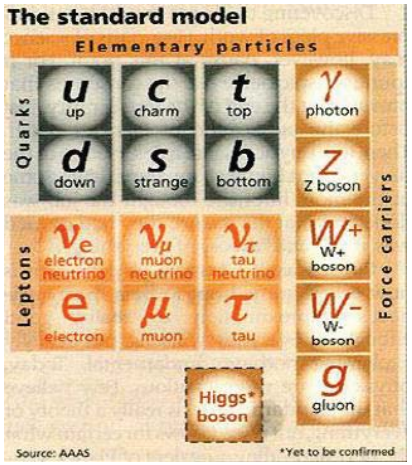
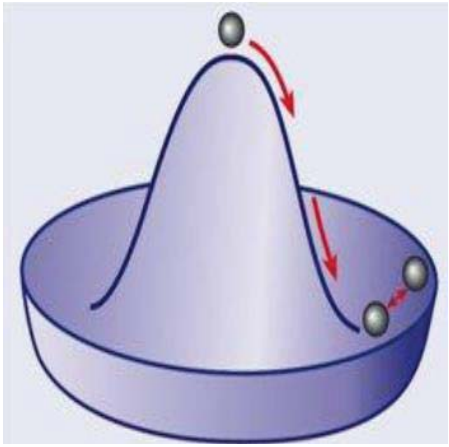
4차원 세계의 파동방정식

최성열 (전북대학교)







KIAS 입자물리 여름캠프 2024

Heinrich Hertz
on Maxwell Equations

One cannot escape the feeling that these mathematical formulae have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than was originally put into them.

이 수학 공식이 독립적 존재와 고유 지성을 지니며, 우리보다 더 지혜롭고 그 발견자들보다 더 현명하단 느낌을 피할 수 없다. 우리가 이로부터 원래 부여한 것보다 훨씬 더 많은 것을 얻어 내고 있다는 것도..

Paul Dirac

God used beautiful mathematics in creating the world.

Spock

Mr. Scott, there are always alternatives

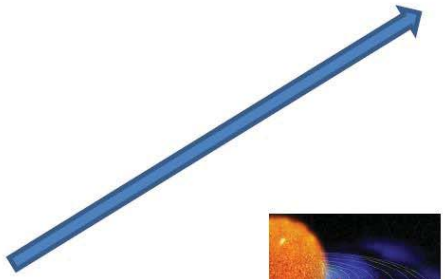
Albert Einstein

Anyone who has never made a mistake has never tried anything new.

Carl Sagan

Science is a way of thinking much more than it is a body of knowledge.

Particle Physics



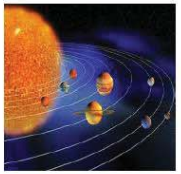






10^{23} m

10^{27} m



10^{20} m

10^{11} m



10^{7} m

Hierarchy of Scales

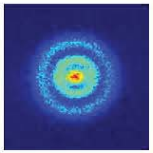
10^{3} m



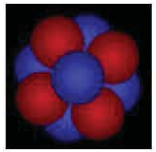
10^{-1} m



10^{-10} m



10^{-15} m



10^{-19} m

?

Particle World

Quantum : particle © wave

$\lambda = h/p \qquad h = 6.6 \times 10^{-34}\,\mathrm{J\,s}$

Relativity : energy © momentum © mass

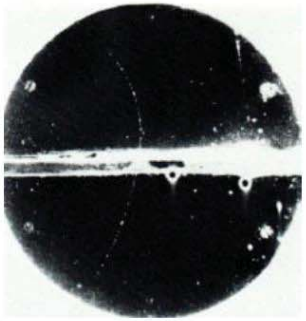
$E^2 = p^2c^2 + m^2c^4 \qquad c = 3 \times 10^8\,\mathrm{m/s}$

Probing Nature at the smallest time/length demands the strongest energy/momentum.

100 GeV ⇔ 10^{-18} m ⇔ 10^{-26} s



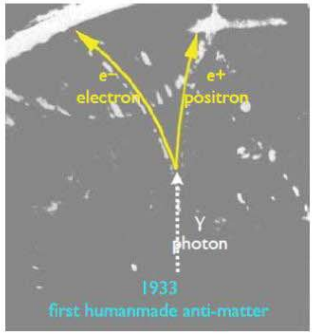
Terascale: 1000 GeV ⇔ 10^{-19} m ⇔ 10^{-27} sec



Relativistic Quantum Field Theory

particle ⇔ anti-particle

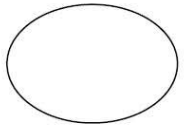
Heisenberg uncertainty principle



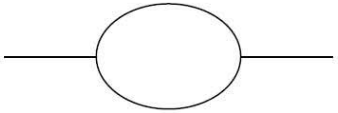
$\Delta E \sim h/\Delta t$

$\Delta p \sim h/\Delta x$

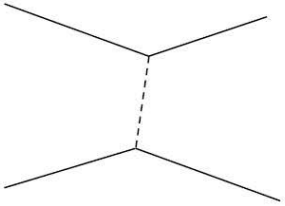
→ Dynamic vacuum



→ Pair creation/annihilation



→ Interactions



Interaction strengths vary with distance

Particle Physics

Study the constituents and the interactions
among them at the smallest time and length scales

What is matter made of?
Why and how do they interact to build things?

Tear things down and see what come out
See how they interact with each other

Necessary to understand the Universe

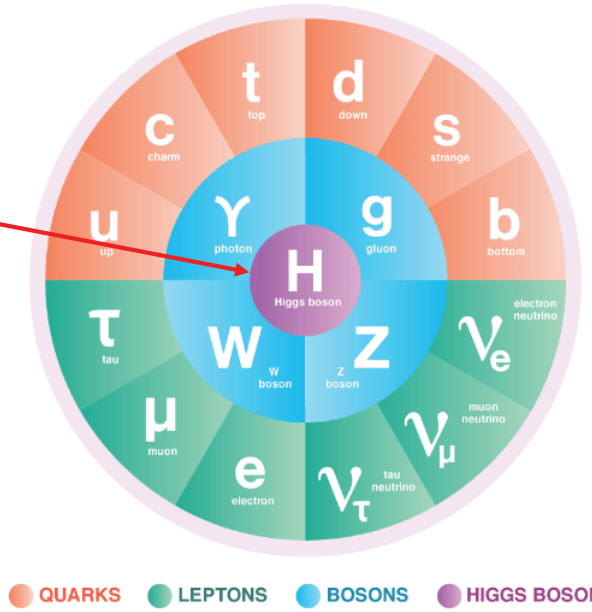
July 4, 2012, CERN : a memorable day in human history





