \frac{\epsilon_0 - \epsilon_\tau}{\epsilon_0} = 1 - \epsilon_\tau/\epsilon_0$$



- Error: evaluated though the prob. of undesired states

$$\epsilon_t := -1/\ln P_u(t) \qquad \epsilon_t \in [0, \infty]$$

$P_u(t) = 0+$ $P_u(t) = 1-$

- Cost: thermodynamic cost

$$C := \frac{1}{\tau}\int_0^\tau dt\, w_{d,u}\, g(\sigma_t/a_t)$$

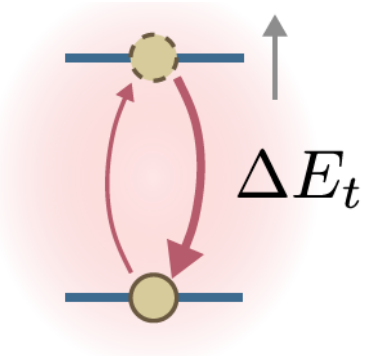
$$g(x) = 1 - \Phi(x)$$

Φ : inverse of $\frac{x-1}{x+1}\ln x$

Numerical demonstrations

- Cooling

$$\partial_t p_1(t) = -w_{2,1}(t)p_1(t) + w_{1,2}(t)p_2(t)$$
$$\partial_t p_2(t) = -w_{1,2}(t)p_2(t) + w_{2,1}(t)p_1(t)$$



- Check of the trade-off relation

$$\tau C \epsilon_\tau \geq \eta_\tau = 1 - \epsilon_\tau/\epsilon_0 \quad \longleftrightarrow \quad C\frac{\epsilon_0\epsilon_\tau}{\epsilon_0 - \epsilon_\tau} \geq \frac{1}{\tau}$$

$$w_{1,2}(t) = w_{\rm tot}\frac{e^{\beta\Delta E_t}}{1+e^{\beta\Delta E_t}}$$
$$w_{2,1}(t) = w_{\rm tot}\frac{1}{1+e^{\beta\Delta E_t}}$$

