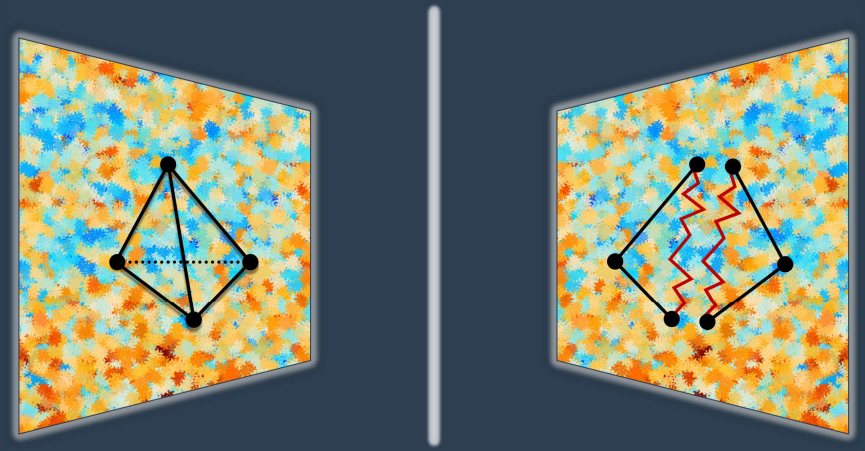
기초과학연구원
Institute for Basic Science

# Mapping and Factorising Cosmological Observables

Yuhang Zhu (IBS, CTPU-CGA)