Higgspendence Day

Standard Model

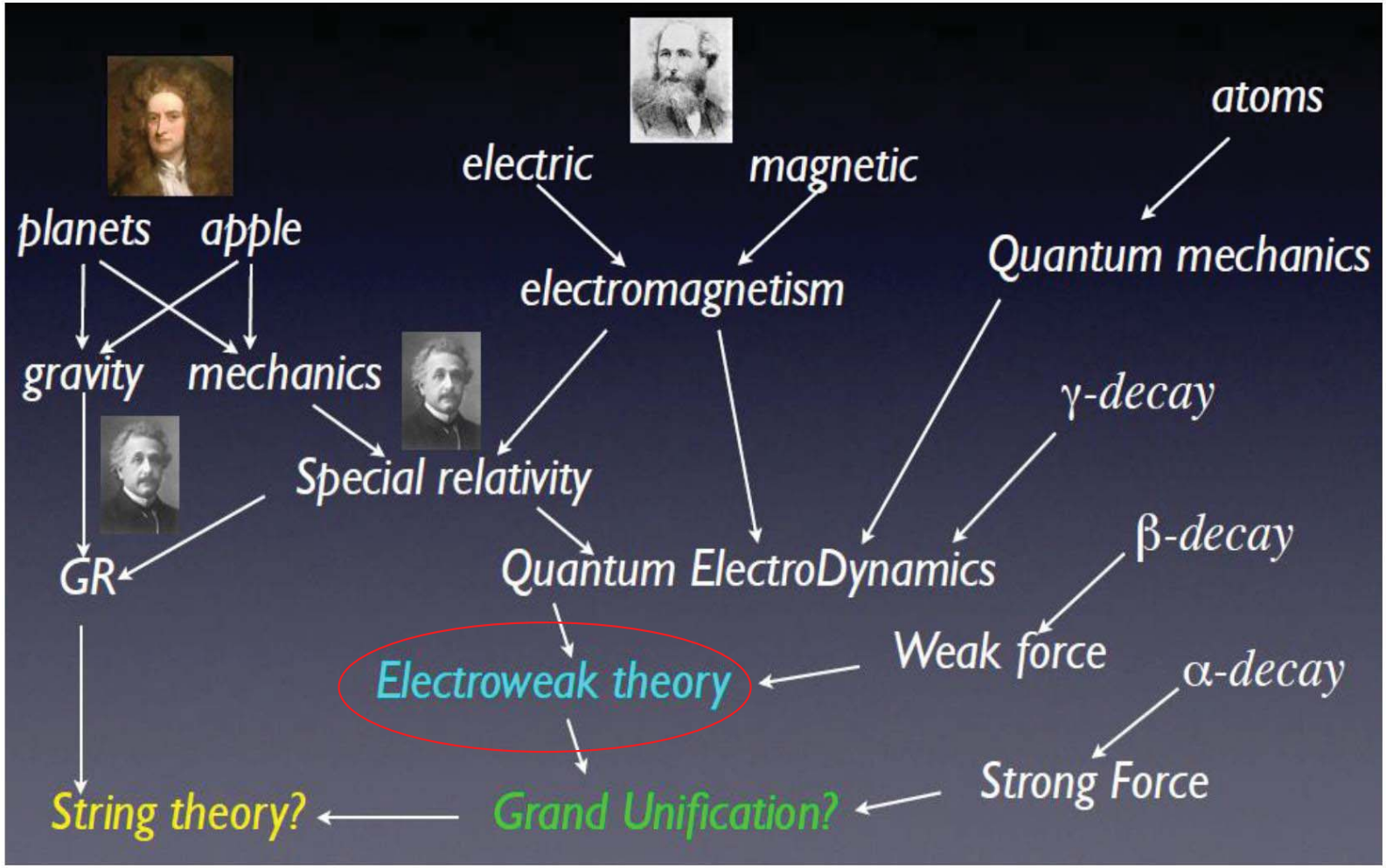


Final Stroke (화룡점정)

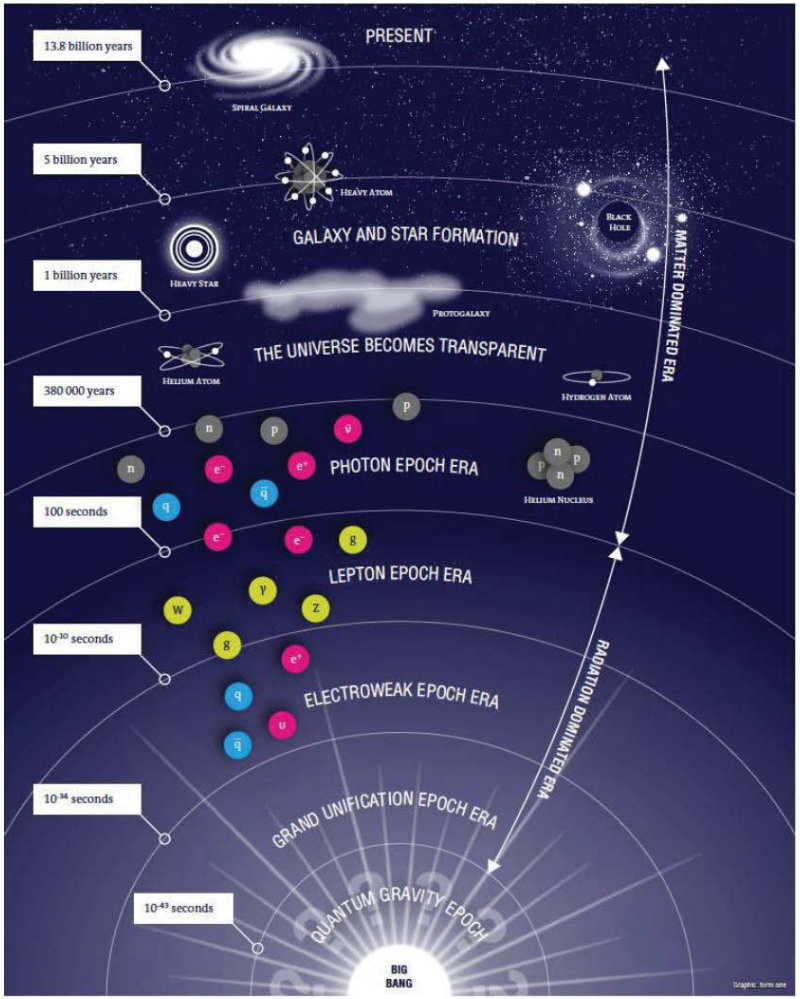


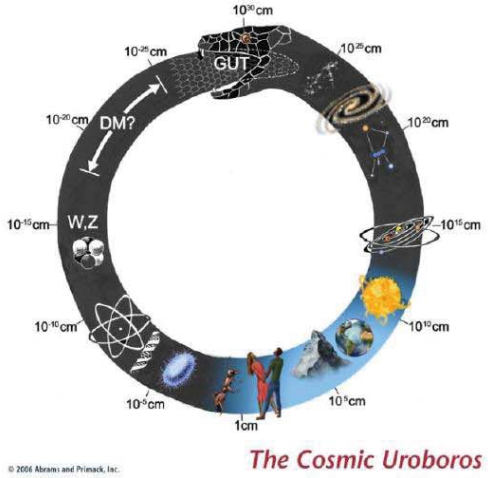
Unification

[H. Murayama]



Cosmic Uroboros

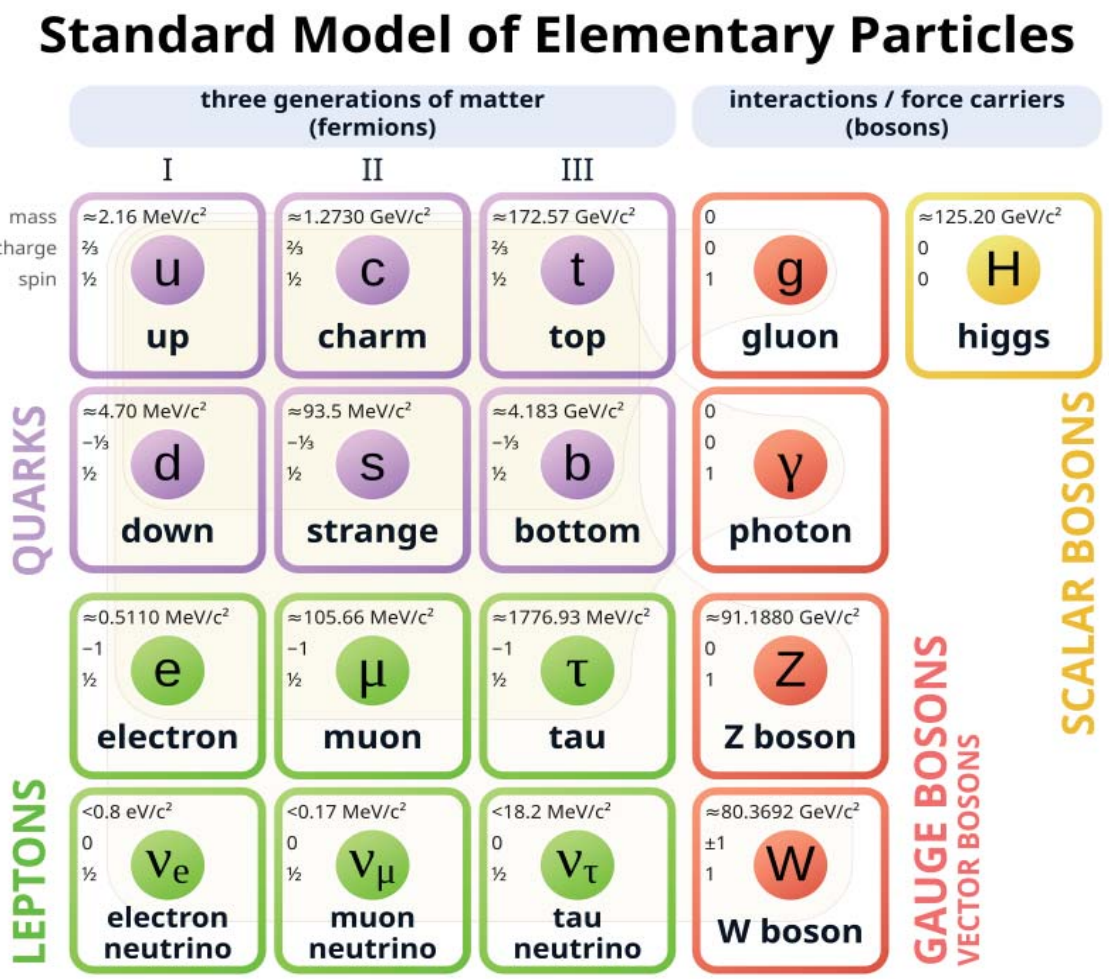




The Cosmic Uroboros

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4-d relativistic wave equations
(4차원 세계의 파동방정식)



Spin 0

S = 0, 1/2, 1

Spin 1/2

Spin 1

Fundamental Framework

Baselines

4-d flat Minkowski spacetime (with gravitation ignored)

$$c = 3 \times 10^8\,\mathrm{m/s} \equiv 1$$

Metric : $g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$

$$x^\mu = (t,\ \vec{r}), \quad x_\mu = g_{\mu\nu} x^\nu = (t, -\vec{r})$$

$$p^\mu = (E, \vec{p}), \quad p_\mu = g_{\mu\nu} p^\nu = (E, -\vec{p})$$

Invariant interval : $dx^2 \equiv dx^\mu dx_\mu = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - d\vec{r}^{\,2}$

Description

(Renormalizable) relativistic quantum field theory

$$\hbar = 6.6 \times 10^{-25}\,\mathrm{GeV\,s} \equiv 1$$

$$[\mathrm{L}] = [\mathrm{T}], \quad [\mathrm{E}] = [\vec{p}] = [\mathrm{M}] = [\mathrm{L}]^{-1}$$

Key concepts

Symmetries $\Leftrightarrow$ Mass, Spin and Charges

Basic strategy

Perturbation theory

Poincaré Group

The 10-parameter Poincaré group is the fundamental spacetime symmetry group of 4-parameter spacetime translations and 6-parameter Lorentz transformations consisting of 3-parameter rotations and 3-parameter boosts.