Randomly generated ΔE_t

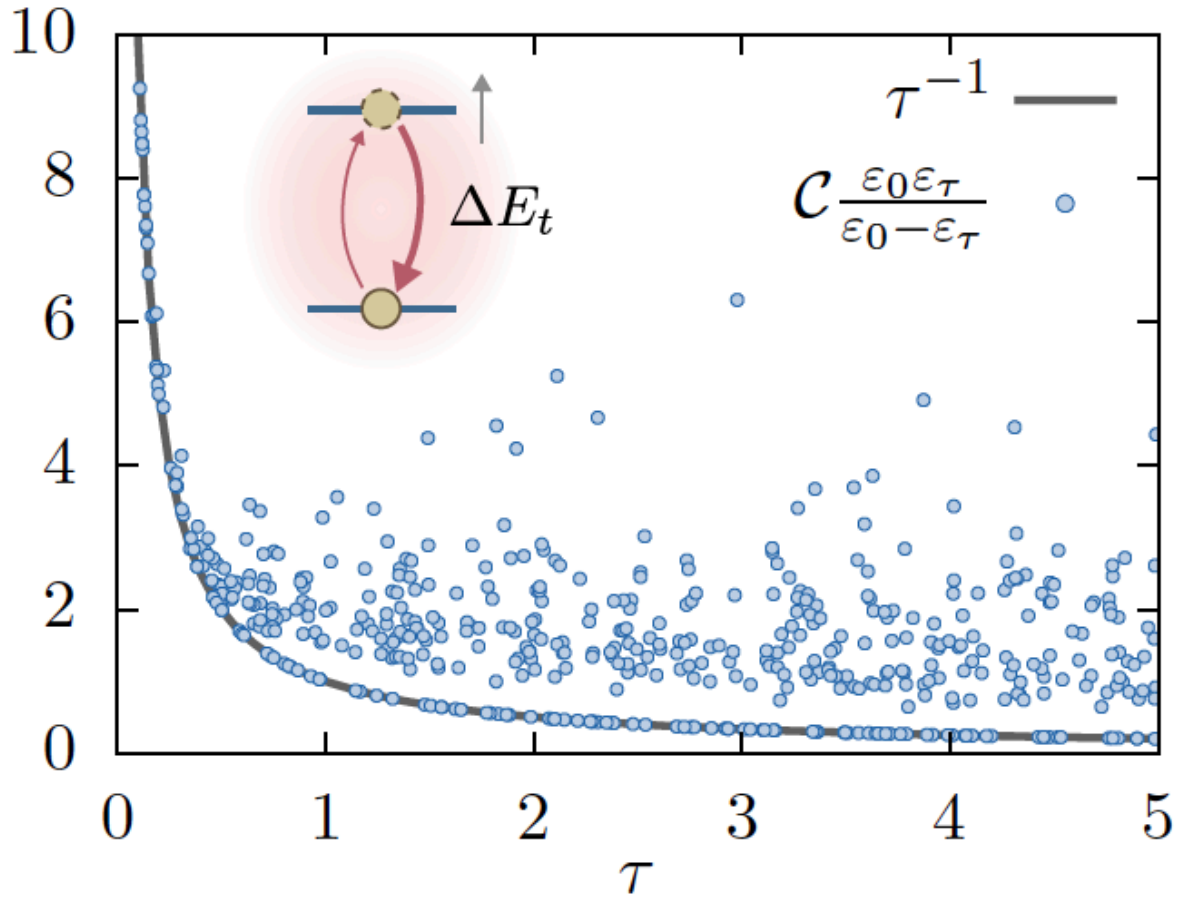
Numerical demonstrations

$$w_{1,2}(t) = w_{\rm tot}\frac{e^{\beta\Delta E_t}}{1+e^{\beta\Delta E_t}}$$

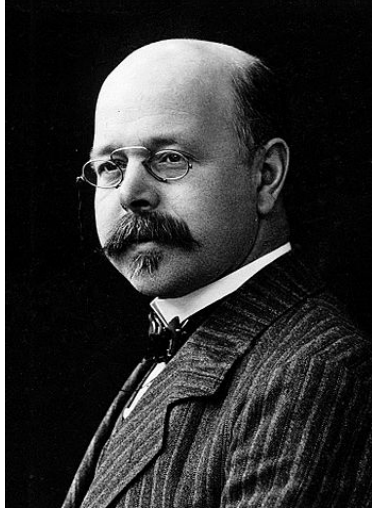
$$w_{2,1}(t) = w_{\rm tot}\frac{1}{1+e^{\beta\Delta E_t}}$$

Randomly generated ΔE_t

Check of $C\dfrac{\epsilon_0\epsilon_\tau}{\epsilon_0-\epsilon_\tau} \geq \dfrac{1}{\tau}$



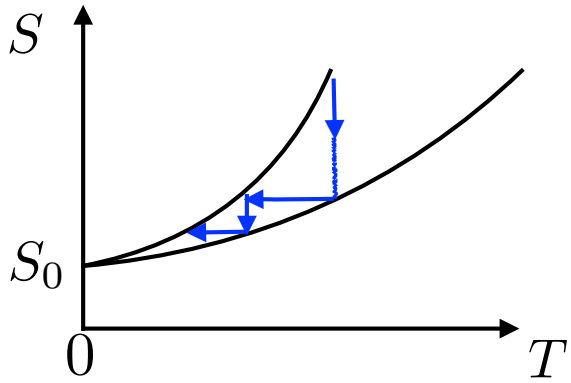
Remark on the connection to the Third law



- The third law consists of the two

(1) Heat theorem: $S \to S_0 \quad (T \to 0)$

(2) Unattainability theorem



"It is impossible for any process, no matter how idealized, to reduce the entropy of a system to its absolute-zero value in a finite number of operations" wikipedia

