Testing statistical isotropy of the universe using morphological statistics

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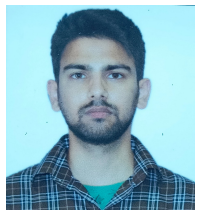
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Plan of talk



- **Minkowski functionals for composite fields**

  Based on PC & **Fazlu Rahman** 2024





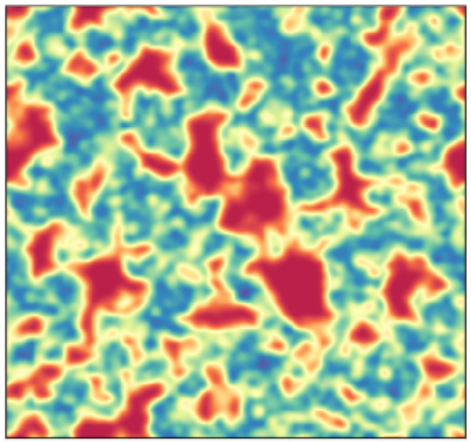
- **Application of morphological statistics to test statistical isotropy**

  Based on **Masroor Bashir** et al. in prep

  Priya Goyal & PC 2021

with Stephen Appleby & Changbom Park.

Morphological statistics



Minkowski functionals: $V_0,\ V_1,\ V_2$

Betti numbers: $b_c,\ b_h$

Minkowski tensors: α

Spectral parameters

Variances: $\sigma_0,\ \sigma_1,\ \sigma_2$

Skewness: $S^{(0)},\ S^{(1)},\ S^{(2)}$

Kurtosis: $K^{(0)},\ K^{(1)},\ K_1^{(2)},\ K_2^{(2)},\ K^{(3)}$

Typical scales: $r_c = \frac{\sigma_0}{\sigma_1},\ R_* = \frac{\sigma_1}{\sigma_2}$,...

Ensemble expectation of morphological statistics are functions of the field level ν, with coefficients given by functions of the spectral variables.

MFs for mildly non-Gaussian fields - d dimensions

Matsubara 1996, 2003, 2011, 2020

$$
\begin{aligned}
S^{(0)} &= \frac{\langle f^3\rangle_c}{\sigma_0^4}, \\
S^{(1)} &= \frac{3}{2}\frac{\langle f|\nabla f|^2\rangle_c}{\sigma_0^2\sigma_1^2}, \\
S^{(2)} &= \frac{-3d}{2(d-1)}\frac{\langle|\nabla f|^2\nabla^2 f\rangle_c}{\sigma_1^4}.
\end{aligned}
\qquad
\begin{aligned}
K^{(0)} &= \frac{\langle f^4\rangle_c}{\sigma_0^6}, \\
K^{(1)} &= 2\frac{\langle 2f^2|\nabla f|^2\rangle_c}{\sigma_0^4\sigma_1^2}, \\
K_1^{(2)} &= \frac{-2d}{(d+2)(d-1)}\frac{(d+2)\langle f|\nabla f|^2\nabla^2 f\rangle_c + \langle|\nabla f|^4\rangle_c}{\sigma_0^2\sigma_1^4}, \\
K_2^{(2)} &= \frac{-2d}{(d+2)(d-1)}\frac{(d+2)\langle f|\nabla f|^2\nabla^2 f\rangle_c + d\langle|\nabla f|^4\rangle_c}{\sigma_0^2\sigma_1^4}, \\
K^{(3)} &= \frac{2d^2}{(d-1)(d-2)}\frac{\langle|\nabla f|^2(\nabla^2 f)^2\rangle_c - \langle|\nabla f|^2 f_{ij}f_{ij}\rangle_c}{\sigma_1^6}.
\end{aligned}
$$

$$
\begin{aligned}
\bar{V}_k^{(d)}(\nu) \simeq A_k e^{-\nu^2/2}\Bigg[& H_{k-1}(\nu) + \left\{\frac{1}{6}S^{(0)}H_{k+2}(\nu) + \frac{k}{3}S^{(1)}H_k(\nu) + \frac{k(k-1)}{6}S^{(2)}H_{k-2}(\nu)\right\}\sigma_0 \\
& + \left\{\frac{1}{72}(S^{(0)})^2 H_{k+5}(\nu) + \left(\frac{1}{24}K^{(0)} + \frac{k}{18}S^{(0)}S^{(1)}\right)H_{k+3}(\nu)\right. \\
& + k\left(\frac{1}{8}K^{(1)} + \frac{k-1}{36}S^{(0)}S^{(2)} + \frac{k-2}{18}(S^{(1)})^2\right)H_{k+1}(\nu) \\
& + k\left(\frac{k-2}{16}K_1^{(2)} + \frac{k}{16}K_2^{(2)} + \frac{(k-1)(k-4)}{18}S^{(1)}S^{(2)}\right)H_{k-1}(\nu) \\
& \left. + k(k-1)(k-2)\left(\frac{1}{24}K^{(3)} + \frac{k-7}{72}(S^{(2)})^2\right)H_{k-3}(\nu)\right\}\sigma_0^2 + \mathcal{O}(\sigma_0^3)\Bigg],
\end{aligned}
$$

