The integrated perturbation theory for cosmological tensor fields

Selected topics from:

T. Matsubara, PRD 110, 063543 [arXiv:2210.10435] (Paper I)
T. Matsubara, PRD 110, 063544 [arXiv:2210.11085] (Paper II)
T. Matsubara, PRD 110, 063545 [arXiv:2304.13304] (Paper III)
T. Matsubara, PRD 110, 063546 [arXiv:2405.09038] (Paper IV)

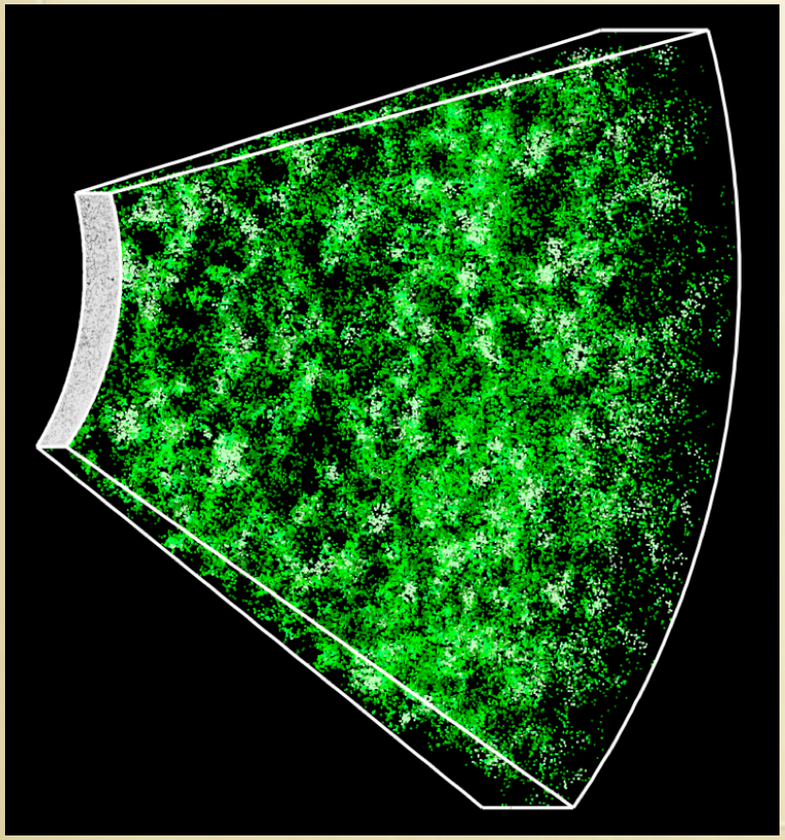
Takahiko Matsubara

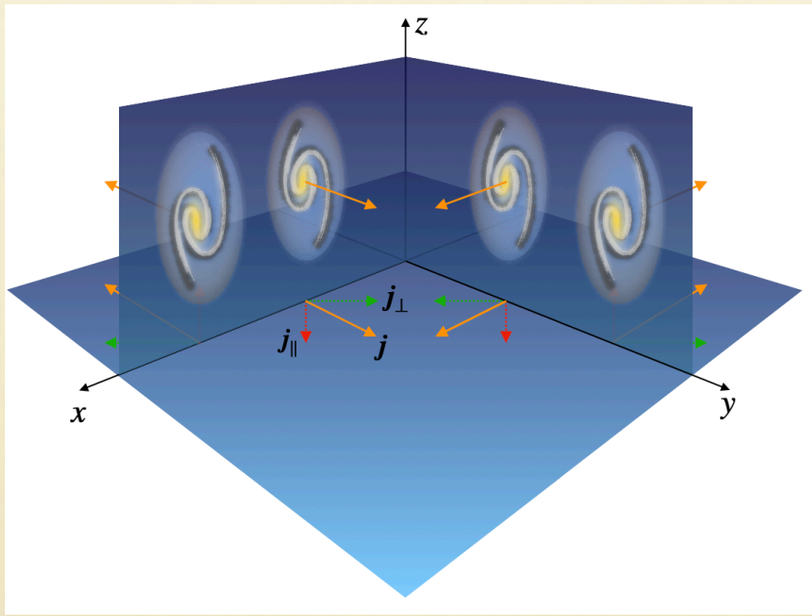
November 31, 2024

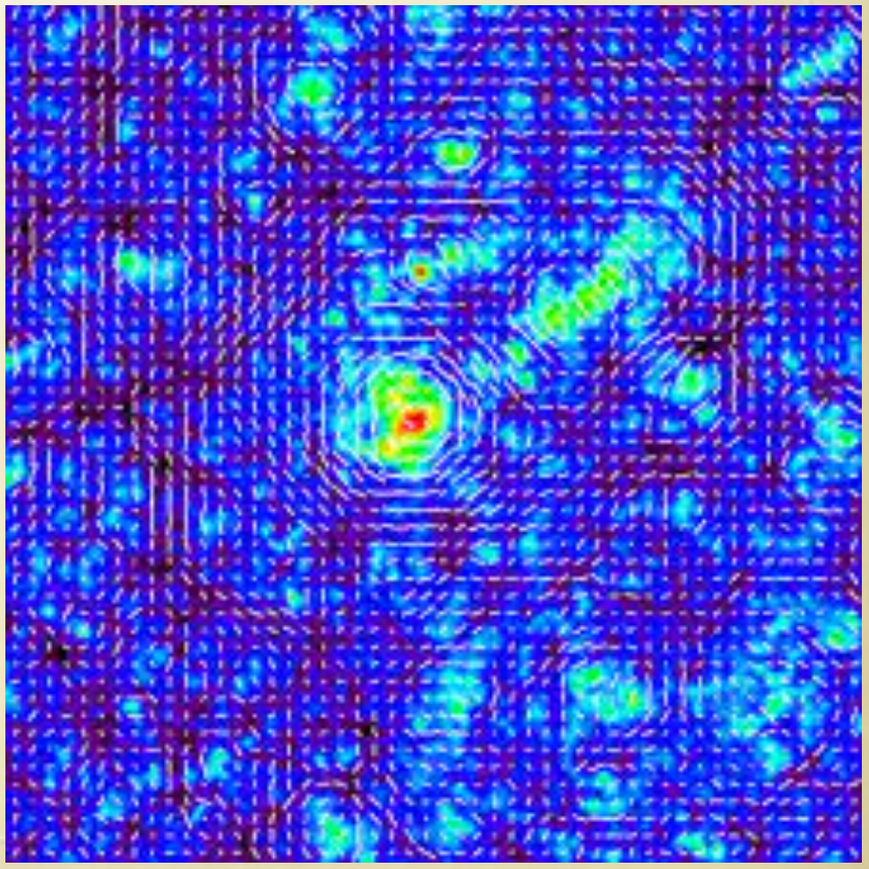
@Gyeongju

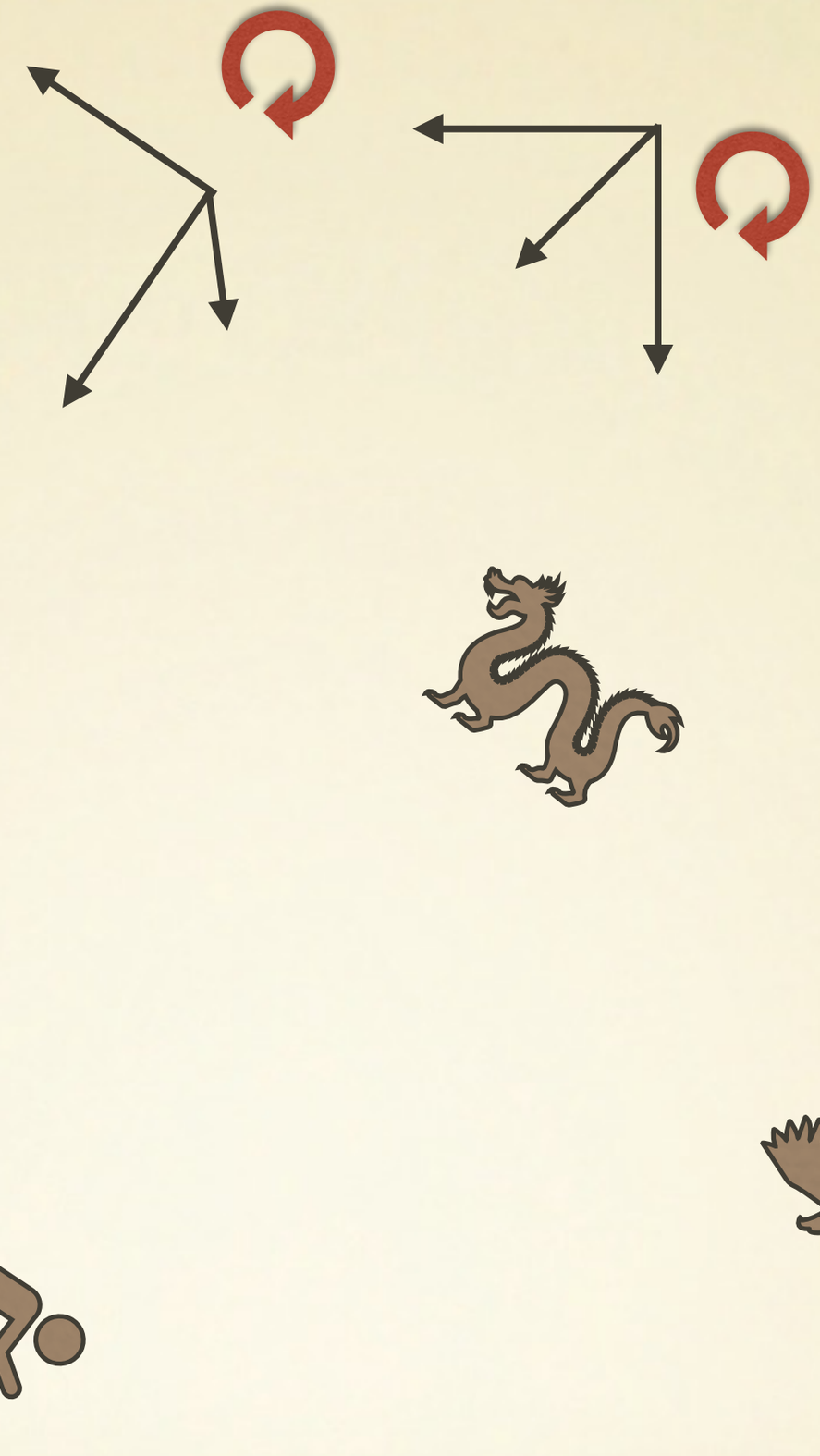
Tensor fields in cosmology

- Large-scale structure (LSS), weak lensing
  - galaxy density field: 3D rank-0 scalar field
  - galaxy angular momentum: 3D rank-1 vector field
  - galaxy shape field: 3D rank-2 (or higher) tensor fields









Moments of galaxy shapes

- Example of higher-rank tensor field
  - higher-order moments of galaxy shape

$$I_{i_1\cdots i_l}(\boldsymbol{x}) = \frac{\int d^3x' \, (x'_{i_1} - x_{i_1}) \cdots (x'_{i_l} - x_{i_l}) \rho(\boldsymbol{x}')}{\int d^3x' \, \rho(\boldsymbol{x}')}$$

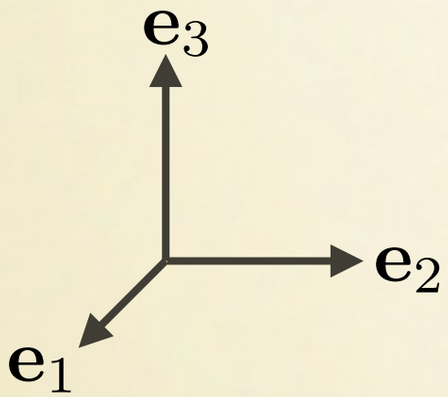
  - The higher-order shape field

$$I_{i_i\cdots i_l}(\boldsymbol{x}) = \sum_{a \in \mathrm{galaxies}} I_{i_1\cdots i_l}(\boldsymbol{x}_a) \delta^3_{\mathrm{D}}(\boldsymbol{x} - \boldsymbol{x}_a)$$

  - (Normalization is arbitrary)

Spherical basis

- Spherical basis in Cartesian coordinates



$$\mathbf{e}^0 \equiv \mathbf{e}_3, \quad \mathbf{e}^{\pm} \equiv \mp \frac{\mathbf{e}_1 \mp i\mathbf{e}_2}{\sqrt{2}}$$

- Traceless spherical tensor basis

$$\text{rank-0} : Y^{(0)} \equiv 1$$