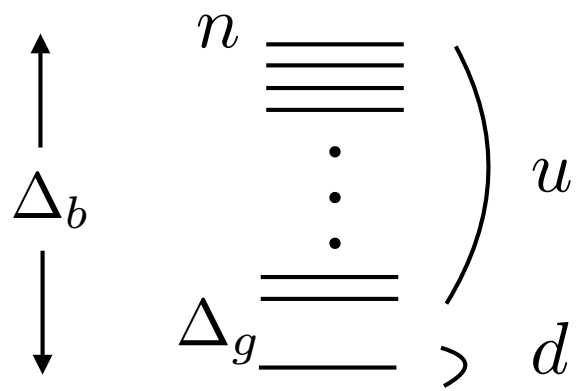
Levy, Alicki, Kosloff, 2012
Masanes, Oppenheim, (2017)
Taranto et al., PRX (2023)

- The present scheme generalizes the unattainability theorem

Ground state	→	Separated states (with errors)
Infinite protocols	→	Thermodynamic cost
quasi-static		Any operation time

The cooling case: Effective temperature

- Cooling to $T = 0$



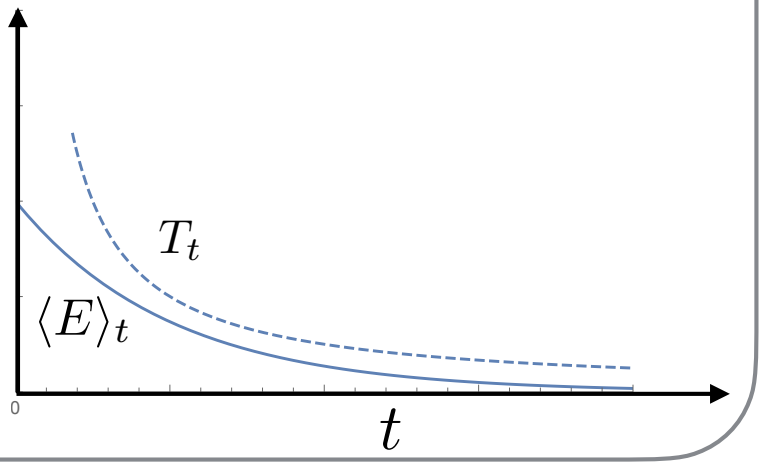
- Effective temperature T_t $\quad \langle E \rangle_t = \sum_i e^{-E_i/T_t} E_i / Z(T_t)$

(1) Sufficiently low temperatures, low-lying two-levels are dominant

$$\epsilon_\tau := -\frac{1}{\ln P_u(\tau)} \sim T_\tau$$

(2) Essential bound of the temperature

$$T_\tau \geq \frac{\Delta_g}{\tau C_\tau + \epsilon_0 + \ln\left[(n-1)\Delta_b/\Delta_g\right]}$$



Quantum extension: cooling case

- GKLS dynamics: Essentially the same inequality is derived

$$\tau \cdot C \cdot \epsilon_\tau \geq \eta_\tau$$

- Pure quantum evolution: System+finite Bath

$$H_{\rm tot}(t) = H_{\rm S}(t) + H_{\rm SB} + H_{\rm B}$$

$$\rho_\tau = U \rho_S \otimes \rho_{B,\rm eq} U^\dagger$$



Entropy production $\Sigma_\tau = D(\rho_\tau || \rho_{\rm S,\tau} \otimes \rho_{\rm B,eq})$

Result $\lambda_{\rm S}(t)$: the smallest eigenvalue in $\rho_{{\rm S},t}$ —— $\lambda_{\rm S}(t)$

$$\epsilon_t := -\frac{1}{\ln \lambda_{\rm S}(t)}$$

$$\beta \Delta Q \geq -\Delta S + \lambda_{\rm S}(0) \psi(\epsilon_\tau^{-1} - \epsilon_0^{-1})$$

$$\psi(x) := \min_{p \in [0,1]} \left(x + \frac{1-p}{p} \ln \frac{1-p}{1-pe^{-x}} \right) \geq \frac{2x}{1+e^{3/x}}$$