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu \, x^\nu + a^\mu \quad \Rightarrow \quad \Delta x'^2 = \Delta x^2 : \text{invariant}$$

Every physical object that lives in the Minkowski space of the 4-dimensional spacetime must belong to some representations of the Poincaré group.

Covariant classical mechanics

A 4-vector velocity of a massive particle can be constructed by differentiation of the spacetime coordinate 4-vector x^μ with respect to the Lorentz invariant proper time τ:

$$u^\mu = \frac{dx^\mu}{d\tau} = (\gamma, \gamma\vec{v}) \qquad \text{with} \quad d\tau = \sqrt{dt^2 - d\vec{r}^2} = dt\sqrt{1-\beta^2} = dt/\gamma$$

satisfying the relation $u^\mu u_\mu = 1$ with the particle speed $\vec{v} = d\vec{r}/dt = \vec{\beta}$ and the boost factor γ. Naturally, the 4-momentum of a free particle of mass m, transforming as a 4-vector, is defined to be

$$p^\mu = mu^\mu = (E, \vec{p}) = (\gamma m, \gamma m\vec{v}) \quad \Rightarrow \quad p^2 = p^\mu p_\mu = E^2 - |\vec{p}|^2 = m^2$$

$$\Rightarrow \ \frac{dp^\mu}{d\tau} = f^\mu \ \rightarrow \ \frac{d\vec{p}}{dt} = \vec{F} \ \oplus \ \frac{dE}{dt} = \vec{F}\cdot\vec{v} \quad \Rightarrow \quad f^\mu_{\rm em} = qF^{\mu\nu}u_\nu \ \text{[Lorentz 1895]}$$

All the massless ($m = 0$) particles always move at the light speed $\beta = 1$, and thus they cannot be distinguished by their speed. Remarkably this puzzle is resolved in the framework of quantum mechanics.

Covariant classical electrodynamics

Let us introduce the partial derivative operators with respect to x^μ and x_μ:

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = (\frac{\partial}{\partial t}, \vec{\nabla}) \quad \text{and} \quad \partial^\mu \equiv \frac{\partial}{\partial x_\mu} = (\frac{\partial}{\partial t}, -\vec{\nabla})$$

Invariant 4-d Laplacian operator : $\Box \equiv \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \nabla^2$ [d'Alembert 1747]

The invariance of electric charge forces the charge density $\rho(x)$ and current density $\vec{j}(x)$ to satisfy the continuity equation, requiring ρ and $\vec{j}$ to form a 4-vector j^μ:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0 \quad \Rightarrow \quad \partial_\mu j^\mu = 0 \quad \text{with} \quad j^\mu = (\rho, \vec{j})$$

In the Lorenz family of gauges the electromagnetic wave equations for the scalar potential ϕ and the vector potential $\vec{A}$ are

$$\Box\phi = \rho \quad \oplus \quad \Box\vec{A} = \vec{j} \quad \text{with} \quad \frac{\partial\phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0$$

Q1

Obviously, Lorentz covariance requires that the potentials ϕ and $\vec{A}$ form a 4-vector potential A^μ, casting the wave equations and the Lorenz condition into the manifestly covariant forms,

$$A^\mu = (\phi, \vec{A}) \quad \Rightarrow \quad \boxed{\Box A^\mu = j^\mu \oplus \partial_\mu A^\mu = 0}$$

Quantum mechanically the wave equation corresponds to the massless spin-1 field equation and the massive spin-1 free field equation is given by

$$\boxed{(\Box + m^2)A^\mu = 0 \oplus \partial_\mu A^\mu = 0} \Rightarrow \quad A^\mu = \epsilon^\mu(p)\, \mathrm{e}^{-ip\cdot x} \text{ : plane wave}$$

with $p^2 = m^2$ and the constraint $p\cdot\epsilon(p) = 0$.

[Proca 1936]

Spin 0 & 1 Wave Equations

Spin-0 scalar field

$$(\Box + m^2)\phi(x) = 0$$ [Klein & Gordon 1926]

Spin-1 vector field

$$(\Box + m^2)A^\mu(x) = 0 \oplus \partial_\mu A^\mu(x) = 0$$ [Proca 1936]

$$m = 0 \Rightarrow A^\mu(x) \to A^\mu + \partial^\mu \lambda(x) \text{ with } \Box\lambda(x) = 0$$

⇒ degrees of freedom: 3 (massive) - 1 = 2 (massless)

Spin-1/2 Wave Equation

Lorentz transformations

The Lorentz transformation to the spacetime coordinate 4-vector x^μ

$$x'^\mu = \Lambda^\mu{}_\nu \, x^\nu$$

leaving the length of the four vector x^μ invariant as

$$x'^2 = g_{\mu\nu} \, x'^\mu x'^\nu = x^2$$

should satisfy the relation:

$$g_{\mu\nu} \, \Lambda^\mu{}_\rho \Lambda^\nu{}_\tau = [\, \Lambda^T \, g \, \Lambda \,]_{\rho\tau} = g_{\rho\tau}$$

The six (anti-symmetric) generators $M_{\rho\sigma}$ of the Lorentz group defined by

$$\Lambda^{\mu}{}_{\nu} = \left[\exp\left(-\frac{i}{2}\omega^{\rho\sigma}M_{\rho\sigma}\right)\right]^{\mu}{}_{\nu} \quad \text{with} \quad [M_{\rho\sigma}]^{\mu}{}_{\nu} = i\left(g^{\mu}_{\rho}g_{\sigma\nu} - g^{\mu}_{\sigma}g_{\rho\nu}\right)$$

satisfy the Lie algebra

$$[M_{\mu\nu},\, M_{\rho\sigma}] = i\,[g_{\mu\sigma}M_{\nu\rho} - g_{\mu\rho}M_{\nu\sigma} + g_{\nu\rho}M_{\mu\sigma} - g_{\nu\sigma}M_{\mu\rho}]$$

Q2

which can be expressed as

$$\begin{aligned}
[J_i,\ \ J_j\,] &= +i\,\epsilon_{ijk}\,J_k \\
[J_i,\ \ K_j] &= +i\,\epsilon_{ijk}\,K_k \\
[K_i,\ K_j] &= -i\,\epsilon_{ijk}\,J_k
\end{aligned}$$

$$\Lambda = \exp\left(-i\vec{\theta}\cdot\vec{J} - i\vec{\eta}\cdot\vec{K}\right)$$

in terms of the generators of rotations and boosts

$$J_i \equiv \frac{1}{2}\,\epsilon_{ijk}M_{jk}, \qquad K_i \equiv M^{0i} = -M_{0i}$$

$$\theta^i = \frac{1}{2}\epsilon^{ijk}\omega_{jk}, \quad \eta^i = -\omega^{0i}$$

$$e^{\eta} = \sqrt{(1+\beta)/(1-\beta)}$$

The mixed algebra of $\{J_i, K_i\}$ can be diagonalized as

$$
\begin{aligned}
&[J_{Li},\ J_{Rj}] = 0 \\
&[J_{Li},\ J_{Lj}] = i\,\epsilon_{ijk}\,J_{Lk} \\
&[J_{Ri},\ J_{Rj}] = i\,\epsilon_{ijk}\,J_{Rk}
\end{aligned}
\quad \Rightarrow \quad \boxed{\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R}
$$

by introducing two independent sets of generators

$$
\begin{aligned}
\mathcal{L}: \quad J_{Li} &\equiv \frac{1}{2}\,(J_i + i\,K_i) \\
\mathcal{R}: \quad J_{Ri} &\equiv \frac{1}{2}\,(J_i - i\,K_i)
\end{aligned}
$$

In terms of J_{Li} and J_{Ri}, we can construct $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ representations of the Lorentz group

$$(j_L, j_R) \quad : \quad \begin{cases} J_L^2 \,|(j_L, j_R)\rangle = j_L(j_L+1)\,|(j_L, j_R)\rangle \\ J_R^2 \,|(j_L, j_R)\rangle = j_R(j_R+1)\,|(j_L, j_R)\rangle \end{cases}$$

with $J_{L,R} = 0, 1/2, \cdots$. The two SU(2) generators are related by Parity P

$$P : \begin{cases} J_i \;\to\; +J_i \\ K_i \;\to\; -K_i \end{cases} \quad \Rightarrow \quad \vec{J}_L \;\leftrightarrow\; \vec{J}_R$$

The handedness is a convention. In the following we shall discuss these two spinor representations in some detail so as to construct the Lorentz covariant Dirac equation for a massive spin-1/2 particle.