$$\text{rank-1} : Y_i^{(0)} = \mathrm{e}^0{}_i,\ Y_i^{(\pm 1)} = \mathrm{e}^{\pm}{}_i$$

$$\text{rank-2} : Y_{ij}^{(0)} = \sqrt{\frac{3}{2}} \left(\mathrm{e}^0{}_i \mathrm{e}^0{}_j - \frac{\delta_{ij}}{3} \right),\ Y_{ij}^{(\pm 1)} = \sqrt{2} \mathrm{e}^0{}_{(i} \mathrm{e}^{\pm}{}_{j)},\ Y_{ij}^{(\pm 2)} = \mathrm{e}^{\pm}{}_{(i,} \mathrm{e}^{\pm}{}_{j)}$$

$$\cdots$$

$$\text{rank-}l : Y_{i_1 \cdots i_l}^{(m)} = \cdots \ (m = 0, \pm 1, \ldots, \pm l)$$

Decomposition of tensor into spherical basis

- Any symmetric tensor can be decomposed into traceless tensors

$$T_{i_1 i_2 \cdots i_l} = T^{(l)}_{i_1 i_2 \cdots i_l} + \frac{l(l-1)}{2(2l-1)} \delta_{(i_1 i_2} T^{(l-2)}_{i_3 \cdots i_l)} + \cdots$$

- Decomposition of traceless tensor field into spherical basis:

$$F^{(l)}_{X i_1 \cdots i_l}(\boldsymbol{x}) = F_{X l m}(\boldsymbol{x}) Y^{(m)}_{i_1 \cdots i_l}$$

Power spectrum of tensor field

- Definition of the power spectrum in Fourier space

$$\langle F_{X_1 l_1 m_1}(\boldsymbol{k}) F_{X_2 l_2 m_2}(\boldsymbol{k}')\rangle = (2\pi)^3 \delta_{\rm D}^3(\boldsymbol{k}+\boldsymbol{k}') P^{(l_1 l_2)}_{X_1 X_2 m_1 m_2}(\boldsymbol{k})$$

- Statistical isotropy
  - When the Universe is statistically isotropic, the power spectrum should be invariant under the coordinate rotation
  - In this case, the power spectrum should take the following form,

$$P^{(l_1 l_2)}_{X_1 X_2 m_1 m_2}(\boldsymbol{k}) = \sum_l (l_1\, l_2\, l)_{m_1 m_2}{}^m\, C_{lm}(\hat{\boldsymbol{k}}) P^{l_1 l_2; l}_{X_1 X_2}(k)$$

3j-symbol

Spherical harmonics
(Racah's normalization)