@High1| 15 Jan 2025

In collaboration with: David Stefanyszyn and Xi Tong



Based on: 2406.00099 & 2309.07769

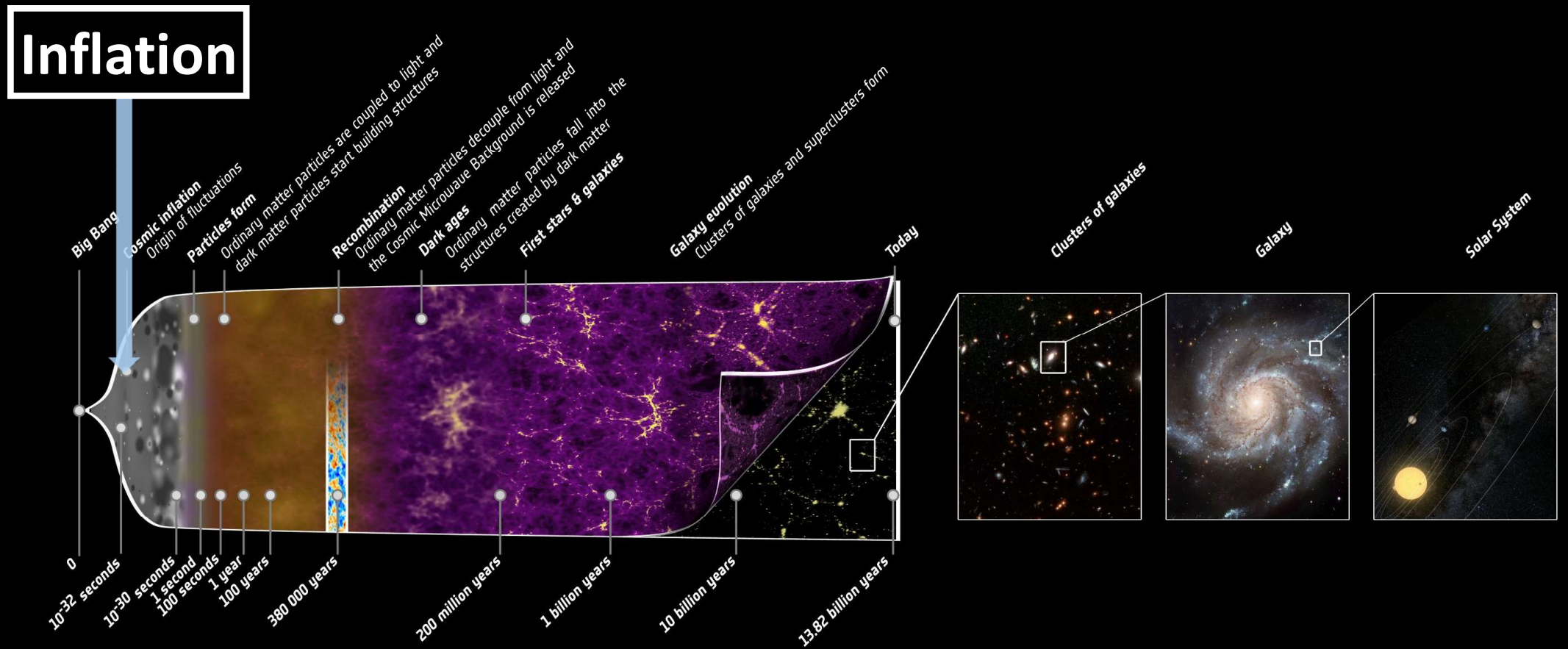
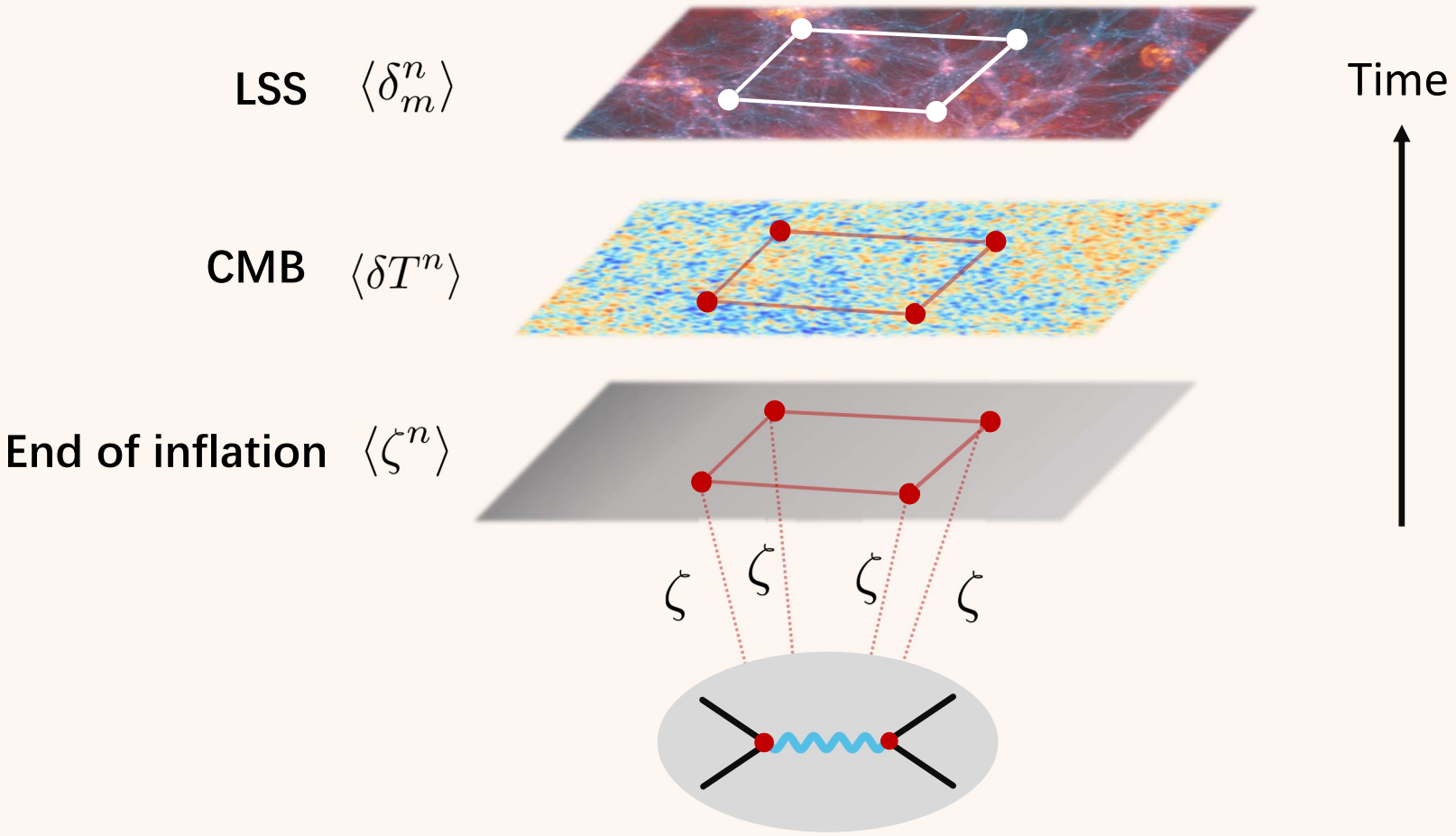