Spinors

We introduce 2–component spinors transforming under the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ group as

$$\left(\frac{1}{2}, 0\right) : \quad \psi_L(x) = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \{\eta_\alpha\} \quad \Rightarrow \quad \psi'_L(x') = \Lambda_L \psi_L(x)$$

$$\left(0, \frac{1}{2}\right) : \quad \psi_R(x) = \begin{pmatrix} \bar{\chi}^1 \\ \bar{\chi}^2 \end{pmatrix} = \{\bar{\chi}^{\dot{a}}\} \quad \Rightarrow \quad \psi'_R(x') = \Lambda_R \psi_R(x)$$

The rotation/boost generators in the L/R-spinor representation read

$$J_k = \frac{1}{2}\,\sigma^k \quad \text{and} \quad K_k = \mp\frac{i}{2}\,\sigma^k \quad \text{for} \quad L/R$$

in terms of the 2×2 Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \sigma^i \sigma^j = \delta_{ij} + i\,\epsilon_{ijk}\sigma^k$$

For the rotation of $\vec{\theta}$ and the boost of $\vec{\eta}$, the Lorentz transformations of the L and R spinor fields are described by the operators

$$\Lambda_L = \exp\left\{-i(\vec{J}\cdot\vec{\theta} + \vec{K}\cdot\vec{\eta})\right\} = \exp\left\{-i\frac{\vec{\sigma}}{2}\cdot(\vec{\theta} - i\vec{\eta})\right\}$$

$$\Lambda_R = \exp\left\{-i(\vec{J}\cdot\vec{\theta} + \vec{K}\cdot\vec{\eta})\right\} = \exp\left\{-i\frac{\vec{\sigma}}{2}\cdot(\vec{\theta} + i\vec{\eta})\right\}$$

The magnitude $\theta = |\vec{\theta}|$ is the rotation angle while the magnitude of the boost vector $\vec{\eta}$ (which is parallel to the velocity vector $\vec{\beta}$) is called the rapidity related to the velocity parameter β as

$$e^{\eta} = \gamma(1+\beta) = \sqrt{(1+\beta)/(1-\beta)}\,; \quad \cosh\eta = \gamma,\ \sinh\eta = \gamma\beta$$

with the boost factor $\gamma = 1/\sqrt{1-\beta^2}$.

Dirac Equation

For a free particle, the only operator appearing in the wave equation is the 4–momentum operator $p_\mu = i\,\partial_\mu$ or in the spinor notation with $\sigma^\mu_\pm = (1, \pm\vec{\sigma})$

$$p_\mu\ \sigma^\mu_{+\alpha\dot{\alpha}} = \begin{pmatrix} p_0 - p_z & -p_x + ip_y \\ -p_x - ip_y & p_0 + p_z \end{pmatrix} \quad \Leftarrow \quad \Lambda_L^{-1}\sigma^\mu_+\,\Lambda_R = \Lambda^\mu{}_\nu\,\sigma^\nu_+$$

$$p_\mu\,\sigma^{\mu\,\dot{\alpha}\alpha}_- = \begin{pmatrix} p_0 + p_z & p_x - ip_y \\ p_x + ip_y & p_0 - p_z \end{pmatrix} \quad \Leftarrow \quad \Lambda_R^{-1}\sigma^\mu_-\,\Lambda_L = \Lambda^\mu{}_\nu\,\sigma^\nu_-$$

Q3

One simple wave equation can be a linear differential relation between the components of spinors, expressed by the operators $p_\mu\sigma^\mu_{+\alpha\dot{\alpha}}$ and $p_\mu\sigma^{\mu\dot{\alpha}\alpha}_-$

$$p_\mu\,\sigma^\mu_{+\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = m\,\eta_\alpha \quad \text{and} \quad p_\mu\,\sigma^{\mu\,\dot{\alpha}\alpha}_-\,\eta_\alpha = m\bar{\chi}^{\dot{\alpha}}$$

The need to use the mass in the wave equation implies the simultaneous consideration of two L- and R-spinors (η_α and $\bar{\chi}^{\dot{\alpha}}$); with only one of these, it would not be possible to construct a relativistically invariant equation containing a dimensional parameter. The above relativistic wave equation is called the Dirac equation having been first derived by Dirac in 1928.

The spinor form of the Dirac equation is the most natural one, in the sense that its relativistic invariance is immediately apparent. In applications of the equation, however, other forms of the wave equation may be more convenient. We denote a four–component Dirac spinor by the symbol ψ. In the spinor representation, it is a bi-spinor consisting of two 2-component spinors

$$\psi = \begin{pmatrix} \eta_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$$

The Dirac equation is put in terms of the 4–component spinor in the form

$p_\mu = i\partial_\mu$ $\quad \boxed{p_\mu \gamma^\mu \psi \equiv \not{p}\psi = m\,\psi \quad \Rightarrow \quad (\not{p} - m)\,\psi = 0}$ [Dirac 1928]

The spinor form of the wave equation with the components of the above bispinor corresponds to the 4×4 matrices γ^μ :

$$\text{Weyl/Chiral} : \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma_+^\mu \\ \sigma_-^\mu & 0 \end{pmatrix}$$

We introduce an additional gamma matrix γ_5 and two chiral projection operators :

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad P_\lambda = \frac{1+\lambda\gamma_5}{2} \qquad \lambda = \pm = \mathrm{R/L}$$

In the general case, the matrices γ^μ need to satisfy only the conditions ensuring that $p^2 = m^2$. To find these conditions, we multiply the Dirac equation by $\not{p}$

$$\not{p}\ \not{p}\psi = \frac{1}{2}p_\mu p_\nu \left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right)\psi = m\ \not{p}\psi = m^2\psi$$

and we must therefore have

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,g^{\mu\nu}$$

which can be checked with the Weyl representations of the gamma matrices given above.

Free–particle Solutions

Let us solve the Dirac equation. We take into account a plane wave solution

$$\psi(x) = u(p)\,\mathrm{e}^{-ip\cdot x} \quad p^2 = m^2 \oplus p^0 > 0 \quad \Rightarrow\ (\gamma^\mu p_\mu - m)u(p) = 0$$

It is easiest to analyze this equation in the rest frame, where $p_0^\mu = (m, \vec{0})$; the solution for a general p can then be found by boosting the rest–frame solution

$$(m\gamma^0 - m)u(p_0) = m \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u(p_0) = 0 \quad \Rightarrow\ u(p_0) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

for any 2–component spinor ξ normalized to be $\xi^\dagger \xi = 1$.

Now that we have the general form of $u(p)$ in the rest frame, we can obtain $u(p)$ in any other frame by boosting. Consider a boost along the 3–direction. Then with the rapidity η the boosted spinor is given by

$$
\begin{aligned}
u(p) &= \exp\left[-\frac{1}{2}\eta\begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix}\right]\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}\\
&= \left[\cosh\frac{\eta}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \sinh\frac{\eta}{2}\begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix}\right]\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}\\
&= \begin{bmatrix}\left[\sqrt{E+p^3}\left(\frac{1-\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1+\sigma^3}{2}\right)\right]\xi\\ \left[\sqrt{E+p^3}\left(\frac{1+\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1-\sigma^3}{2}\right)\right]\xi\end{bmatrix}
\end{aligned}
$$

The last line can be simplified to give

$$u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma_+}\,\xi \\ \sqrt{p \cdot \sigma_-}\,\xi \end{pmatrix}$$

where it is understood that in taking the square root of a matrix we take the positive root of each eigenvalue. This expression for $u(p)$ is not only more compact, but is also valid for any arbitrary direction of $\vec{p}$.