Invariant spectrum

Symmetries of invariant spectrum

- Complex conjugate

$$P^{l_1 l_2; l\,*}_{X_1 X_2}(k) = P^{l_1 l_2; l}_{X_1 X_2}(k) \quad \text{i.e., real function}$$

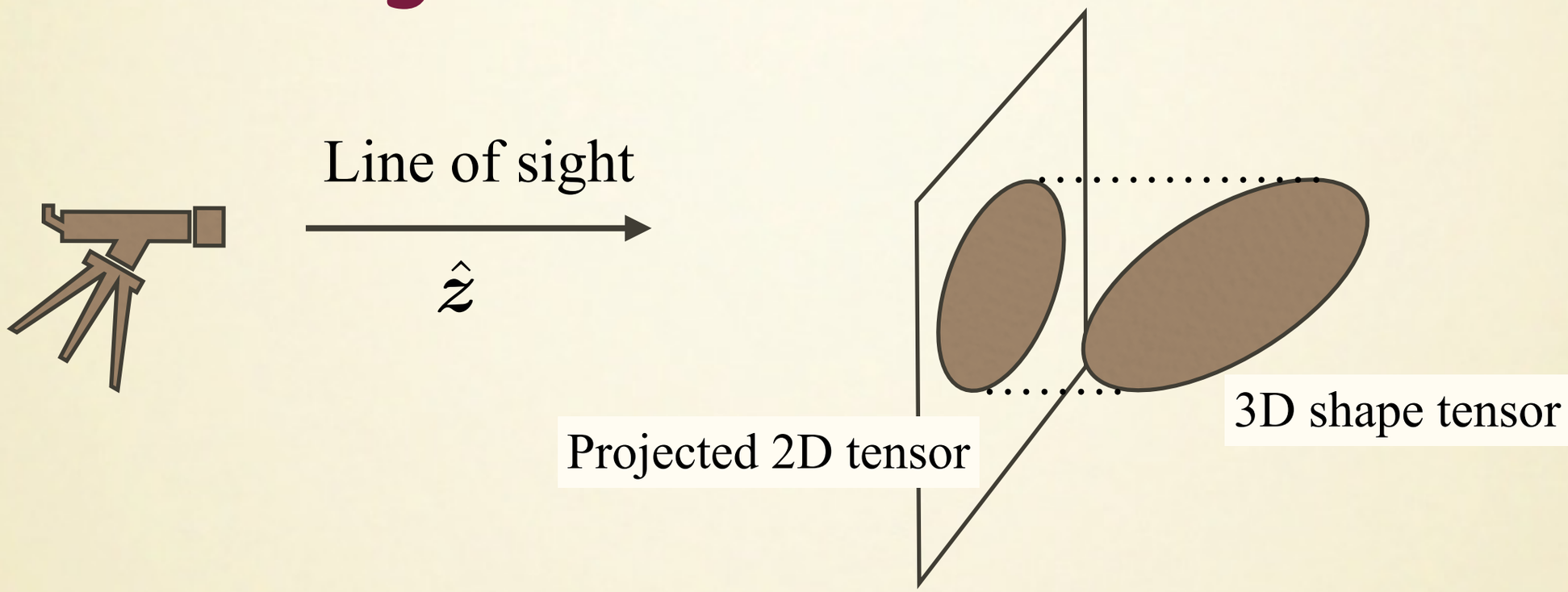
- Parity

$$P^{l_1 l_2; l}_{X_1 X_2}(k) \xrightarrow{\mathbb{P}} (-1)^{p_{X_1} + p_{X_2} + l_1 + l_2 + l} P^{l_1 l_2; l}_{X_1 X_2}(k)$$

- interchange

$$P^{l_2 l_1; l}_{X_2 X_1}(k) = (-1)^{l_1 + l_2} P^{l_1 l_2; l}_{X_1 X_2}(k)$$

Projection effects



- Measurable tensors in realistic observations
  - 2D projected tensor on the sky

$$f_{X i_1 \cdots i_s}(\boldsymbol{x}) = \mathcal{P}_{i_1 j_1} \cdots \mathcal{P}_{i_s j_s} F_{X j_1 \cdots j_s}(\boldsymbol{x})$$

$$\mathcal{P}_{ij} = \delta_{ij} - \hat{z}_i \hat{z}_j \quad \text{(projection tensor)}$$

(distant-observer approximation applied)

2D irreducible decomposition

- 2D spherical basis

$$\mathbf{m}^{\pm} \equiv \mp \frac{\mathbf{e}_1 \mp i\mathbf{e}_2}{\sqrt{2}}$$

- Decomposition of 2D traceless tensor into 2D spherical basis

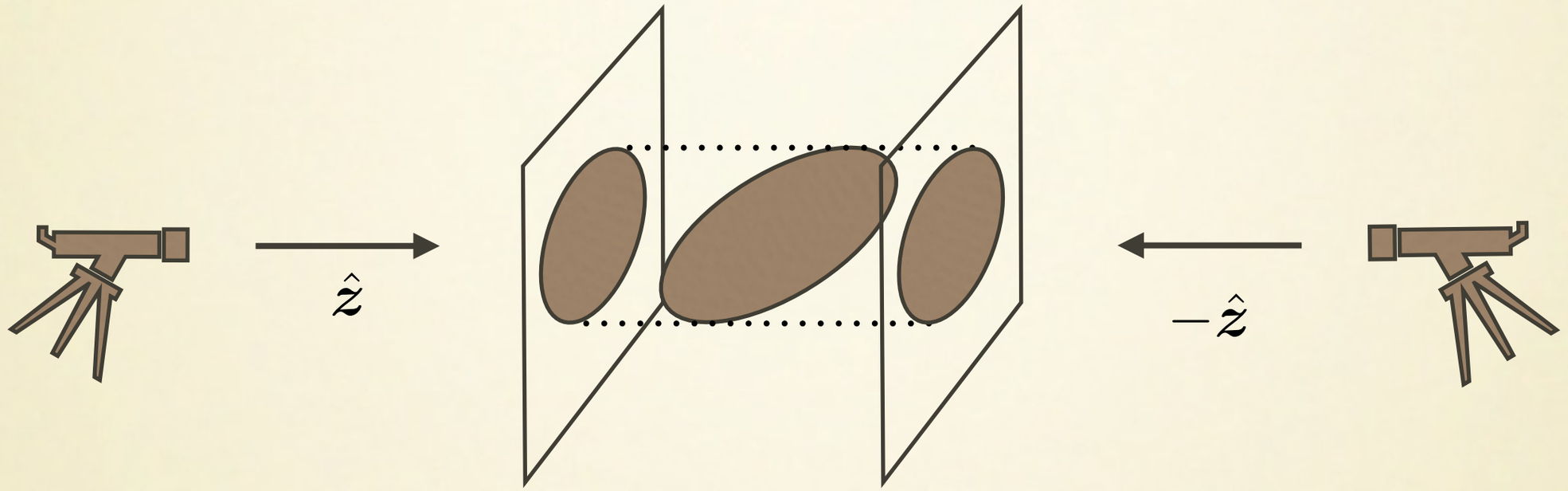
$$f^{(s)}_{Xi_1\cdots i_s}(\boldsymbol{x}) = f^{(+s)}_X(\boldsymbol{x})\mathrm{m}^+_{i_1}\cdots\mathrm{m}^+_{i_s} + f^{(-s)}_X(\boldsymbol{x})\mathrm{m}^-_{i_1}\cdots\mathrm{m}^-_{i_s}$$

- Relation between 3D & 2D irreducible tensors

$$f^{(\pm s)}_X(\boldsymbol{x}) = i^s\sqrt{\frac{s!}{(2s-1)!}}\underbrace{F_{Xs,\pm s}(\boldsymbol{x})}$$
$$= F_{Xlm}(\boldsymbol{x})|_{l=s,m=\pm s}$$