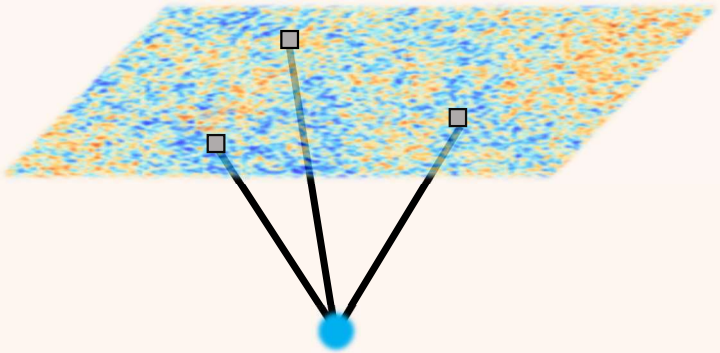

Image from Planck team

# Cosmological correlators

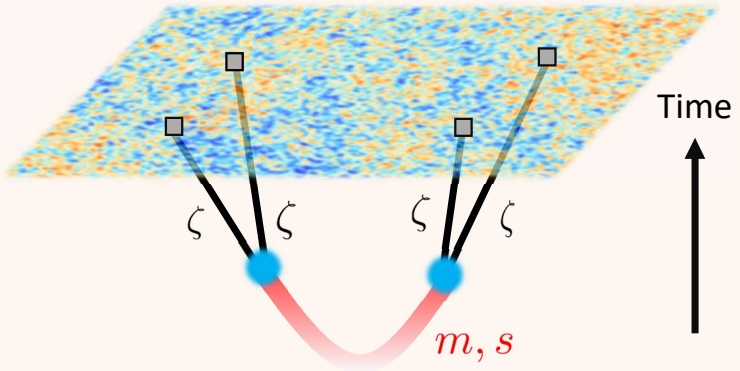


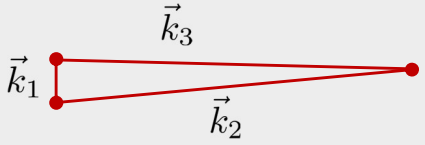
# Cosmological Correlators

$\langle \zeta^3 \rangle$



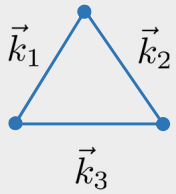
$\langle \zeta^4 \rangle$



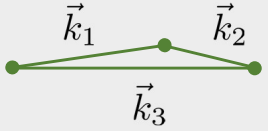

Soft limit

Inflationary particle spectrum




Equilateral limit