The amplitude of the plane wave contains one arbitrary two–component quantity ξ. Thus, for a given momentum, there are two different independent states, corresponding to the two possible values of the spin component. But, in the relativistic theory the orbital angular momentum $\vec{l}$ and the spin $\vec{s}$ of a moving particle are not separately conserved. Only the total angular momentum $\vec{j} = \vec{l} + \vec{s}$ is conserved. The component of the spin in any fixed direction is therefore also not conserved. However, the component of the spin in the direction of the momentum is conserved; since $\vec{l} = \vec{r} \times \vec{p}$ the product $\hat{p} \cdot \vec{s}$ is equal to the conserved product $\hat{p} \cdot \vec{j}$. This quantity is called the helicity. Helicity states correspond to plane waves in which $\xi = \xi_\lambda$ is an eigenvalue of the operator $\hat{p} \cdot \hat{\sigma}$:

$$\frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|} \, \xi_\lambda(p) = \lambda \, \xi_\lambda(p) \qquad \lambda = \pm$$

If we write

$$\frac{\vec{p}}{|\vec{p}|} = (n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

then the normalized helicity eigenstates can be expressed as

$$\xi_+(p) = \frac{1}{\sqrt{2(1+n_z)}} \begin{pmatrix} 1+n_z \\ n_x + in_y \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\phi} \end{pmatrix}$$

$$\xi_-(p) = \frac{1}{\sqrt{2(1+n_z)}} \begin{pmatrix} -n_x + in_y \\ 1+n_z \end{pmatrix} = \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

Q4

apart from arbitrary phases. In the helicity basis we can express the Dirac spinor as

$$u(p,\lambda) = \begin{bmatrix} (E-\lambda p)^{1/2}\,\xi_\lambda(p) \\ (E+\lambda p)^{1/2}\,\xi_\lambda(p) \end{bmatrix} \equiv \begin{pmatrix} u_-(p,\lambda) \\ u_+(p,\lambda) \end{pmatrix}$$

The subscripts are so chosen as to satisfy

$$P_+ u(p,\lambda) = \frac{1+\gamma_5}{2} u(p,\lambda) = \begin{pmatrix} 0 \\ u_+(p,\lambda) \end{pmatrix}$$

$$P_-(p,\lambda) = \frac{1-\gamma_5}{2} u(p,\lambda) = \begin{pmatrix} u_-(p,\lambda) \\ 0 \end{pmatrix}$$

Free Antiparticle Solutions

In addition to the positive–frequency plane wave solution $u(p)$, there exists a negative–frequency plane wave solution

$$\psi(x) = v(p)\, \mathrm{e}^{+ip\cdot x}$$

satisfying the same Dirac equation. The easiest way to find this solution is to consider charge conjugation represented by a unitary matrix C:

$$C\gamma_\mu^T C^\dagger = -\gamma_\mu \qquad CC^\dagger = 1 \qquad \Rightarrow \qquad C = -i\gamma^2\gamma^0$$

$$v(p,\lambda) = C\bar{u}^T(p,\lambda) = (-i\gamma^2\gamma^0)\,\gamma^0 u^*(p,\lambda) = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} u^*(p,\lambda)$$

which can be derived through the following procedure

$$
\begin{aligned}
&(i\gamma^\mu \partial_\mu - m)\,\psi = 0 \\
&\psi^\dagger \left(-i\,\overleftarrow{\partial_\mu}\,\gamma^{\mu\dagger} - m\right) = 0 \\
&\psi^\dagger\gamma^0 \left(-i\,\overleftarrow{\partial_\mu}\,\gamma^0\gamma^{\mu\dagger}\gamma^0 - m\right) = 0 \\
&\bar{\psi} \left(-i\,\overleftarrow{\partial_\mu}\,\gamma^\mu - m\right) = 0 \\
&\left(-i\gamma^{\mu T}\partial_\mu - m\right)\bar{\psi}^T = 0 \\
&\left(-iC\gamma^{\mu T}C^\dagger\partial_\mu - m\right)C\bar{\psi}^T = 0
\end{aligned}
$$

Noting that $i\,\sigma^2\,\xi^*_\lambda = -\lambda\xi_{-\lambda}\ \ [\lambda = \pm]$ we can obtain the negative–frequency solution in the helicity basis as

$$
v(p,\lambda) = \lambda \begin{bmatrix} -(E+\lambda p)^{1/2}\,\xi_{-\lambda}(p) \\ (E-\lambda p)^{1/2}\,\xi_{-\lambda}(p) \end{bmatrix} \equiv \begin{pmatrix} v_-(p,\lambda) \\ v_+(p,\lambda) \end{pmatrix}
$$

It is an easy exercise to check that the helicity spinors $u(p,\lambda)$ and $v(p,\lambda)$ satisfy the following relations

$$\bar{u}(p,\lambda)u(p,\lambda') = +2m\delta_{\lambda\lambda'} \quad \text{and} \quad \bar{v}(p,\lambda)v(p,\lambda') = -2m\delta_{\lambda\lambda'}$$

For a massive spin-1/2 particle with 4-momentum $p^\mu = (E,\vec{p})$, the spin 4-vector is defined as

$$s^\mu = \lambda\,(|\vec{p}|, E\hat{p})/m$$

where $\lambda = \pm 1$ is twice the spin-1/2 particle helicity. Note that in the rest frame, $s = \lambda(0,\hat{p})$, while in the high energy limit with $E \gg m$, $s = \lambda p/m$. The helicity spinors satisfying the Dirac equation are eigenstates of $\gamma_5 \not{s}$ with unit eigenvalue. From these results, we can derive the helicity projection operators for a massive spin-1/2 particle. [Bouchiat & Michel 1958] Q5

$$u(p,\lambda)\bar{u}(p,\lambda) = \frac{1}{2}(1+\gamma_5 \not{s})(\not{p}+m) \quad \Rightarrow \quad \sum_{\lambda=\pm} u(p,\lambda)\bar{u}(p,\lambda) = \not{p}+m$$

$$v(p,\lambda)\bar{v}(p,\lambda) = \frac{1}{2}(1+\gamma_5 \not{s})(\not{p}-m) \quad \Rightarrow \quad \sum_{\lambda=\pm} v(p,\lambda)\bar{v}(p,\lambda) = \not{p}-m$$

We have seen that the necessity of two spinors (η, χ) to describe a particle with spin $1/2$ is due to the mass of the particle. This necessity disappears if the mass is zero. The wave equation which describes such a particle can be derived from a single spinor, say the undotted spinor

$$p_\mu \, \sigma_-^{\mu \, \dot{\alpha}\alpha} \eta_\alpha = 0 \quad \Rightarrow \quad (E + \vec{p} \cdot \vec{\sigma}) \, \eta = 0$$

The energy and momentum of a particle with $m = 0$ are related by $E = |\vec{p}|$ so that we have

$$(\hat{p} \cdot \vec{\sigma}) \eta(p) = -\eta(p) \; : \quad \text{helicity } \lambda = -1/2 = -$$

On the other hand the dotted spinor $\bar{\chi}$ satisfies [Weyl 1929]

$$(\hat{p} \cdot \vec{\sigma}) \bar{\chi}(p) = +\bar{\chi}(p) \; : \quad \text{helicity } \lambda = +1/2 = +$$

Consequently, states of the massless particle with a definite momentum are necessarily helicity states, for which the spin component in the direction of motion has a definite value. If the particle spin is opposite to the momentum (helicity $-1/2$), the antiparticle spin is along the momentum (helicity $+1/2$). The neutrinos in the Standard Model were supposed to be such particles possessing these properties.

Solutions to the Spin-1 Wave Equation

Polarization vectors of a spin-1 massive vector boson with a non-zero momentum can be obtained by boosting the polarization vectors in the rest frame

$$p_R^\mu = (m, \vec{0}) \quad \Rightarrow \quad p^\mu = (E, p\hat{n})$$

$$\epsilon^\mu(p,0) = \frac{1}{m}(p, E\hat{n}), \quad \epsilon^\mu(p,\pm 1) = \frac{1}{\sqrt{2}}(0, \mp\hat{\theta} - i\hat{\phi})$$

in terms of the orthonomal vectors given with the polar & azimuthal angles θ & ϕ by

$$\begin{aligned}
\hat{n} &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \\
\hat{\theta} &= (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \\
\hat{\phi} &= (-\sin\phi, \cos\phi, 0)
\end{aligned}$$

The polarization vectors satisfy the orthogonal conditions

$$p\cdot\epsilon(p,\lambda) = 0, \quad \epsilon(p,\lambda)\cdot\epsilon^*(p,\lambda') = -\delta_{\lambda\lambda'} \quad \text{with} \quad \lambda,\lambda' = \pm 1, 0$$

Note that the longitudinal ($\lambda = 0$) mode is unphysical, i.e. absent in the $m = 0$ case.