  - (The last eq. shows that the projected tensor from a 3D traceless tensor remains traceless also in the projected 2D space)

Flip symmetry of projected field



- Flip symmetry: invariance under $\hat{\boldsymbol{z}} \xrightarrow{\mathbb{F}} -\hat{\boldsymbol{z}}$

$$f_X^{(\pm s)}(\boldsymbol{x}) \xrightarrow{\mathbb{F}} f_X'^{(\pm s)}(\boldsymbol{x}) = e^{\pm 2is\phi} f_X^{(\mp s)}(-\boldsymbol{x}), \quad [\boldsymbol{x} : (x, \theta, \phi)]$$

- Parity+flip

$$f_X^{(\pm s)}(\boldsymbol{x}) \xrightarrow{\mathbb{PF}} f_X'^{(\pm s)}(\boldsymbol{x}) = (-1)^{p_X+s} e^{\pm 2is\phi} f_X^{(\mp s)}(\boldsymbol{x})$$

  - Parity+flip in distant-observer limit is more similar to the “parity” in full-sky spherical coordinates

E/B decomposition of projected field

- In Fourier space,

$$f_X^{(\pm s)}(\boldsymbol{k}) = (\mp i)^s e^{\pm is\phi}\left[f_X^{\mathrm{E}(s)}(\boldsymbol{k}) \pm i f_X^{\mathrm{B}(s)}(\boldsymbol{k})\right]$$

- PF symmetry is simply given in the E/B modes

$$f_X^{\mathrm{E}(s)}(\boldsymbol{k}) \xrightarrow{\mathbb{PF}} (-1)^{p_X+s} f_X^{\mathrm{E}(s)}(\boldsymbol{k})$$
$$f_X^{\mathrm{B}(s)}(\boldsymbol{k}) \xrightarrow{\mathbb{PF}} (-1)^{p_X+s+1} f_X^{\mathrm{B}(s)}(\boldsymbol{k})$$

  - When p_X+s=even, E mode is parity even, B mode is parity odd
  - When p_X+s=odd, E mode is parity odd, B mode is parity even