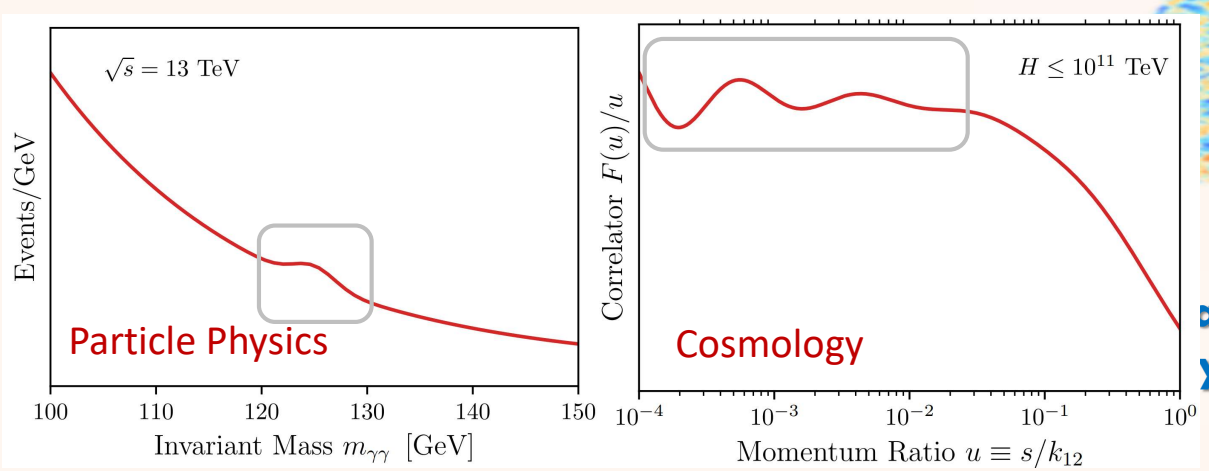
Higher dimension interactions



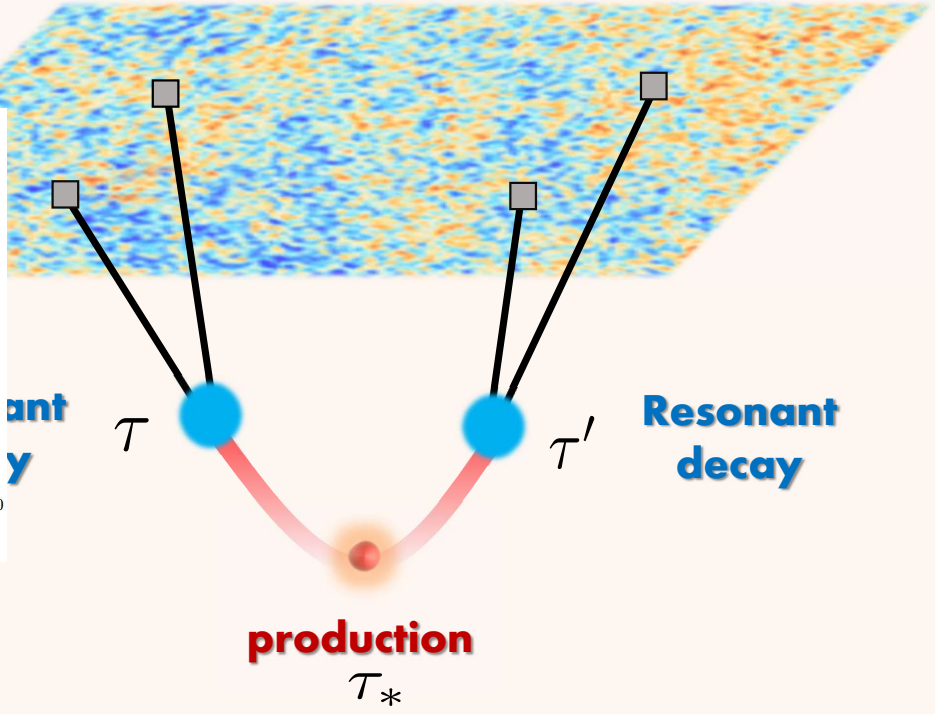

Collinear limit

Probe Initial state

# Inflation as the Cosmological Collider (CC)




Credit: Denis Werth



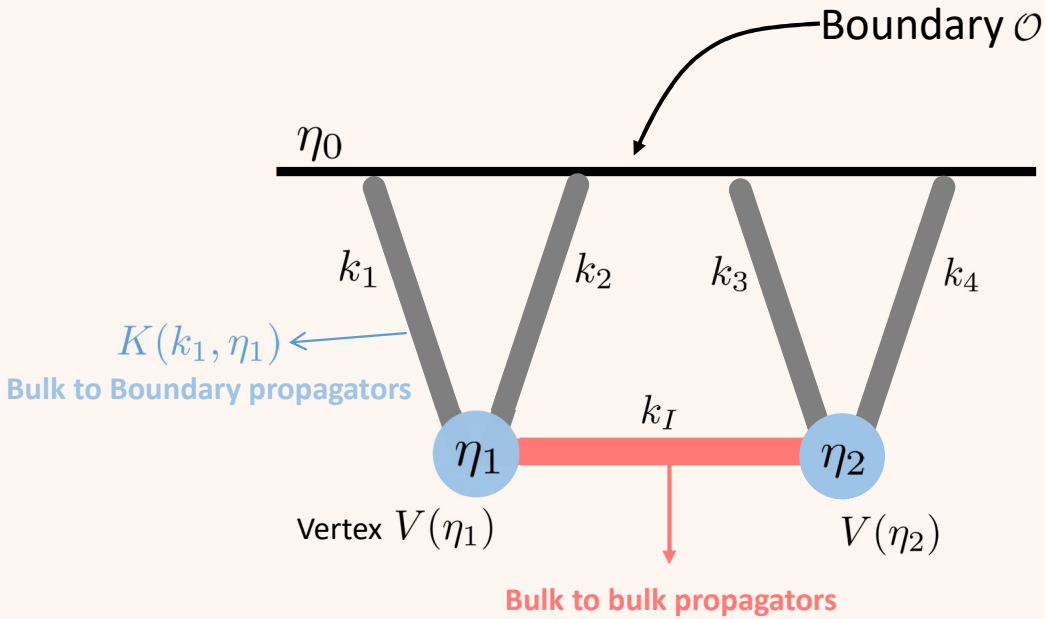

$$S \sim \mathcal{A}(\lambda, m)\, e^{-\pi\mu} \left(\frac{k_3}{k_1}\right)^{1/2} \sin\left[\mu \log\left(\frac{k_3}{k_1}\right) + \vartheta\right] P_s(\cos\theta)$$