$$A_k \propto r_c^{-k}$$

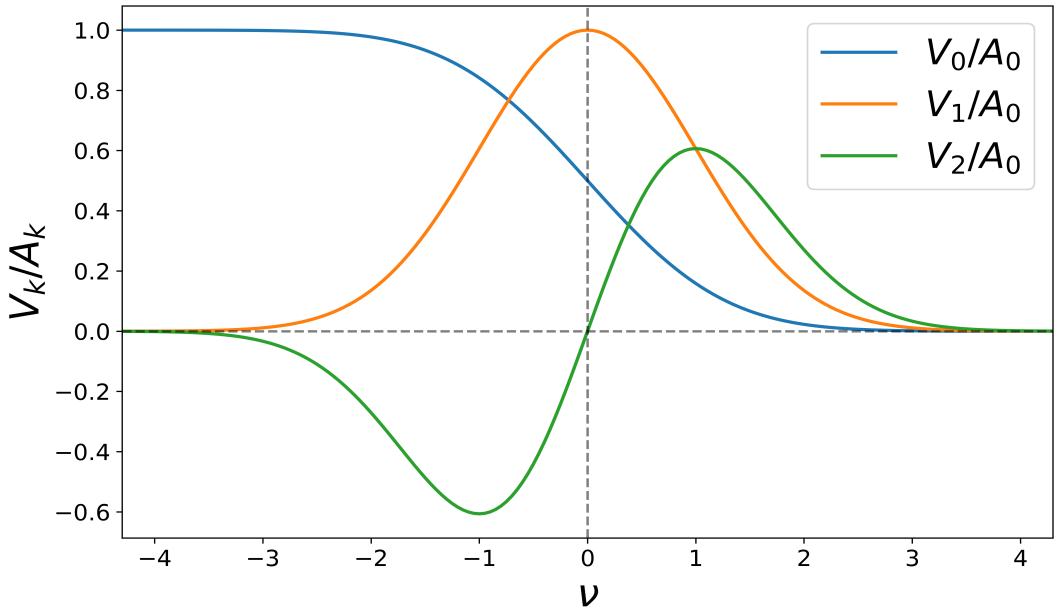
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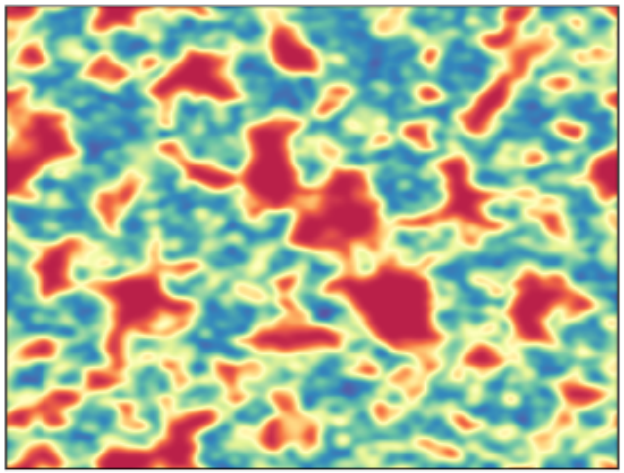
$$
\bar{V}_k^{(d)}(\nu) \simeq A_k e^{-\nu^2/2}\left[H_{k-1}(\nu) + \left\{\frac{1}{6}S^{(0)}H_{k+2}(\nu) + \frac{k}{3}S^{(1)}H_k(\nu) + \frac{k(k-1)}{6}S^{(2)}H_{k-2}(\nu)\right\}\sigma_0\right.
$$



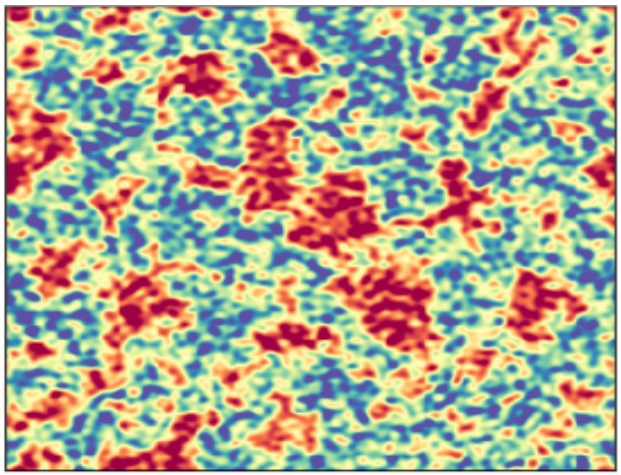
Composite field

$$\text{Observed data} = \text{true signal} + \underbrace{\text{noise} + \text{contaminating signal} + \ldots.}_{\text{secondary field} \equiv v}$$