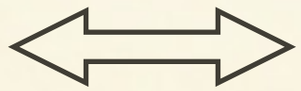
E/B decomposition



E mode

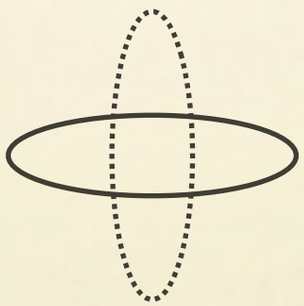
B mode

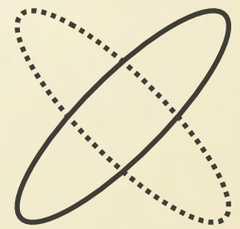
$s = 1$



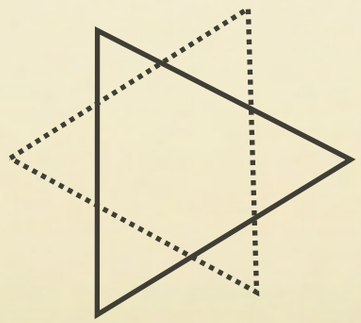


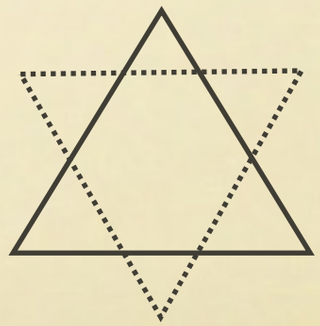
$s = 2$





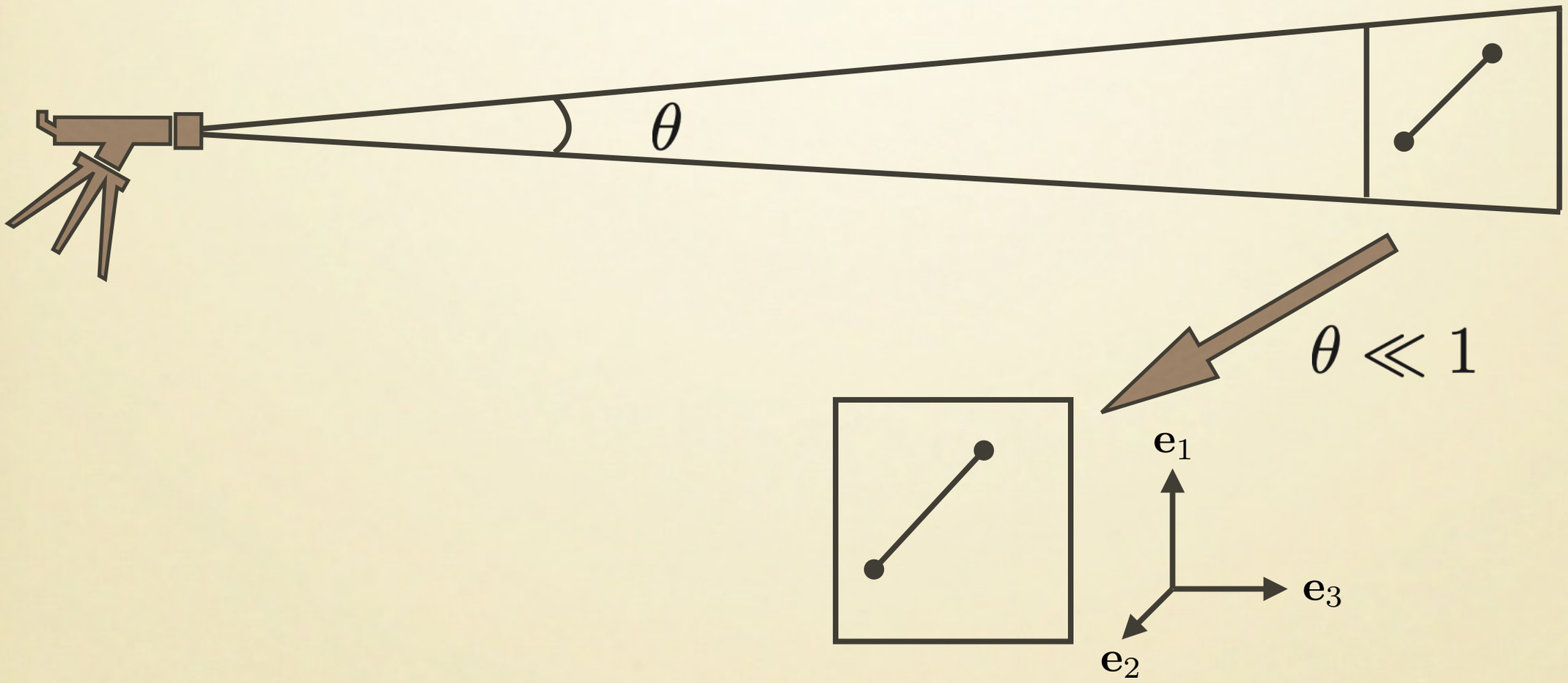
$s = 3$





Distant-observer approximation

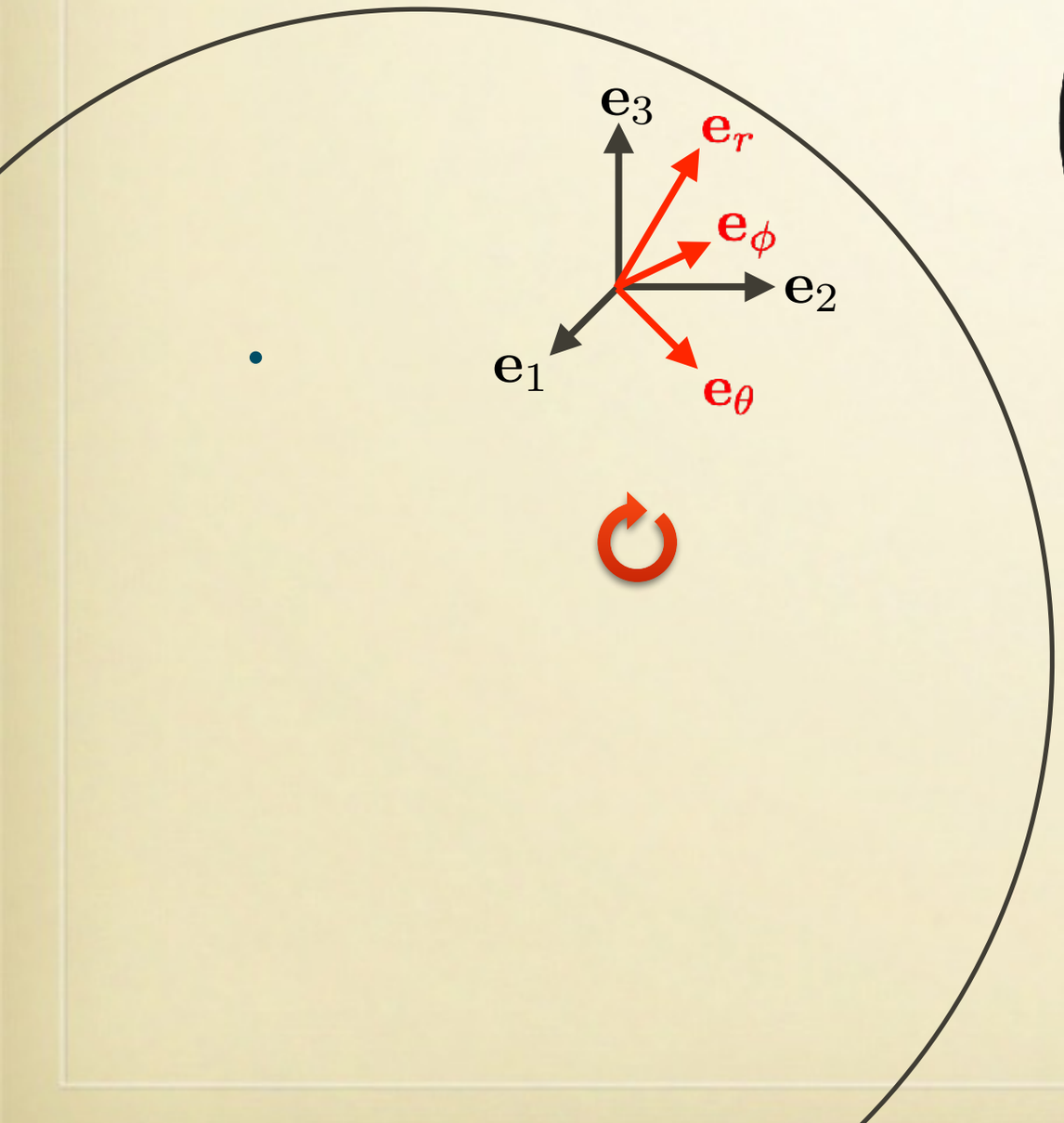
- DOA: clustering scales are much smaller than the distance to the objects



Spherical coordinates → Cartesian Coordinates

Full-sky formulation: spherical coordinates

- Spherical basis in spherical coordinates

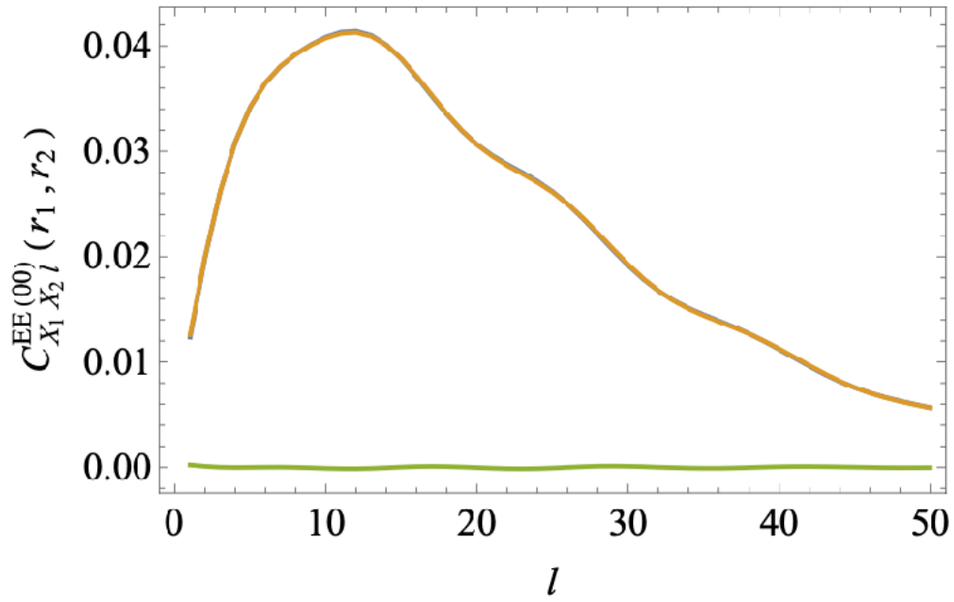


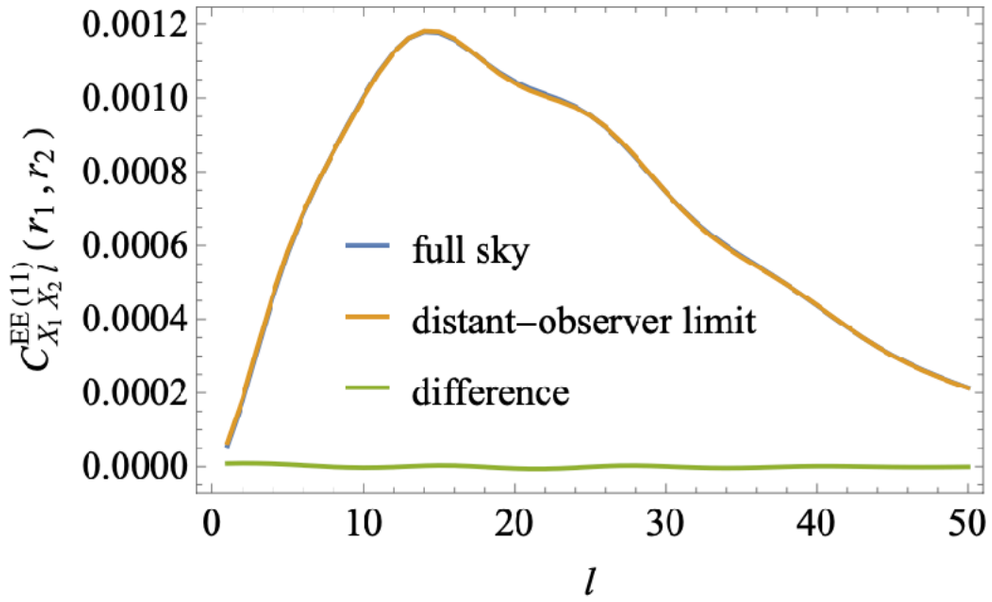
$$\begin{pmatrix} \mathbf{e}_\theta(\theta,\phi) \\ \mathbf{e}_\phi(\theta,\phi) \\ \mathbf{e}_r(\theta,\phi) \end{pmatrix} = \begin{pmatrix} \cos\theta\,\cos\phi & \cos\theta\,\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\,\cos\phi & \sin\theta\,\sin\phi & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