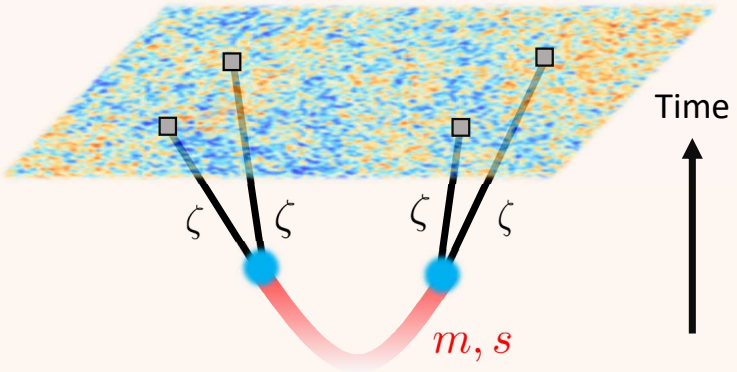
mass

spin

$$\mu = \begin{cases} \sqrt{\dfrac{m^2}{H^2} - \dfrac{9}{4}} & s = 0 \\ \sqrt{\dfrac{m^2}{H^2} - \left(s - \dfrac{1}{2}\right)^2} & s \neq 0 \end{cases}$$

# Analytical Calculation of cosmological correlators




$$\langle\zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\zeta_{k_4}\rangle' = \int d\eta_1 \int d\eta_2 V(\eta_1)V(\eta_2)G(k_I,\eta_1,\eta_2)K(k_1,\eta_1)K(k_2,\eta_1)K(k_3,\eta_2)K(k_4,\eta_2)$$

**Too difficult to solve analytically!**

(1) Nested Time integral

$$G \sim \theta(\eta_1-\eta_2)v_k(\eta_1)v_k^*(\eta_2) + \theta(\eta_2-\eta_1)v_k^*(\eta_1)v_k(\eta_2)$$

(2) Mode functions are complicate

$$v_k \sim H_{i\mu} \text{ or } W_{i\kappa,i\mu}$$

BUT, the calculation of cosmological correlators is really HARD!



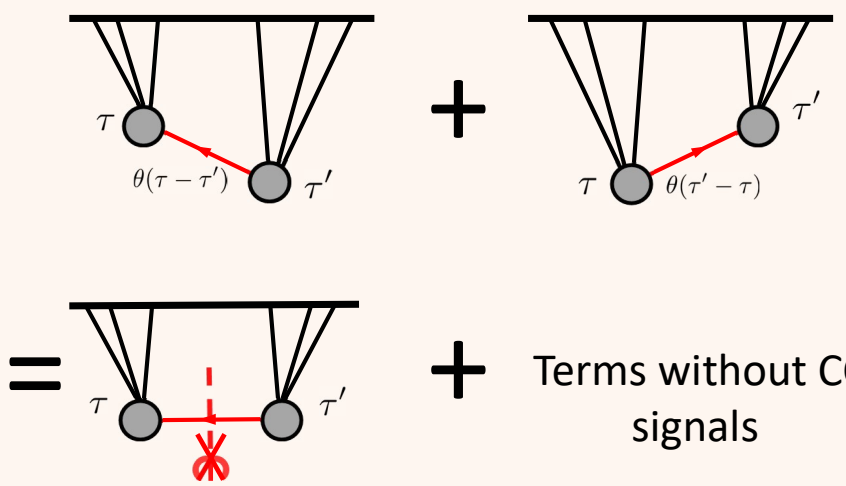
| **Exact** | **Approximation** |
| --- | --- |
| • Sometimes, too complicated | • Sometimes, too naïve |
| • Hard to understand all information | • Hard to include all information |
| ..... | ..... |

**Trade-off**

# Recent Progress: **Approximation Method**

*Cosmological collider cutting rule*

Tong, Wang and **YHZ** [2112.03448]

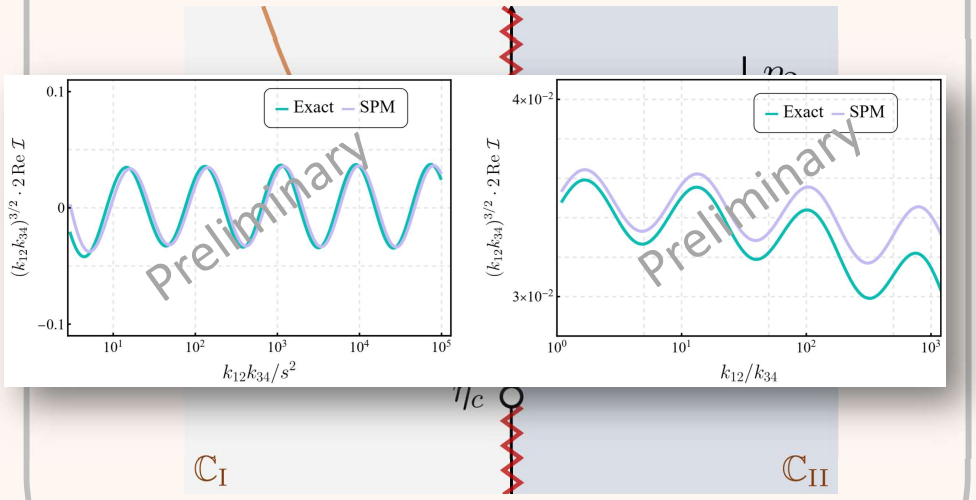



Efficient way to extract CC signals

*WKB + Saddle point method*

Renaux-Petel, Tong, Werth,**YHZ** [2502.XXXX]