u



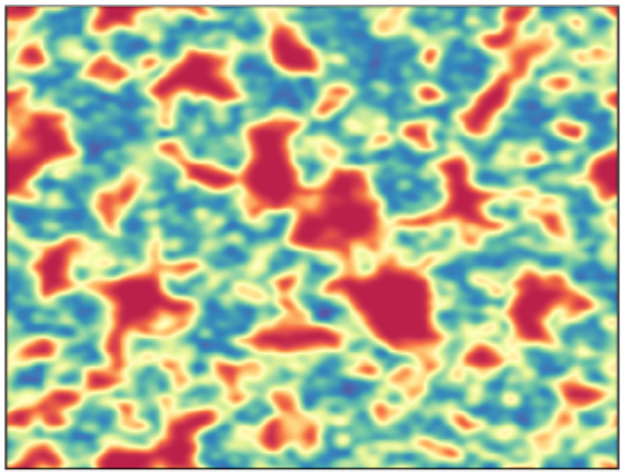
$f = u + v$



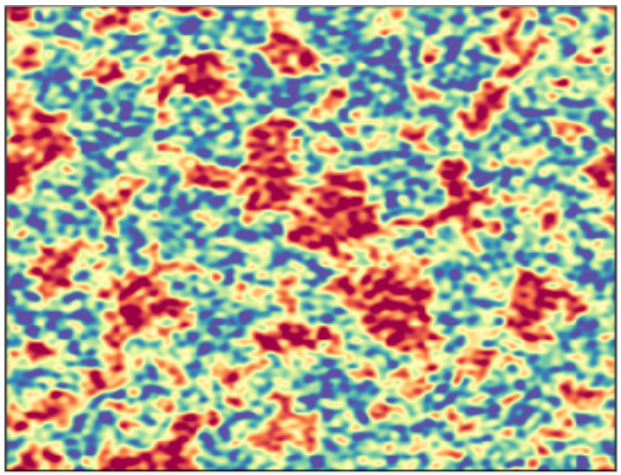
Composite field

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u



$f = u + v$



Question

Can we quantify the effect of secondary fields on the morphological properties of the true signal?

Answer: Yes*.

Strategy

1. Factorize as Minkowski functionals of u $\times$ bias factor due to v.

2. Introduce ratios of spectral parameters of u and v:

$$\epsilon \equiv \frac{\sigma_0^v}{\sigma_0^u}, \qquad p \equiv \frac{r_c^u}{r_c^v} = \epsilon^{-1}\frac{\sigma_1^v}{\sigma_1^u}.$$

$$c^{uv} = \frac{\langle uv \rangle}{\sigma_0^u \sigma_0^v}, \qquad c_1^{uv} = \frac{\langle \nabla u \cdot \nabla v \rangle}{\sigma_1^u \sigma_1^v}.$$

By definition, $0 < \epsilon < \infty, \quad 0 < p < \infty, \quad |c^{uv}| \leq 1, \quad |c_1^{uv}| \leq 1.$

Effect on amplitudes:

$$\left(r_c^f\right)^{-k} \quad = \quad = \left(r_c^u\right)^{-k} \left[\frac{1+\epsilon^2 p^2 + 2p\,\epsilon\, c_1^{uv}}{1+\epsilon^2+2\epsilon c^{uv}}\right]^{k/2}.$$

For $c^{uv}=0$ and $c_1^{uv}=0$,

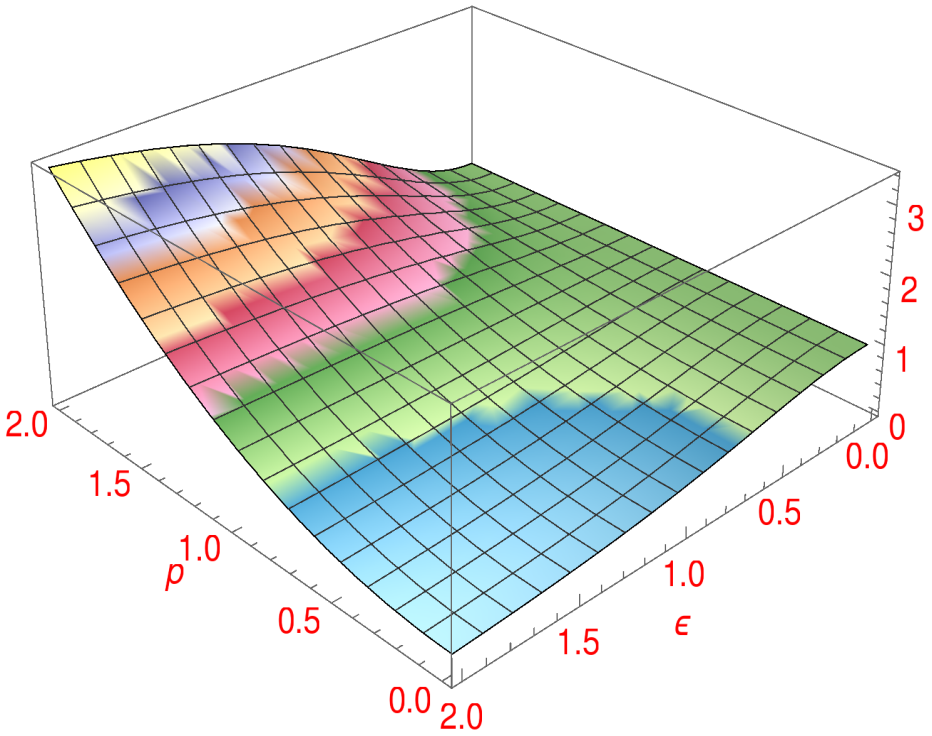
$$\left(r_c^f\right)^{-k} \quad = \quad \left(r_c^u\right)^{-k} \left[\frac{1+\epsilon^2 p^2}{1+\epsilon^2}\right]^{k/2}.$$

Effect on Skewness for non-Gaussian u and Gaussian v:

$c^{uv}=0$ and $c_1^{uv}=0$

$$\begin{aligned}
S^{(0)\,f}\sigma_0^f &= S^{(0)\,u}\sigma_0^u \,\frac{1}{\left(1+\epsilon^2\right)^{3/2}},\\
S^{(1)\,f}\sigma_0^f &= S^{(1)\,u}\sigma_0^u \,\frac{1}{\left(1+\epsilon^2\right)^{1/2}\left(1+\epsilon^2 p^2\right)},\\
S^{(2)\,f}\sigma_0^f &= S^{(2)\,u}\sigma_0^u \,\frac{\left(1+\epsilon^2\right)^{1/2}}{\left(1+\epsilon^2 p^2\right)^2}.
\end{aligned}$$

$$B(\epsilon, p) = \left[\frac{1+\epsilon^2 p^2}{1+\epsilon^2}\right]$$



1. $\epsilon \ll 1,\ \epsilon p \ll 1$
   $\Rightarrow r_c^f \to r_c^u,$
2. $\epsilon = 1$, p unconstrained,
   $\Rightarrow \left(r_c^f\right)^{-2} = \frac{1}{2}\left(\frac{1}{(r_c^u)^2} + \frac{1}{(r_c^v)^2}\right)$
3. $p = 1$, ϵ unconstrained,
   $\Rightarrow r_c^f = r_c^u = r_c^v.$
4. $p < 1$, ϵ unconstrained,
   $\Rightarrow r_c^f > r_c^u.$
5. $p > 1$, ϵ unconstrained
   $\Rightarrow r_c^f < r_c^u.$

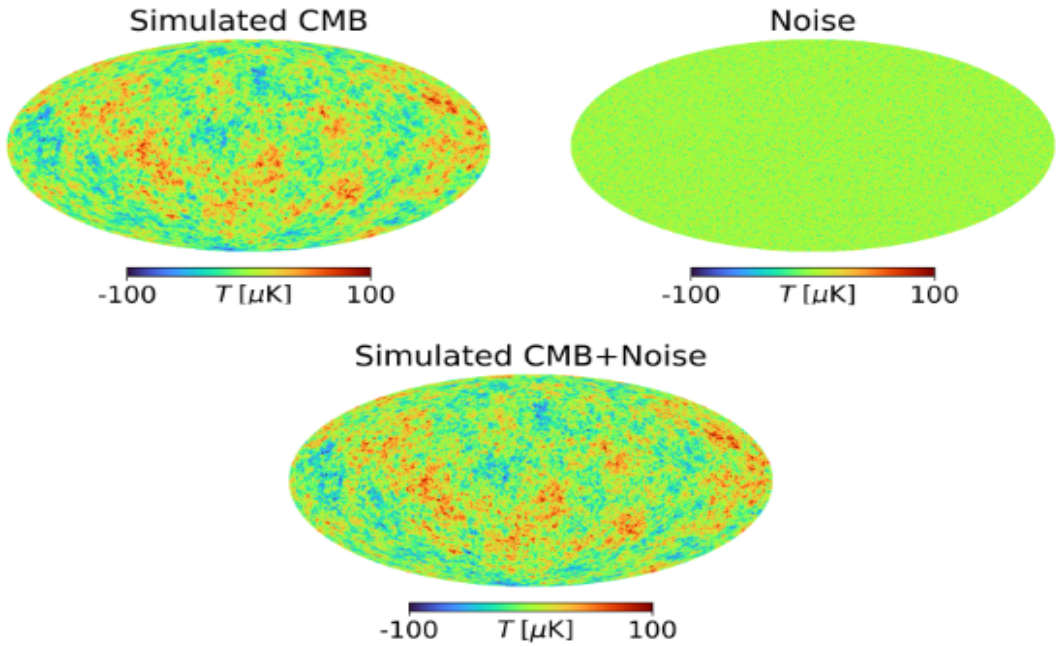
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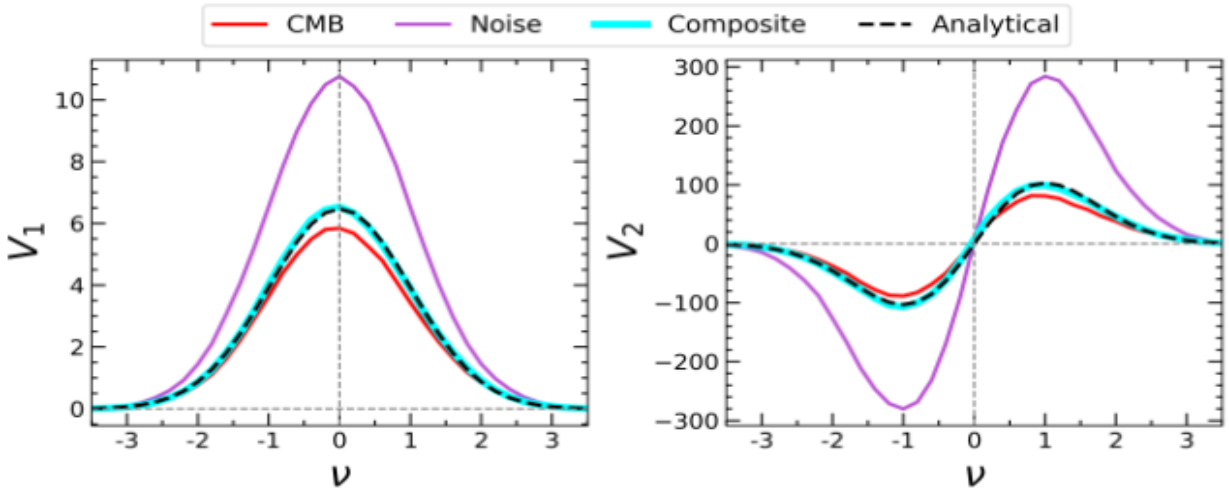
Example 1: sum of Gaussian signal and Gaussian noise

$\boxed{\epsilon = 0.40}$

$\boxed{p = 1.35}$

.