Relativistic Quantum Fields

real

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E}} \left[a(p)\, e^{-ip\cdot x} + a^\dagger(p)\, e^{ip\cdot x}\right]$$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E}} \sum_{\lambda=\pm 1/2} \left[c_\lambda(p)\, u(p,\lambda)\, e^{-ip\cdot x} + d^\dagger_\lambda(p)\, v(p,\lambda)\, e^{ip\cdot x}\right]$$

real

$$A^\mu(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E}} \sum_{\lambda=\pm 1,0} \left[a_\lambda(p)\, \epsilon^\mu(p,\lambda)\, e^{-ip\cdot x} + a^\dagger_\lambda(p)\, \epsilon^{\mu *}(p,\lambda)\, e^{ip\cdot x}\right]$$

$$c_\lambda(p) = d_\lambda(p) \quad \Rightarrow \quad \psi(x) = C\psi^T(x)$$ [Majorana 1937]

Spin-0 scalar & pseudo-scalar bilinears

$$\bar{\psi}_1\psi_2 = \psi_1^\dagger\gamma^0\psi_2$$
$$\bar{\psi}_1\gamma_5\psi_2 = \psi_1^\dagger\gamma^0\gamma_5\psi_2$$

Spin-1 vector & axial-vector bilinears

$$\bar{\psi}_1\gamma^\mu\psi_2 = \psi_1^\dagger\gamma^0\gamma^\mu\psi_2$$
$$\bar{\psi}_1\gamma^\mu\gamma_5\psi_2 = \psi_1^\dagger\gamma^0\gamma^\mu\gamma_5\psi_2$$

Free Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

$$\mathcal{L}_{1/2} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

$$\mathcal{L}_1 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2A_\mu A^\mu$$

Field-strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \cdots$$

Summary

Lagrangians

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$
$$\mathcal{L}_{1/2} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$
$$\mathcal{L}_1 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2A_\mu A^\mu$$

Wave equations

$$(\Box + m^2)\phi(x) = 0$$
$$(i\partial\!\!\!/ - m)\psi(x) = 0$$
$$(\Box + m^2)A^\mu(x) = 0 \quad \text{with} \quad \partial_\mu A^\mu(x) = 0$$

Plane-wave solutions

$$\varphi(p) = 1$$
$$u(p,\lambda) = \begin{bmatrix} (E-\lambda p)^{1/2}\xi_\lambda(p) \\ (E+\lambda p)^{1/2}\xi_\lambda(p) \end{bmatrix}$$
$$v(p,\lambda) = \lambda \begin{bmatrix} -(E+\lambda p)^{1/2}\xi_{-\lambda}(p) \\ (E-\lambda p)^{1/2}\xi_{-\lambda}(p) \end{bmatrix}$$
$$\epsilon^\mu(p,0) = \frac{1}{m}(p, E\hat{n})$$
$$\epsilon^\mu(p,\pm1) = \frac{1}{\sqrt{2}}(0, \mp\hat{\theta} - i\hat{\phi})$$

Quantum fields

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E}} \left[a(p)\, e^{-ip\cdot x} + a^\dagger(p)\, e^{ip\cdot x}\right]$$
$$\psi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E}} \sum_{\lambda=\pm1/2} \left[c_\lambda(p)\, u(p,\lambda)\, e^{-ip\cdot x} + d^\dagger_\lambda(p)\, v(p,\lambda)\, e^{ip\cdot x}\right]$$
$$A^\mu(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E}} \sum_{\lambda=\pm1,0} \left[a_\lambda(p)\, \epsilon^\mu(p,\lambda)\, e^{-ip\cdot x} + a^\dagger_\lambda(p)\, \epsilon^{\mu*}(p,\lambda)\, e^{ip\cdot x}\right]$$

Lorentz-invariant gauge theories



Prof. Jae Sik Lee's Lectures

(Gauge) Interactions

- Gauge interactions with matter are obtained by the replacement

$$\partial_\mu \quad \Rightarrow \quad D_\mu = \partial_\mu + igT^a A^a_\mu$$

- Gauge interactions preserve fermion chirality

$$\bar{\psi}\gamma^\mu(v - a\gamma_5)\psi Z_\mu = \left[(v+a)\bar{\psi}_L\gamma^\mu\psi_L + (v-a)\bar{\psi}_R\gamma^\mu\psi + R\right] Z_\mu$$

  Chirality breaking is entirely due to a mass term or a Yukawa interaction (apart from anomalies).

- Non-renormalizable effective gauge interactions can be constructed from D_μ and $F_{\mu\nu}$ such as

$$\bar{\psi}\sigma_{\mu\nu}\psi F^{\mu\nu}, \quad \bar{\psi}\gamma_\mu D_\nu \psi F^{\mu\nu}, \quad \cdots$$

Prescription of Model Building

- Fix the gauge group
  - Gauge bosons are determined
  - Parameters : gauge couplings

- Fix the representations of fermions and scalars
  - Gauge interactions of matter particles are fixed (no new parameters)
  - The total fermion representation must be anomaly–free

- Impose global symmetries if needed

- Write down all possible mass terms and interactions compatible with the symmetries
  - scalar potential parameters $(\varphi^2, \varphi^3, \varphi^4)$
  - fermion masses
  - Yukawa couplings

Solutions

Albert Einstein

It is the supreme art of the teacher to awaken joy in creative expression and knowledge.

The only source of knowledge is experience.

Learn from yesterday, live for today, hope for tomorrow.
The important thing is not to stop questioning.

We cannot solve problems by using the same kind of thinking we used when we created them.

If you cannot explain it simply, you do not understand it well enough.

Everything should be made as simple as possible, but not simpler.

Maxwell equations in the Lorenz gauge

We introduce the four-potential A^μ containing the scalar and vector potentials, $\phi(x)$ and $\vec{A}(x)$, as

$$A^\mu = (\phi, \vec{A})$$

defining the field-strength tensor consisting of the electric and magnetic fields, $\vec{E}$ and $\vec{B}$, as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$

which enable us to combine the four (2 inhomogeneous + 2 homogeneous) Maxwell equations into

$$\partial_\mu F^{\mu\nu} = j^\nu = (\rho, \vec{j}) \quad \oplus \quad \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu} = 0$$

where the homogeneous equations are automatically satisfied by the definition of the field-strength tensor. Imposing the Lorenz gauge condition casts the Maxwell equations into an extremely compact form

$$\partial_\mu A^\mu = 0 \quad \Rightarrow \quad \Box A^\mu = j^\mu$$

Lie algebra of Lorentz group

Any infinitesimal Lorentz transformation from the identity element can be represented as

$$\Lambda^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$$

Inserting the form into the defining identity for the Lorentz transformation leads to the relation

$$\omega^{\mu\nu} + \omega^{\nu\mu} = 0$$

so that the element $\omega^{\mu}{}_{\nu}$ can be represnted in terms of the six antisymmetric generators as

$$\omega^{\mu}{}_{\nu} = -\frac{i}{2}\omega^{\rho\sigma}[M_{\rho\sigma}]^{\mu}{}_{\nu} \quad \text{with} \quad [M_{\rho\sigma}]^{\mu}{}_{\nu} = i\left(g^{\mu}_{\rho}g_{\sigma\nu} - g^{\mu}_{\sigma}g_{\rho\nu}\right)$$

The six generators $M_{\rho\sigma}$ satisfy the commutaion relations of the Lorentz group

$$[M_{\mu\nu},\, M_{\rho\sigma}] = i[g_{\mu\sigma}M_{\nu\rho} - g_{\mu\rho}M_{\nu\sigma} + g_{\nu\rho}M_{\mu\sigma} - g_{\nu\sigma}M_{\mu\rho}]$$

Spin-1/2 spinor rotation

For a rotation of $\vec{\theta} = \theta\hat{n}$ with respect to the axis denoted by a unit vector $\hat{n}$, the Lorentz transformations of the L & R spinor fields are described by the rotatoion operator $\mathcal{R}$

$$\mathcal{R} = \Lambda_L = \Lambda_R = \exp\left[-i\frac{\theta}{2}(\hat{n}\cdot\vec{\sigma})\right]$$

Using the following four identities of the Pauli matrices with the normalization $\hat{n}\cdot\hat{n} = 1$,

$$\vec{\sigma}(\hat{n}\cdot\vec{\sigma}) + (\hat{n}\cdot\vec{\sigma})\vec{\sigma} = 2\hat{n}, \quad \vec{\sigma}(\hat{n}\cdot\vec{\sigma}) - (\hat{n}\cdot\vec{\sigma})\vec{\sigma} = 2i(\hat{n}\times\vec{\sigma})$$
$$(\hat{n}\cdot\vec{\sigma})(\hat{n}\cdot\vec{\sigma}) = 1, \qquad (\hat{n}\cdot\vec{\sigma})\vec{\sigma}(\hat{n}\cdot\vec{\sigma}) = 2\hat{n}(\hat{n}\cdot\vec{\sigma}) - \vec{\sigma}$$

we can check that the Pauli martrices $\vec{\sigma}$ are transformed by the rotation R as

$$\begin{aligned}\mathcal{R}^{-1}\vec{\sigma}\,\mathcal{R} &= \cos\theta\,\vec{\sigma} + (1-\cos\theta)\,\hat{n}(\hat{n}\cdot\vec{\sigma}) + \sin\theta\,(\hat{n}\times\vec{\sigma}) \\ &= \vec{\sigma}_{\parallel} + \cos\theta\,\vec{\sigma}_{\perp} + \sin\theta\,(\hat{n}\times\vec{\sigma}_{\perp})\end{aligned}$$

with $\vec{\sigma}_{\parallel} = \hat{n}(\hat{n}\cdot\vec{\sigma})$ and $\vec{\sigma}_{\perp} = \hat{n}\times(\vec{\sigma}\times\hat{n})$, as expected for a 3-vector.