$$\left[\frac{\partial^2}{\partial\eta^2} + w^2(s,\eta)\right]\sigma(s,\eta) = 0$$

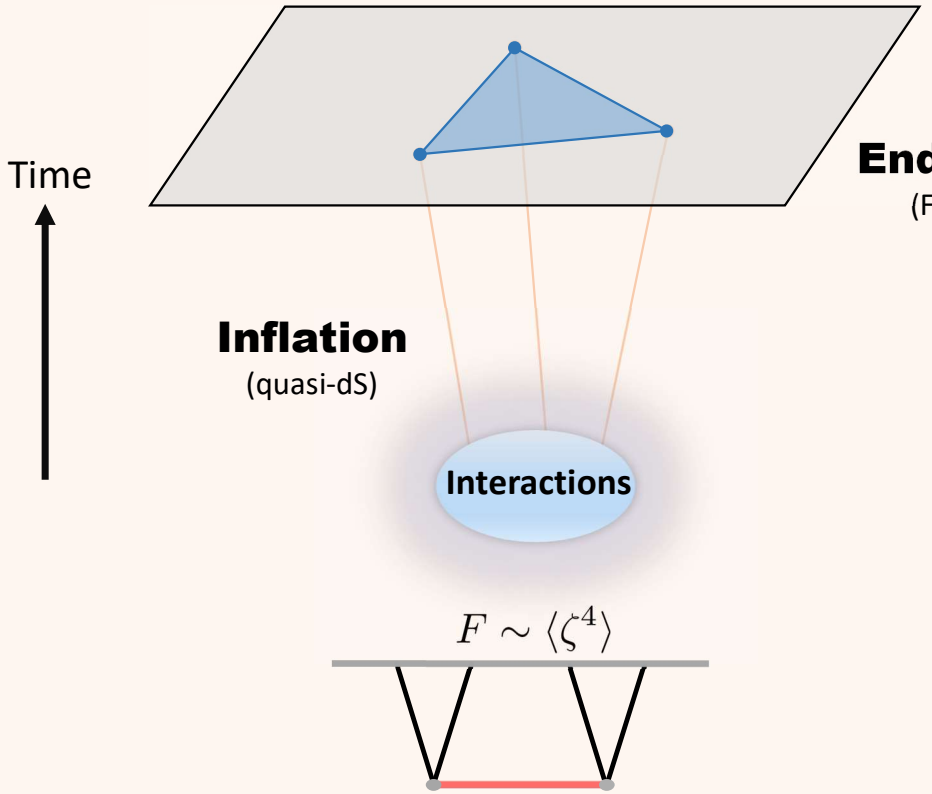



Good way to **understand** physics

# Recent Progress: **Exact Method**

## dS Bootstrap

Arkani-Hamed, Baumann, Lee, Pimentel 2018
Baumann, Duaso Pueyo, Joyce, Lee, Pimentel 2019 2020
.....




**dS Symmetry :** $ds^2 = \dfrac{-d\eta^2 + d\mathbf{x}^2}{H\eta^2}$

Translation: $P_i = \partial_i$

Rotation: $J_{ij} = x_i\partial_j - x_j\partial_i$

Dilation: $D = -\eta\partial_\eta - x_i\partial_i$

dS boosts: $K_i = 2x_i\eta\partial_\eta + \left(2x^j x_i + (\eta^2 - x^2)\delta_i^j\right)\partial_j$

*Conformal Ward identities* :