Example 2: sum of NG CMB with $f_{\rm NL}$ and Gaussian noise

PDFs of u and v: **Ganesan+ 2016**

$$P_u(\nu) = \frac{1}{\sqrt{2\pi(\sigma_0^u)^2}} e^{-\nu^2/2}\left[1 + f_{\rm NL}\sigma_0^u\left\{\nu^3 - 3\nu\right\}\right]$$

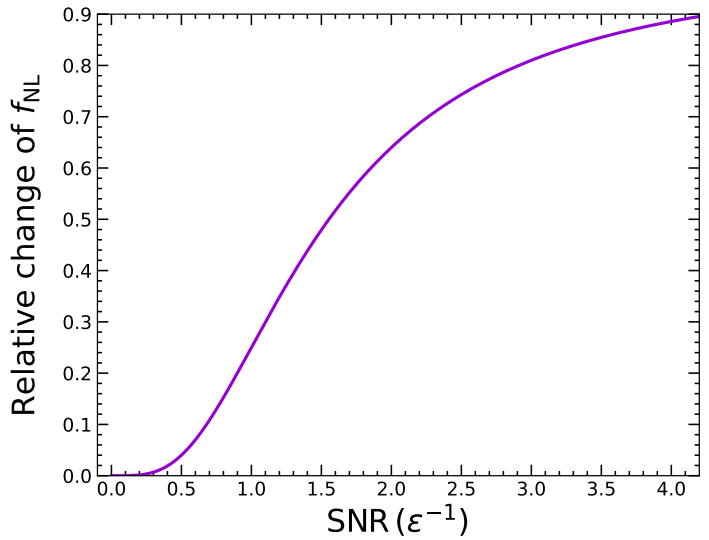
$$P_v(x) = \frac{1}{\sqrt{2\pi(\sigma_0^v)^2}} \exp\left\{-\frac{x^2}{2(\sigma_0^v)^2}\right\}.$$

PDF of f:

$$P_f(\nu) = \frac{1}{\sqrt{2\pi(\sigma_0^f)^2}} e^{-\nu^2/2}\left[1 + \widetilde{f}_{\rm NL}\sigma_0^f\left\{\nu^3 - 3\nu\right\}\right].$$

where

$$\widetilde{f}_{\rm NL} = \left(\frac{\sigma_0^u}{\sigma_0^f}\right)^4 f_{\rm NL} = \frac{1}{(1+\epsilon^2)^2} f_{\rm NL}$$



Non-Gaussian deviations of Minkowski functionals

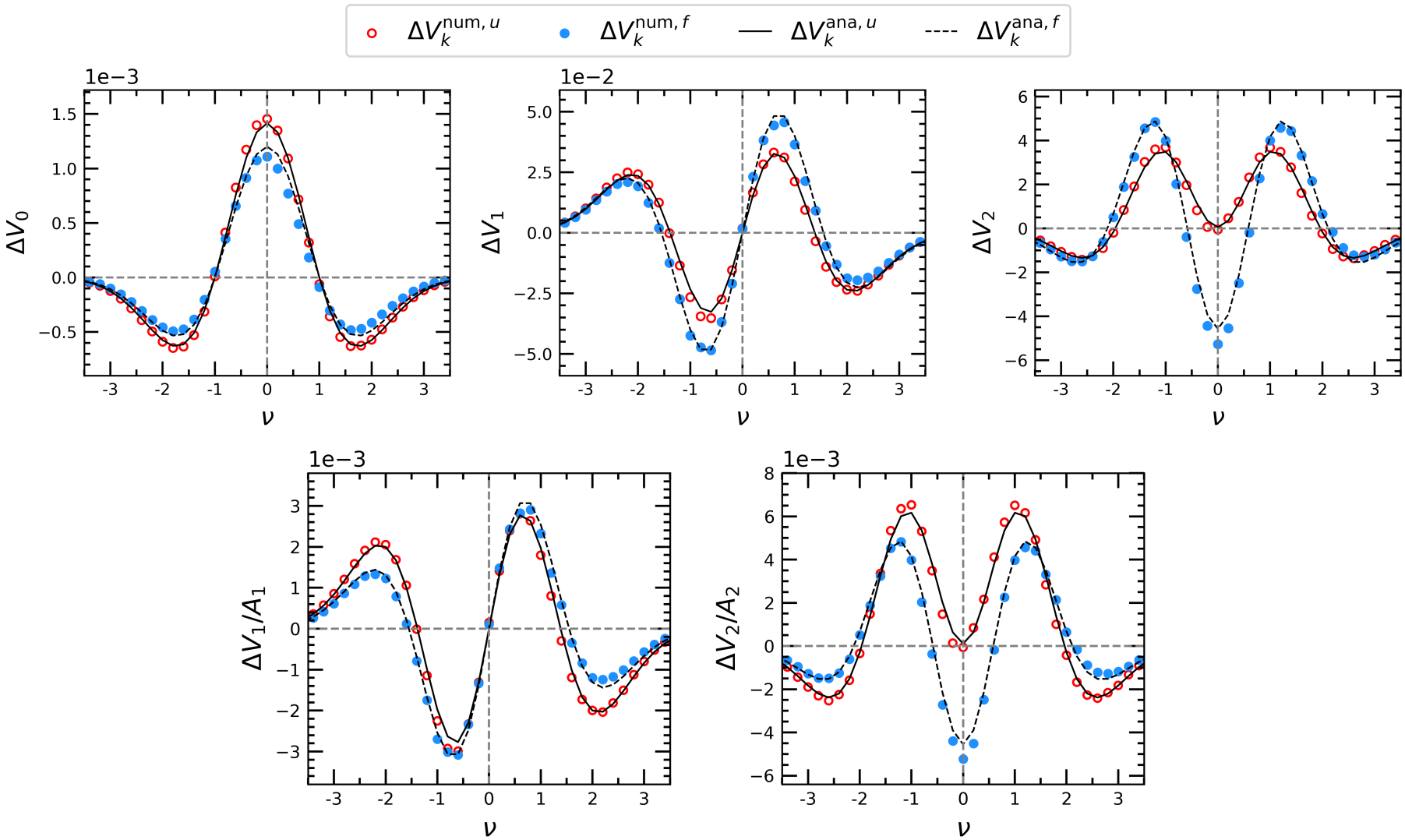
$$\Delta V_k^{\rm ana} \;=\; V_k - V_k^G \;\simeq\; A_k\, e^{-\nu^2/2}\left(v_k^{(1)}\sigma_0 + v_k^{(2)}\sigma_0^2\right)$$