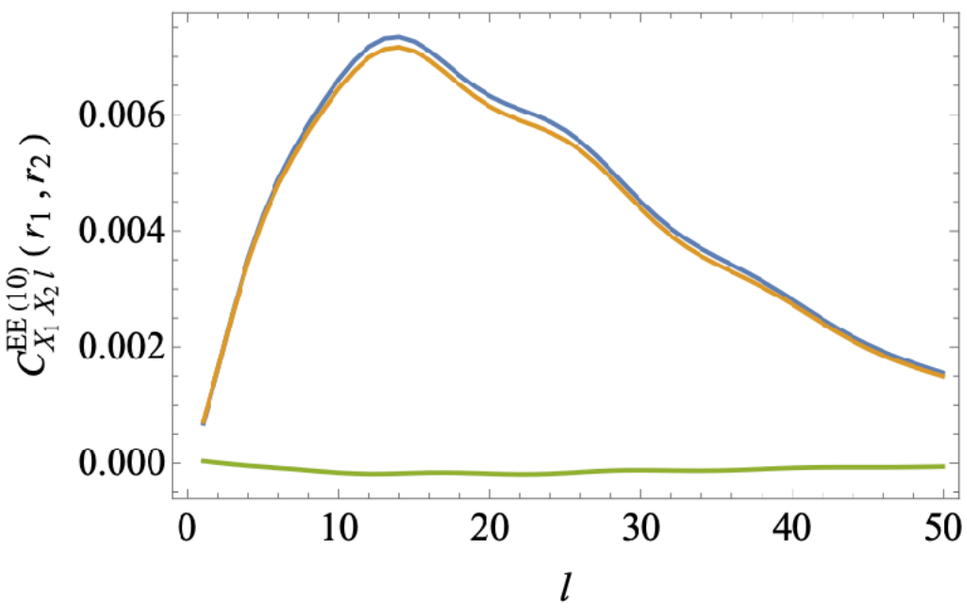
$$\rightarrow S_{ij}(\theta,\phi)$$

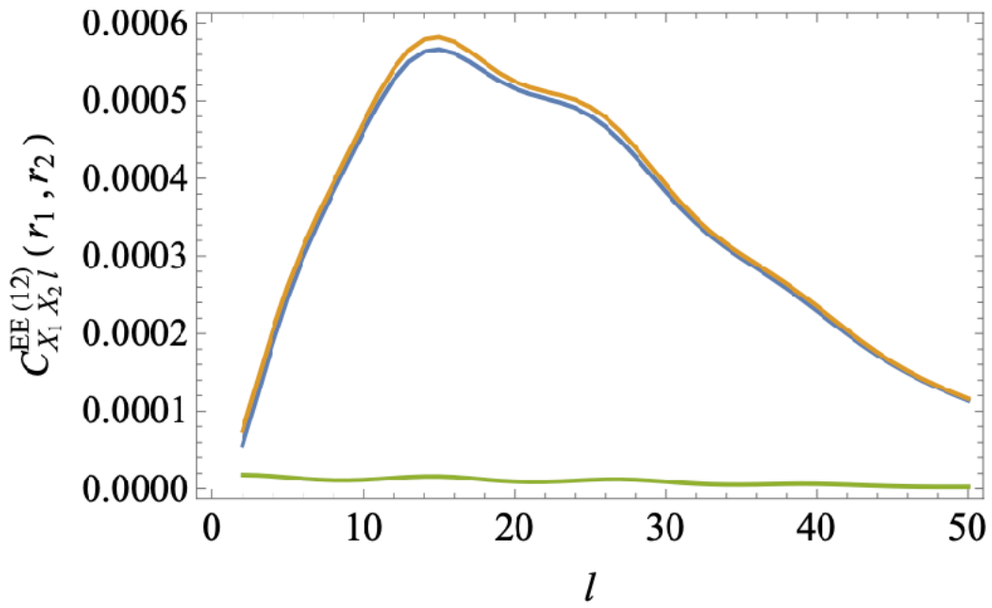
$$\begin{aligned} \tilde{\mathsf{Y}}^{(\sigma)}_{i_1\cdots i_s}(\theta,\phi) &= \mathsf{Y}^{(\sigma)}_{i_1\cdots i_s}\big|_{\mathbf{e}_1\to\mathbf{e}_\theta,\mathbf{e}_2\to\mathbf{e}_\phi,\mathbf{e}_3\to\mathbf{e}_r} \\ &= \mathsf{Y}^{(\sigma)}_{j_i\cdots j_s} S_{j_1 i_1}(\theta,\phi)\cdots S_{j_s i_s}(\theta,\phi) \\ &= \sum_{\sigma'} \mathsf{Y}^{(\sigma')}_{i_1\cdots i_s} D^{\sigma' *}_{(s)\sigma}(\phi,\theta,0) \end{aligned}$$