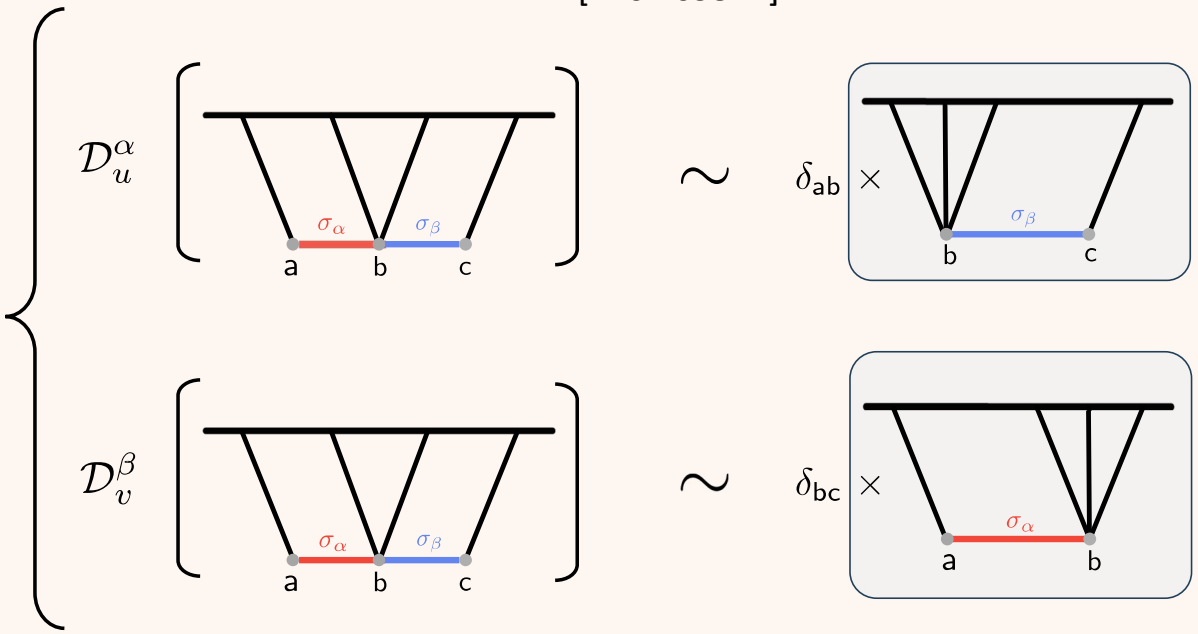
$$\left[u^2(1-u^2)\partial_u^2 - 2u^3\partial_u + \mu^2 + \frac{1}{4}\right] F = g^2 \frac{uv}{u+v}$$

+ **Boundary conditions** = **fix the correlators**

# Recent Progress: **Exact Method**

## ***Double*** **Massive Exchange**

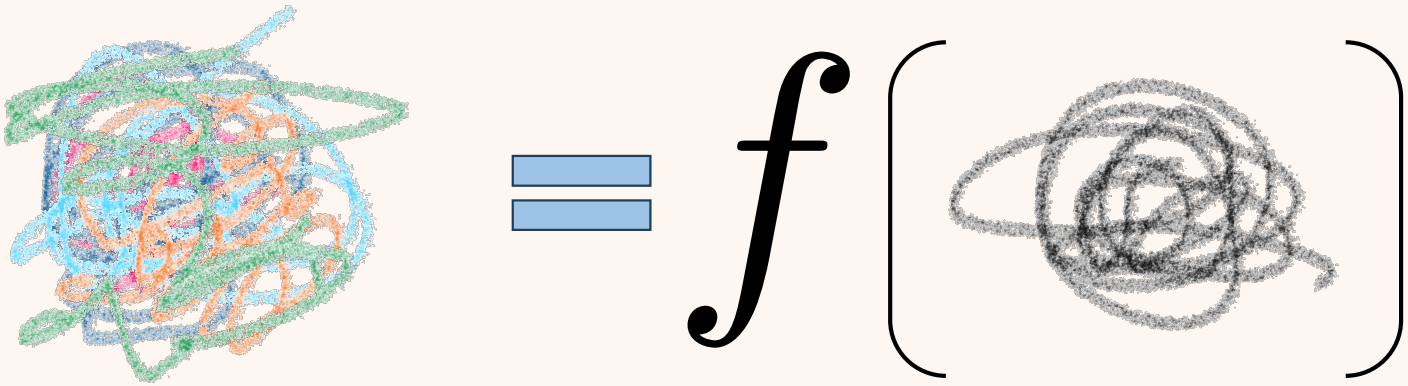
Aoki, Pinol, Sano, Yamaguchi and **YHZ**
[2404.09547]