$$\begin{aligned}
\Delta V_0^{{\rm ana},x}(\nu) &= A_0\, e^{-\nu^2/2}\,\frac{S^{(0)x}}{6}H_2(\nu)\,\sigma_0^x,\\
\Delta V_1^{{\rm ana},x}(\nu) &= A_1^x\, e^{-\nu^2/2}\left[\frac{S^{(0)x}}{6}H_3(\nu) + \frac{S^{(1)x}}{3}H_1(\nu)\right]\sigma_0^x,\\
\Delta V_2^{{\rm ana},x}(\nu) &= A_2^x\, e^{-\nu^2/2}\left[\frac{S^{(0)x}}{6}H_4(\nu) + \frac{2S^{(1)x}}{3}H_2(\nu) + \frac{S^{(2)x}}{3}H_0(\nu)\right]\sigma_0^x
\end{aligned}$$

$$\begin{aligned}
S^{(0)\,f}\sigma_0^f &= S^{(0)\,u}\sigma_0^u\;\frac{1}{(1+\epsilon^2)^{3/2}},\\
S^{(1)\,f}\sigma_0^f &= S^{(1)\,u}\sigma_0^u\;\frac{1}{(1+\epsilon^2)^{1/2}\,(1+\epsilon^2 p^2)},\\
S^{(2)\,f}\sigma_0^f &= S^{(2)\,u}\sigma_0^u\;\frac{(1+\epsilon^2)^{1/2}}{(1+\epsilon^2 p^2)^2}.
\end{aligned}$$

Non-Gaussian deviations of Minkowski functionals

$$\epsilon = 0.40, \quad p = 2.47$$

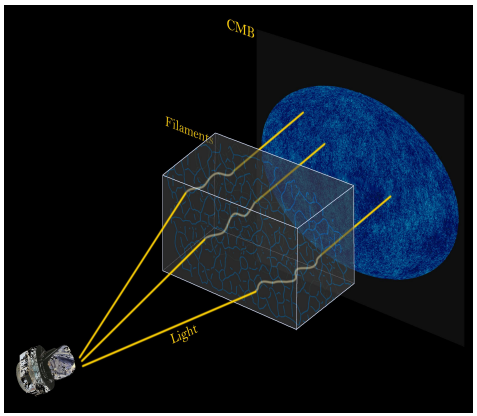


Summary

- Demonstrated that bias on morphology of a true signal field due to secondary fields can be understood in analytically controlled way.
- Useful for understanding bias due to residual foregrounds and noise .