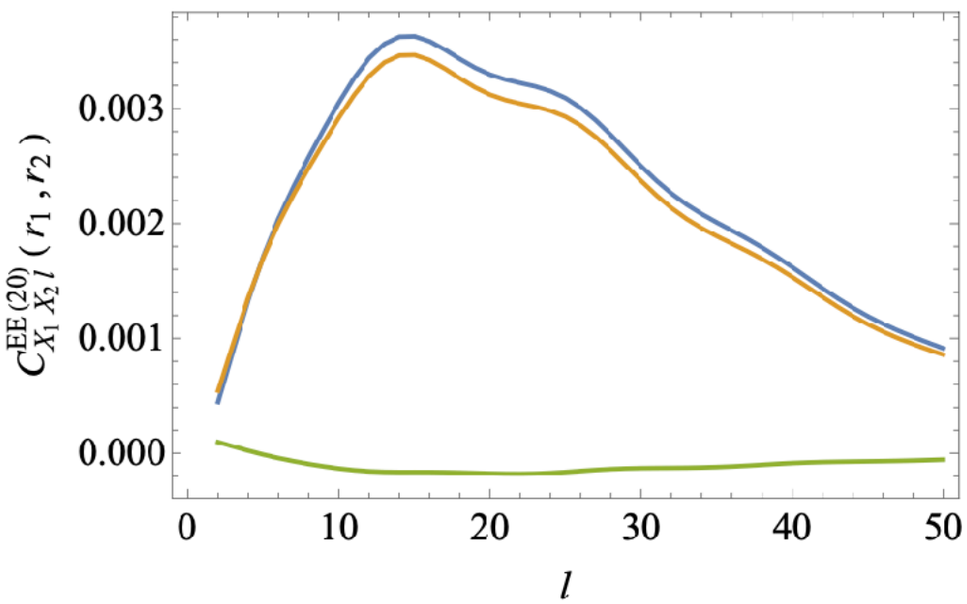
Wigner's rotation matrix

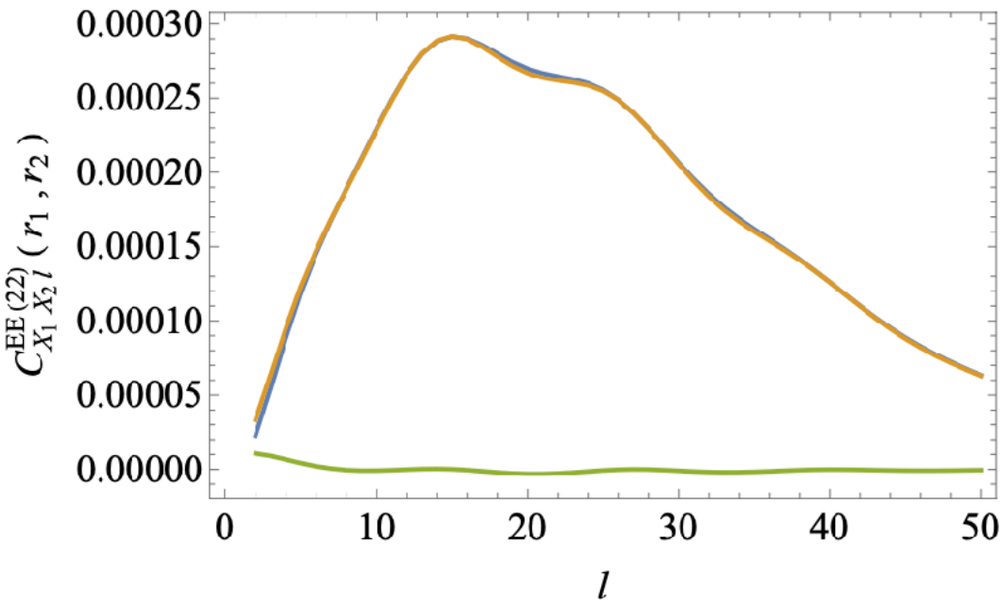

0.04
0.03
0.02
0.01
0.00
0
10
20
30
40
50
l
$C^{\mathrm{EE}\,(00)}_{X_1 X_2 l}(r_1, r_2)$

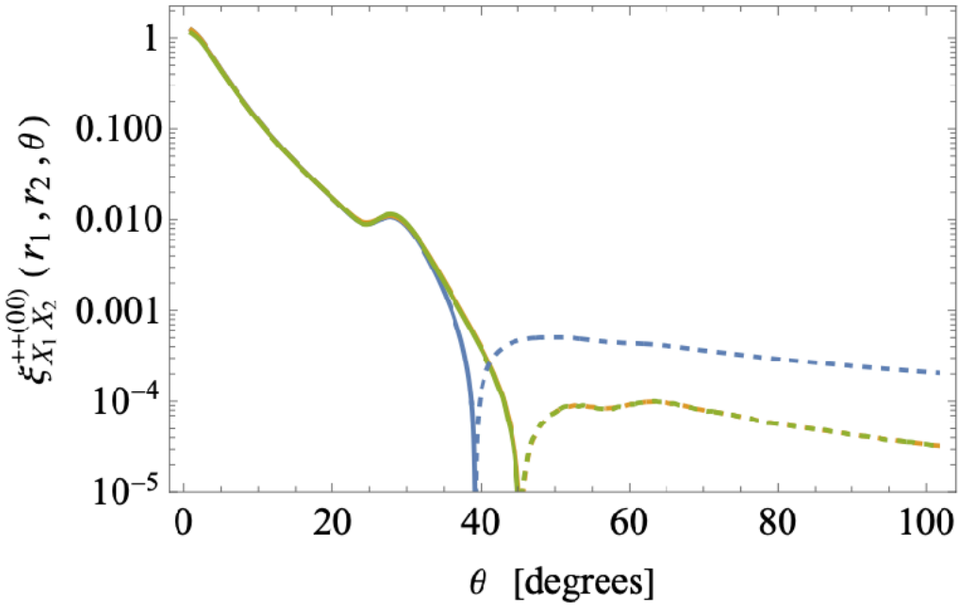

0.0012
0.0010
0.0008
0.0006
0.0004
0.0002
0.0000
full sky
distant–observer limit
difference
0
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20
30
40
50
l
$C^{\mathrm{EE}\,(11)}_{X_1 X_2 l}(r_1, r_2)$

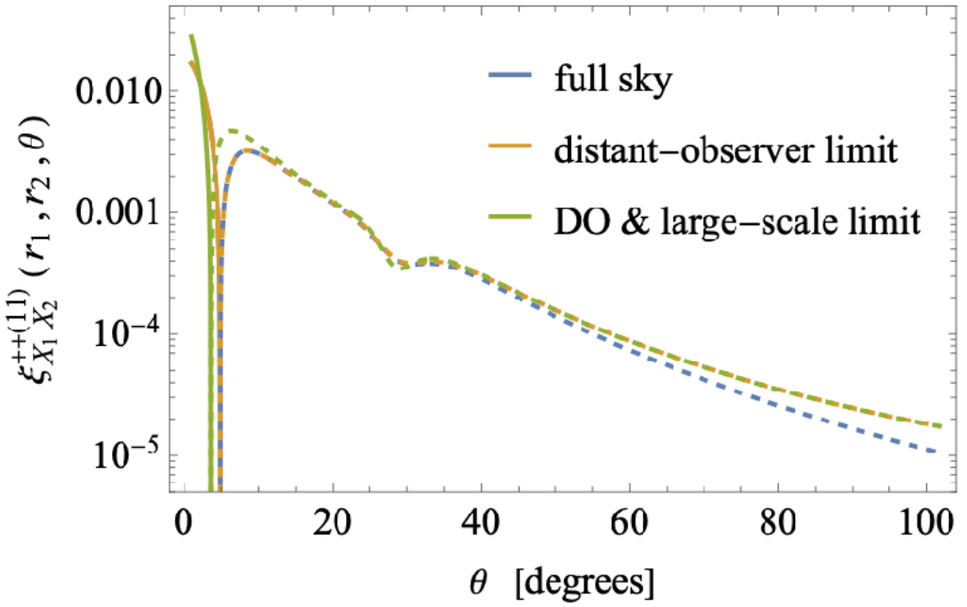

0.006
0.004
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0
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l
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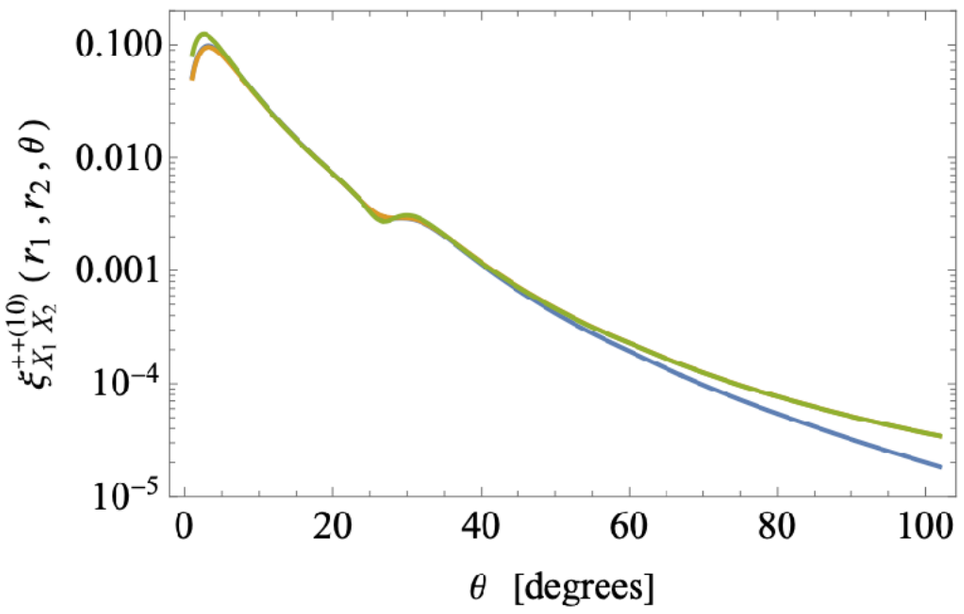