Spin-1/2 spinor boost

For a boost of $\vec{\eta} = \eta\hat{n}$ along the direction denoted by a unit vector $\hat{n}$, the Lorentz transformations of the L and R spinor fields are described by the boost operator $\mathcal{B}$

$$\mathcal{B} = \Lambda_L^{-1} = \Lambda_R = \exp\left[\frac{\eta}{2}(\hat{n}\cdot\vec{\sigma})\right]$$

Using the three identities of the parameter η in relation to β and $\gamma = 1/\sqrt{1-\beta^2}$

$$\cosh^2(\eta/2) = (\gamma+1)/2, \quad \sinh^2(\eta/2) = (\gamma-1)/2, \quad \cosh(\eta/2)\sinh(\eta/2) = \gamma\beta/2$$

we can check that the boost $\mathcal{B}$ transforms the identity matrix and Pauli matrices as

$$\begin{aligned} \mathcal{B}1\mathcal{B} &= \gamma + \gamma\beta(\hat{n}\cdot\vec{\sigma}) \\ \mathcal{B}\vec{\sigma}\mathcal{B} &= \hat{n}[\gamma(\hat{n}\cdot\vec{\sigma}) + \gamma\beta] + \vec{\sigma} - \hat{n}(\hat{n}\cdot\vec{\sigma}) \end{aligned}$$

as expected for a 4-vector $\sigma_+^\mu = (1, \vec{\sigma})$.

Spin-1 polarization vectors

Consider the spin-1 angular-momentum operators $\vec{S} = (S_1, S_2, S_3)$ in the adjoint repesentation with $(S_i)_{jk} = -i\epsilon_{ijk}$, satisfying $[S_i, S_j] = i\epsilon_{ijk}S_k$. The eigenvectors of the matrix $\hat{n} \cdot \vec{S}$ with eigenvalues $\lambda = \pm 1, 0$

$$\vec{\epsilon}(p_R, \pm 1) = \frac{1}{\sqrt{2}}(\mp\vec{\theta} - i\vec{\phi}) \quad \text{and} \quad \vec{\epsilon}(p_R, 0) = (0, \vec{n})$$

$$Y_{1,\pm 1} = \sqrt{\frac{3}{8\pi}}\,(\mp\hat{n}_1 - i\hat{n}_2)$$
$$Y_{1,\ 0} = \sqrt{\frac{3}{4\pi}}\,\hat{n}_3$$

with the momentum $p_R = (m, \vec{0})$ in the rest frame and the three orthonormal vectors

$$\hat{n} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta),$$
$$\hat{\theta} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), \quad \hat{\phi} = (-\sin\phi, \cos\phi, 0)$$

In the frame with the particle momentum $p = (E, p\hat{n})$, the longitudinal-polarization vector is generated by a Lorentz boost as

$$\epsilon^\mu(p, 0) = \frac{1}{m}(p,\ E\hat{n})$$

while the transverse-polarization vectors remain intact.

$$\epsilon^\mu(p, \pm 1) = \frac{1}{\sqrt{2}}(0, \mp\vec{\theta} - i\vec{\phi})$$

Proca wave equation

The so-called Proca wave equations describing a spin-1 massive vector boson follow from the folloiwng Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu) + \frac{m^2}{2}A_\mu A^\mu$$

Taking the following two derivatives and using them,

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + \partial^\nu A^\mu = -F^{\mu\nu}, \quad \frac{\partial \mathcal{L}}{\partial A_\nu} = m^2 A^\nu$$

we can show that the Euler-Lagrange equation for A^ν leads to the Prica equation as

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{\partial \mathcal{L}}{\partial A_\nu} \quad \Rightarrow \quad \partial_\mu F^{\mu\nu} + m^2 A^\mu = \Box A^\mu - \partial^\mu(\partial_\nu A^\nu) + m^2 A^\mu = 0$$

Applying ∂_μ to the last expression leads to the constraint $\partial_\mu A^\mu = 0$. For the massless case, this constraint is simply a choice of gauge (Lorenz gauge), but for the massive case, the constraint must be satisfied. Therefore, the Proca equation is equivalent to

$$(\Box + m^2)A^\mu = 0 \quad \text{and} \quad \partial_\mu A^\mu = 0$$

Spin 0 & 1 complex wave equations

A complex spin-0 field $\phi^{\pm}$ can be decomposed as a linear combination of two real spin-0 fields $\phi_{1,2}$ as

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \quad \text{and} \quad \phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$$

Therefore, the free Lagrangian of a massive scalar complex field ϕ is given by

$$\mathcal{L}_0 = \frac{1}{2}[\partial_\mu\phi_1\partial^\mu\phi_1 - m^2\phi_1\phi_1] + \frac{1}{2}[\partial_\mu\phi_2\partial^\mu\phi_1 - m^2\phi_1\phi_2] \quad \Rightarrow \quad \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi$$

Similarily, a complex spin-1 field $A^{\pm}$ is a linear combination of two real spin-1 fields A_1 and A_2 as

$$A^- = \frac{1}{\sqrt{2}}(A_1 + iA_2) \quad \text{and} \quad A^+ = \frac{1}{\sqrt{2}}(A_1 - iA_2)$$

of which the free Lagrangian with the field-strength tensors $F^{\mu\nu}_{1,2} = \partial^\mu A^\nu_{1,2} - \partial^\nu A^\mu_{1,2}$ is given by

$$\mathcal{L}_1 = -\frac{1}{4}F_{1\mu\nu}F_1^{\mu\nu} + \frac{1}{2}m^2A_{1\mu}A_1^\mu - \frac{1}{4}F_{1\mu\nu}F_1^{\mu\nu} + \frac{1}{2}m^2A_{1\mu}A_1^\mu \quad \Rightarrow \quad -\frac{1}{2}F^+_{\mu\nu}F^{-\mu\nu} + m^2A^+_\mu A^{-\mu}$$

Relativistic Complex Quantum Fields

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E}} \left[a(p)\, e^{-ip\cdot x} + b^\dagger(p)\, e^{ip\cdot x}\right]$$

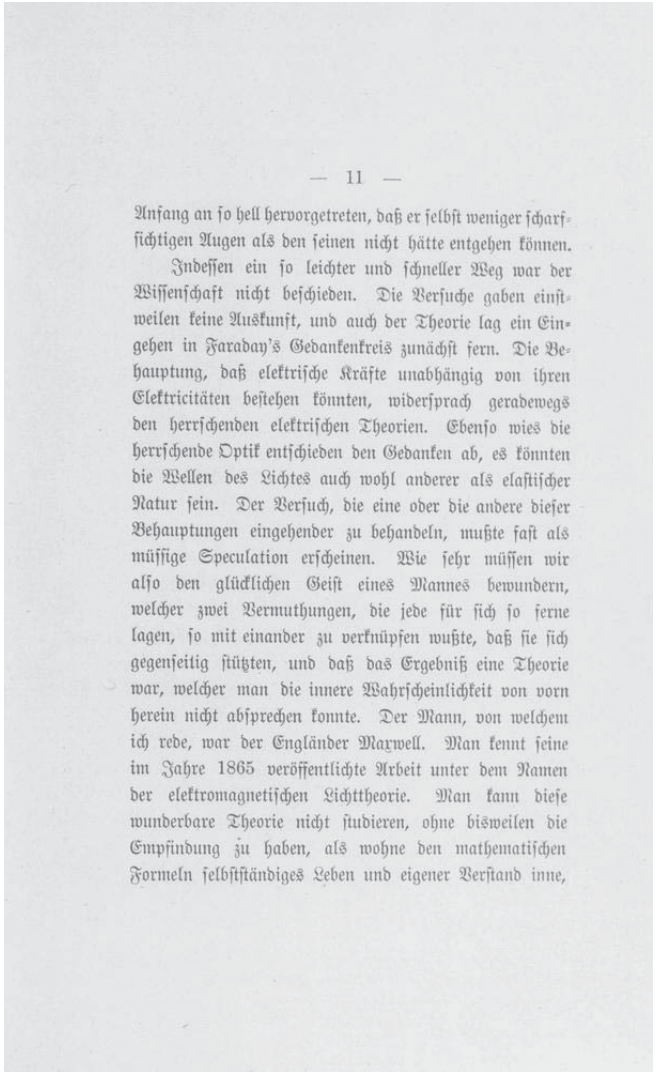
$$\psi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E}} \sum_{\lambda=\pm 1/2} \left[c_\lambda(p)\, u(p,\lambda)\, e^{-ip\cdot x} + d^\dagger_\lambda(p)\, v(p,\lambda)\, e^{ip\cdot x}\right]$$

$$A^\mu(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E}} \sum_{\lambda=\pm 1,0} \left[a_\lambda(p)\, \epsilon^\mu(p,\lambda)\, e^{-ip\cdot x} + b^\dagger_\lambda(p)\, \epsilon^{\mu *}(p,\lambda)\, e^{ip\cdot x}\right]$$

The spin-0, -1/2 and -1 operators satisfy their corresponding non-trivial (anti-)commutation relations

$$[a(p), a^\dagger(p')] = [b(p), b^\dagger(p')] = (2\pi)^3\delta(\vec{p} - \vec{p}')$$
$$\{c_\lambda(p), c^\dagger_{\lambda'}(p')\} = \{d_\lambda(p), d^\dagger_{\lambda'}(p')\} = (2\pi)^3\delta(\vec{p} - \vec{p}')\delta_{\lambda\lambda'}$$
$$[a_\lambda(p), a^\dagger_{\lambda'}(p')] = [b_\lambda(p), b^\dagger_{\lambda'}(p')] = (2\pi)^3\delta(\vec{p} - \vec{p}')\delta_{\lambda\lambda'}$$

for guranteeing internal consistency of the quantum field theory.