Testing statistical isotropy with convergence map from ACT

Dark matter mass map from CMB lensing



He et al & Planck/ESA

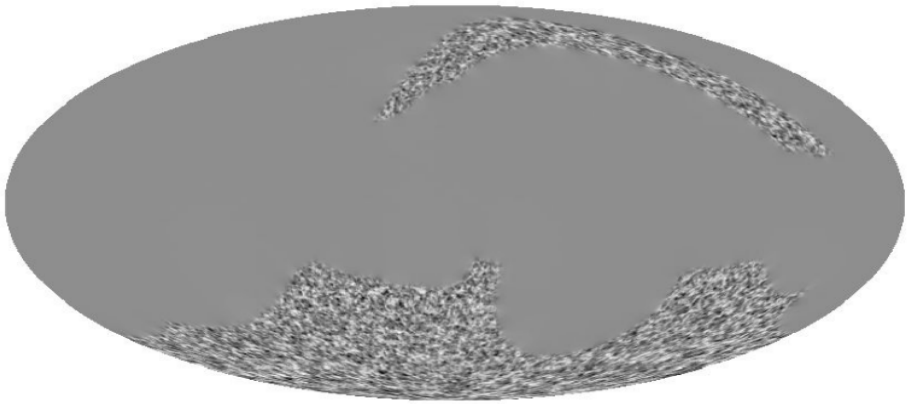
$$
\begin{aligned}
T^{\mathrm{L}}(\hat{n}') &= T^{\mathrm{UL}}(\hat{n}+\vec{d}) \\
(Q+iU)^{\mathrm{L}}(\hat{n}') &= (Q+iU)^{\mathrm{UL}}(\hat{n}+\vec{d}) \\
\vec{d}(\theta,\phi) &= \nabla_{\hat{n}}\Phi
\end{aligned}
$$

$$\kappa(\hat{n}) = -\frac{1}{2}\nabla^2\Phi(\hat{n})$$

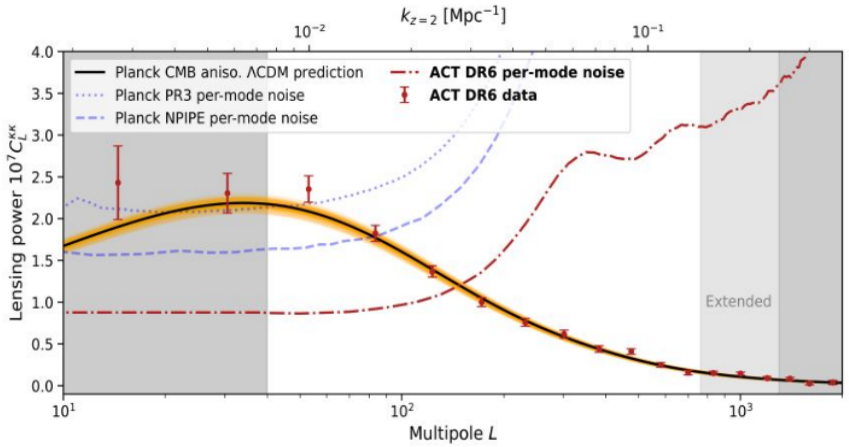
ACT data DR6 - Convergence κ map

Madhavacheril et al. 2023

Skyfrac $\simeq$ 23%



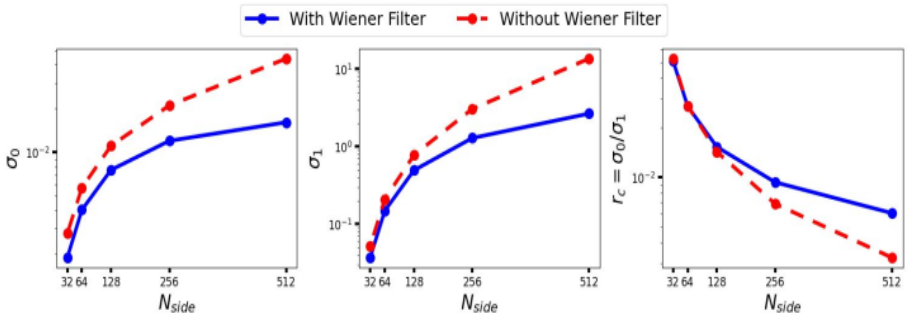
Qu et al. 2023



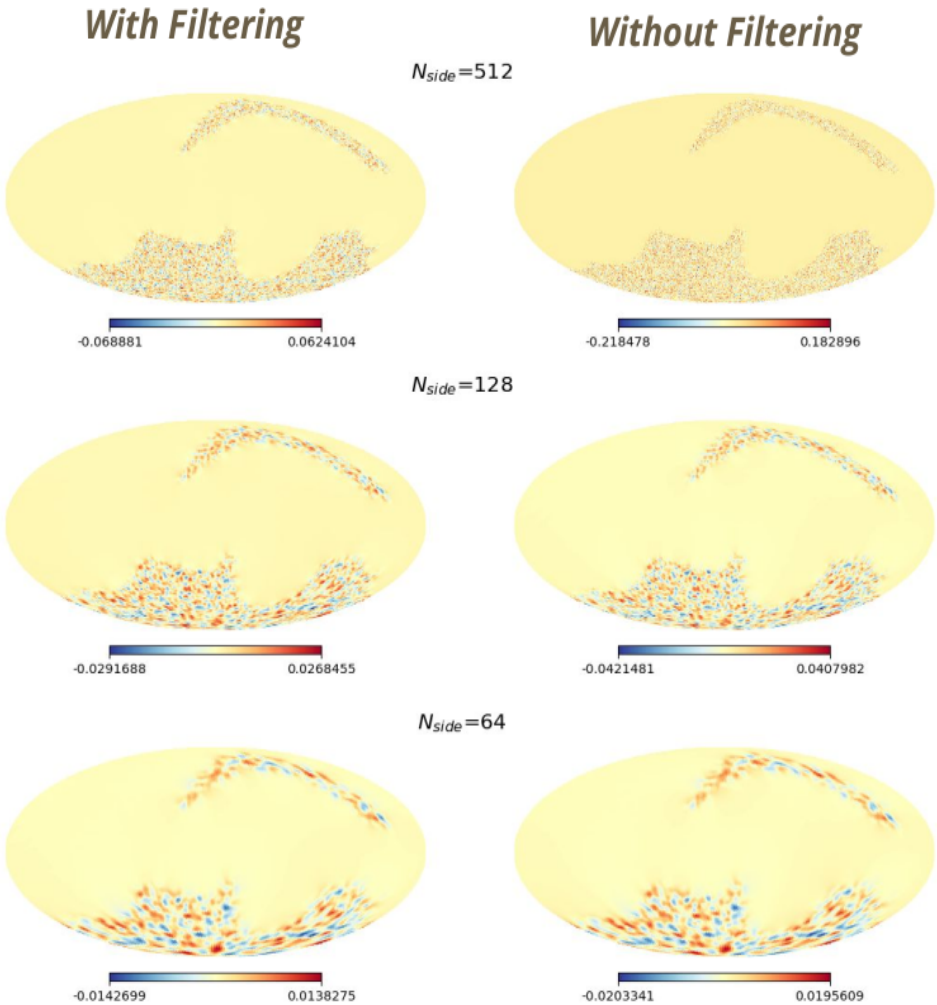
400 realistic simulations provided.