0.0006
0.0005
0.0004
0.0003
0.0002
0.0001
0.0000
0
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l
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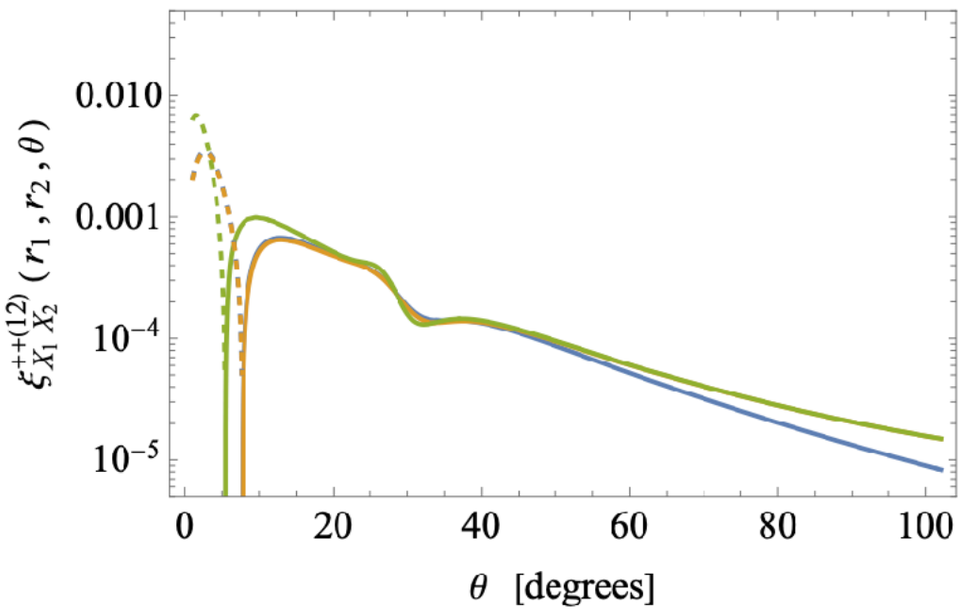

0.003
0.002
0.001
0.000
0
10
20
30
40
50
l
$C^{\mathrm{EE}\,(20)}_{X_1 X_2 l}(r_1, r_2)$

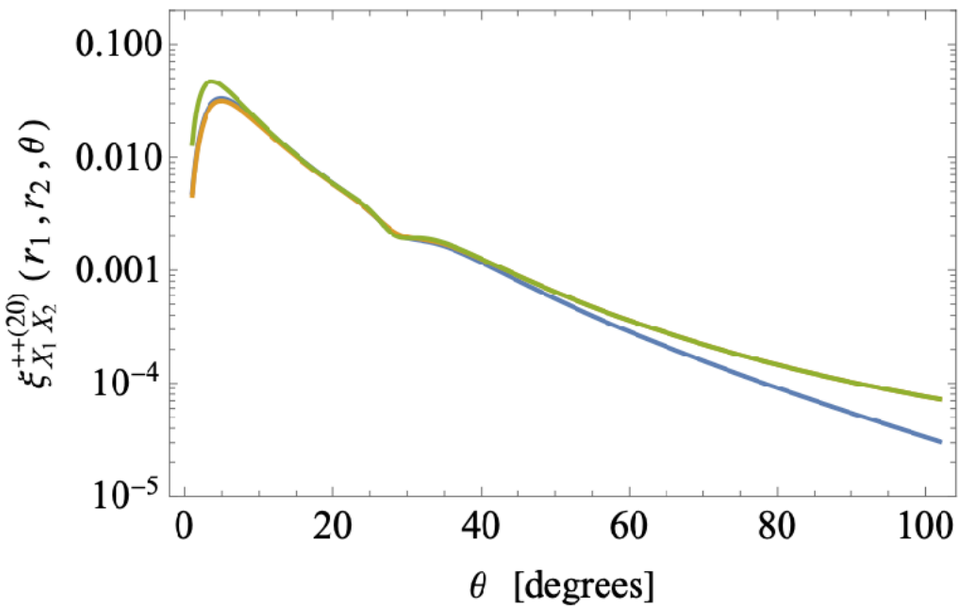

0.00030
0.00025
0.00020
0.00015
0.00010
0.00005
0.00000
0
10
20
30
40
50
l
$C^{\mathrm{EE}\,(22)}_{X_1 X_2 l}(r_1, r_2)$

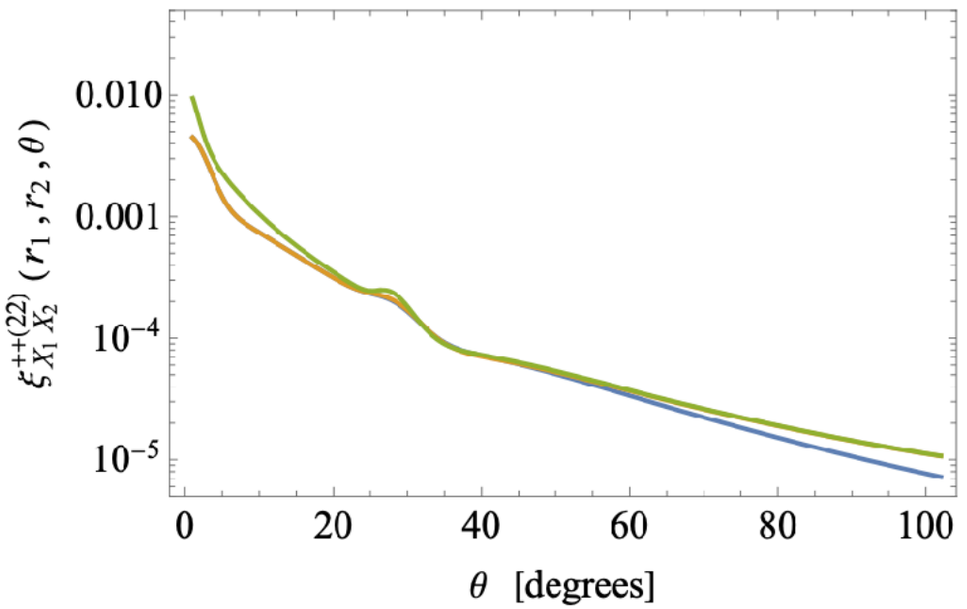

1
0.100
0.010
0.001
10^{-4}
10^{-5}
$\xi^{++(00)}_{X_1X_2}(r_1, r_2, \theta)$
0
20
40
60
80
100
θ [degrees]


0.010
0.001
10^{-4}
10^{-5}
$\xi^{++(11)}_{X_1X_2}(r_1, r_2, \theta)$
full sky
distant−observer limit
DO & large−scale limit
0
20
40
60
80
100
θ [degrees]


0.100
0.010
0.001
10^{-4}
10^{-5}
$\xi^{++(10)}_{X_1X_2}(r_1, r_2, \theta)$
0
20
40
60
80
100
θ [degrees]


0.010
0.001
10^{-4}
10^{-5}
$\xi^{++(12)}_{X_1X_2}(r_1, r_2, \theta)$
0
20
40
60
80
100
θ [degrees]


0.100
0.010
0.001
10^{-4}
10^{-5}
$\xi^{++(20)}_{X_1X_2}(r_1, r_2, \theta)$
0
20
40
60
80
100
θ [degrees]


0.010
0.001
10^{-4}
10^{-5}
$\xi^{++(22)}_{X_1X_2}(r_1, r_2, \theta)$
0
20
40
60
80
100
θ [degrees]

Application of the perturbation theory

- Theoretical predictions from perturbation theory
  - In Paper I, systematic methods to derive the invariant spectrum from the “integrated Perturbation Theory” (iPT, Matsubara 2011) are formulated
  - In Paper II, further methods to calculate nonlinear corrections in perturbation theory are developed
  - In Paper III, the iPT is applied to the formulas of projection effects
  - In Paper IV, the formulas are generalized to those in the full-sky, without assuming the flat-sky, distant-observer limit.