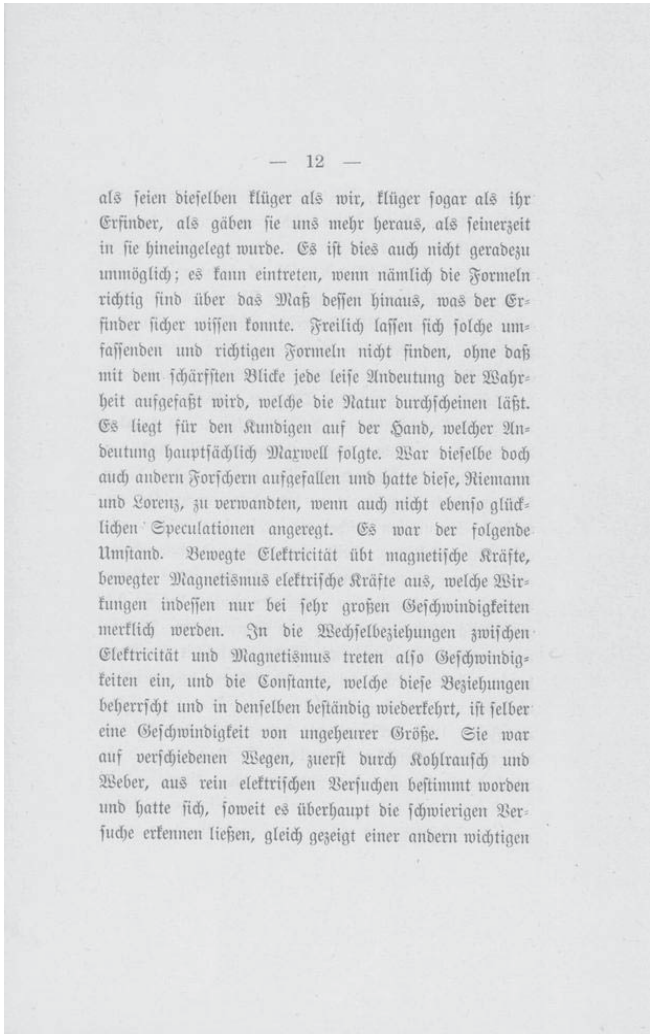
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Anfang an so hell hervorgetreten, daß er selbst weniger scharfsichtigen Augen als den seinen nicht hätte entgehen können.

Indessen ein so leichter und schneller Weg war der Wissenschaft nicht beschieden. Die Versuche gaben einstweilen keine Auskunft, und auch der Theorie lag ein Eingehen in Faraday's Gedankenkreis zunächst fern. Die Behauptung, daß elektrische Kräfte unabhängig von ihren Elektricitäten bestehen könnten, widersprach geradewegs den herrschenden elektrischen Theorien. Ebenso wies die herrschende Optik entschieden den Gedanken ab, es könnten die Wellen des Lichtes auch wohl anderer als elastischer Natur sein. Der Versuch, die eine oder die andere dieser Behauptungen eingehender zu behandeln, mußte fast als müssige Speculation erscheinen. Wie sehr müssen wir also den glücklichen Geist eines Mannes bewundern, welcher zwei Vermuthungen, die jede für sich so ferne lagen, so mit einander zu verknüpfen wußte, daß sie sich gegenseitig stützten, und daß das Ergebniß eine Theorie war, welcher man die innere Wahrscheinlichkeit von vorn herein nicht absprechen konnte. Der Mann, von welchem ich rede, war der Engländer Maxwell. Man kennt seine im Jahre 1865 veröffentlichte Arbeit unter dem Namen der elektromagnetischen Lichttheorie. Man kann diese wunderbare Theorie nicht studieren, ohne bisweilen die Empfindung zu haben, als wohne den mathematischen Formeln selbstständiges Leben und eigener Verstand inne,

Heinrich Hertz 1890

처음에는 눈이 예민하지 않은 사람도 놓칠 수 없을 정도로 눈부시게 빛났다. 그러나 과학은 그렇게 쉽고 빠른 길을 허락 받지 못했습니다. 실험은 당시 어떤 정보도 제공하지 않았으며 이론은 처음에는 패러데이의 사고 범위에 들어 가지 않았습니다. 전기력이 전기와 독립적으로 존재할 수 있다는 주장은 널리 퍼져 있는 전기 이론과 직접적으로 모순됩니다. 마찬가지로, 지배적인 광학계는 빛의 파동이 본질적으로 탄성이 아닐 수 있다는 생각을 단호하게 거부했습니다. 이러한 주장 중 하나 이상을 더 자세히 다루려는 시도는 거의 유휴 추측처럼 보였습니다. 그러므로 우리는 두 가지 추측을 결합하는 방법을 아는 사람의 행복한 마음을 얼마나 존경해야 합니까? 두 추측은 너무 멀리 떨어져 있어서 서로를 뒷받침하고 그 결과는 내적 개연성을 부여할 수 있는 이론이 되었습니다.' 처음부터 동의하지 마세요. 내가 말하고 있는 사람은 영국인 맥스웰이었습니다. 1865년에 출판된 그의 연구는 빛의 전자기 이론으로 알려져 있습니다. **수학 공식이 독립적인 생명과 그 자체의 정신을 가지고 있다는 느낌을 받지 않고는 이 놀라운 이론을 연구할 수 없습니다.**



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als seien dieselben klüger als wir, klüger sogar als ihr Erfinder, als gäben sie uns mehr heraus, als seinerzeit in sie hineingelegt wurde. Es ist dies auch nicht geradezu unmöglich; es kann eintreten, wenn nämlich die Formeln richtig sind über das Maß dessen hinaus, was der Erfinder sicher wissen konnte. Freilich lassen sich solche umfassenden und richtigen Formeln nicht finden, ohne daß mit dem schärfsten Blicke jede leise Andeutung der Wahrheit aufgefaßt wird, welche die Natur durchscheinen läßt. Es liegt für den Kundigen auf der Hand, welcher Andeutung hauptsächlich Maxwell folgte. War dieselbe doch auch andern Forschern aufgefallen und hatte diese, Riemann und Lorenz, zu verwandten, wenn auch nicht ebenso glücklichen Speculationen angeregt. Es war der folgende Umstand. Bewegte Elektricität übt magnetische Kräfte, bewegter Magnetismus elektrische Kräfte aus, welche Wirkungen indessen nur bei sehr großen Geschwindigkeiten merklich werden. In die Wechselbeziehungen zwischen Elektricität und Magnetismus treten also Geschwindigkeiten ein, und die Constante, welche diese Beziehungen beherrscht und in denselben beständig wiederkehrt, ist selber eine Geschwindigkeit von ungeheurer Größe. Sie war auf verschiedenen Wegen, zuerst durch Kohlrausch und Weber, aus rein elektrischen Versuchen bestimmt worden und hatte sich, soweit es überhaupt die schwierigen Versuche erkennen ließen, gleich gezeigt einer andern wichtigen

마치 그들이 우리보다 더 똑똑한 것처럼, 그들의 발명가보다도 더 똑똑한 것처럼, 그들이 당시 그들에게 투입된 것보다 더 많은 것을 우리에게 주는 것처럼 말입니다.
이것이 완전히 불가능한 것은 아닙니다. 발명가가 확실하게 알 수 있었던 것 이상으로 공식이 정확하면 이런 일이 발생할 수 있습니다. 물론, 자연이 빛나게 하는 진실의 모든 작은 힌트를 가장 예리한 눈으로 파악하지 않고서는 그러한 포괄적이고 정확한 공식을 찾을 수 없습니다. 맥스웰이 어떤 제안을 주로 따랐는지는 지식이 있는 사람에게는 명백하다. 이는 또한 다른 연구자들에 의해 발견되었고 리만(Riemann)과 로렌츠(Lorenz)가 똑같이 운이 좋지는 않더라도 관련된 추측을 하도록 영감을 주었습니다. 다음과 같은 상황이었습니다. 움직이는 전기는 자기력을 발휘하고, 움직이는 자기는 전기력을 발휘하며, 그 효과는 매우 빠른 속도에서만 눈에 띄게 나타납니다. 속도는 전기와 자기 사이의 상호 관계에 들어가며, 이러한 관계를 지배하고 그 관계에서 끊임없이 반복되는 상수는 그 자체로 엄청난 규모의 속도입니다. 그것은 처음에는 Kohlrausch와 Weber에 의해 순수한 전기적 실험을 통해 다양한 방식으로 결정되었으며, 어려운 실험을 통해 볼 수 있듯이 그것은 또 다른 중요한 실험과 유사한 것으로 나타났습니다