Noise mitigation: Wiener filtering

$$\kappa_{LM}^{WF} = \frac{C_L^{\kappa\kappa}}{C_L^{\kappa\kappa} + N_L^{\kappa}} \kappa_{LM}.$$



$$\sigma_0^2 = \langle u^2 \rangle \qquad \sigma_1^2 = \langle |\nabla u|^2 \rangle$$



Methodology

- Measure all morphological statistics from observed κ.
- Measure from 400 simulations. Estimate the mean (median) values and uncertainties from them.
- Compare observed value against the 400 values of each statistic at each threshold.
- Hierarchichal multiscale and multi-resolution analysis by dividing sky into local patches and pixel resolutions set by

$$N_{\rm side} = 512,\ 256,\ 64.$$

Anomalous behaviour: higher than 2σ deviation that persists in

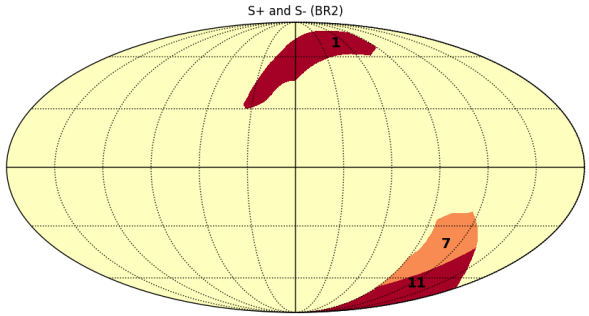
(a) threshold range,

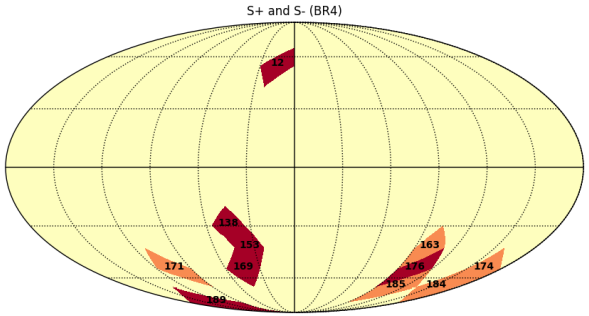
(b) spatial resolution.

Anomalous patches

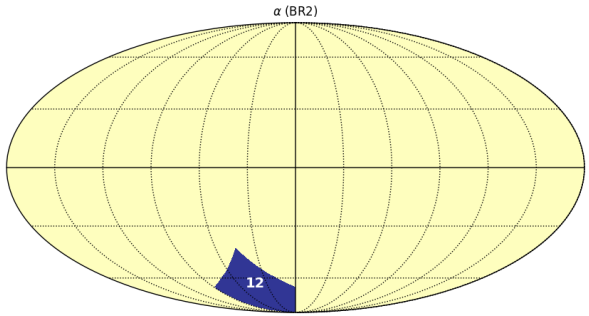
Base res 2 **Base res 4**

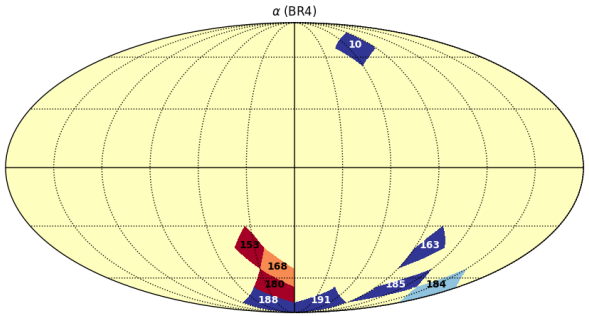
V_0, V_1, V_2, b_c, b_h $\Rightarrow$





α $\Rightarrow$





Summary

- Identified sky regions having anomalous morphological properties in ACT convergence data.
- 3 patches show persistent anomaly across threshold and resolutions for all statistics. One of them being in the region of CMB cold spot.
- **Future** - use larger sky fraction, investigate deeper the origin of anomaly, correlate with large scale structures.