$$
\begin{aligned}
&\mathcal{I}_{+++}(r_1,r_3)\\
&=\sum_{a,b=\pm}\sum_{m=0}^{\infty}\pi^2 e^{\pi(a\mu_\alpha+b\mu_b)}\operatorname{csch}(2\pi a\mu_\alpha)\operatorname{csch}(2\pi b\mu_\beta)\left(\frac{r_1}{2}+\frac{r_3}{2}-1\right)^3 r_1^{-\frac{1}{2}+m+ia\mu_\alpha} r_3^{-\frac{1}{2}+ib\mu_\beta}\\
&\quad\times\Gamma\left[\begin{matrix}\frac{1}{2}+m+ia\mu_\alpha\\ 1+m,1+m+2ia\mu_\alpha\end{matrix}\right]{}_2\mathcal{F}_1\left[\begin{matrix}\frac{1}{2}+ib\mu_\beta,2+m+ia\mu_\alpha+ib\mu_\beta\\ 1+2ib\mu_\beta\end{matrix}\middle| r_3\right]\\
&+\left\{\sum_{m=0}^{\infty}\frac{\pi^2}{2}\left[\operatorname{csch}(\pi\mu_\alpha)-\operatorname{sech}(\pi\mu_\alpha)\right]\operatorname{sech}(\pi\mu_\beta)\left(\frac{r_1}{2}+\frac{r_3}{2}-1\right)^3 r_1^{-\frac{1}{2}+m-i\mu_\alpha}\right.\\
&\quad\times\Gamma\left[\begin{matrix}\frac{1}{2}+m-i\mu_\alpha\\ 1+m,1+m-2i\mu_\alpha\end{matrix}\right]{}_3\mathcal{F}_2\left[\begin{matrix}1,1,\frac{5}{2}+m-i\mu_\alpha\\ \frac{3}{2}-i\mu_\beta,\frac{3}{2}+i\mu_\beta\end{matrix}\middle| r_3\right]+(\mu_\alpha\to-\mu_\alpha)\Big\}\\
&+\left\{\sum_{m=0}^{\infty}\frac{\pi^2}{2}\left[\operatorname{csch}(\pi\mu_\beta)-\operatorname{sech}(\pi\mu_\beta)\right]\operatorname{sech}(\pi\mu_\alpha)\left(\frac{r_1}{2}+\frac{r_3}{2}-1\right)^3 r_1^m r_3^{-\frac{1}{2}-i\mu_\beta}\right.\\
&\quad\times\Gamma\left[\begin{matrix}m+1\\ \frac{3}{2}+m-i\mu_\alpha,\frac{3}{2}+m+i\mu_\alpha\end{matrix}\right]{}_2\mathcal{F}_1\left[\begin{matrix}\frac{1}{2}-i\mu_\beta,\frac{5}{2}+m-i\mu_\beta\\ 1-2i\mu_\beta\end{matrix}\middle| r_3\right]+(\mu_\beta\to-\mu_\beta)\Big\}\\
&+\sum_{m=0}^{\infty}\pi^2\operatorname{sech}(\pi\mu_\beta)\operatorname{sech}(\pi\mu_\alpha)\left(\frac{r_1}{2}+\frac{r_3}{2}-1\right)^3 r_1^m\\
&\quad\times\Gamma\left[\begin{matrix}1+m\\ \frac{3}{2}+m-i\mu_\alpha,\frac{3}{2}+m+i\mu_\alpha\end{matrix}\right]{}_3\mathcal{F}_2\left[\begin{matrix}1,1,3+m\\ \frac{3}{2}-i\mu_\beta,\frac{3}{2}+i\mu_\beta\end{matrix}\middle| r_3\right],
\end{aligned}
$$

It becomes incredibly difficult
to make further progress…

Without knowing the analytical results,
can we still find something useful?



## Maldacena's consistency relation

$$\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \rangle \sim - \langle \zeta_{\vec{k}_1} \zeta_{-\vec{k}_1} \rangle k \frac{d}{dk} \langle \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \rangle =$$
$$= - n_s \langle \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \rangle \langle \zeta_{\vec{k}_1} \zeta_{-\vec{k}_1} \rangle \ , \qquad k_1 \ll k_2, k_3$$

(Maldacena, 2002)

## Suyama-Yamaguchi Inequality

$$\tau_{\rm NL} \geq \left( \frac{6}{5} f_{\rm NL} \right)^2 ,$$

(Suyama, Yamaguchi 2007)

## Unitarity Cutting Rule

$$\psi_4'(k_1, k_2, k_3, k_4, s, t, u) + [\psi_4'(-k_1, -k_2, -k_3, -k_4, s, t, u)]^* =$$
$$P_\sigma(s) \left[ \psi_3'^{\phi\phi\sigma}(k_1, k_2, s) - \psi_3'^{\phi\phi\sigma}(k_1, k_2, -s) \right] \left[ \psi_3'^{\phi\phi\sigma}(k_3, k_4, s) - \psi_3'^{\phi\phi\sigma}(k_3, k_4, -s) \right]$$

(Goodhew, Jazayeri, Pajer 2020 )

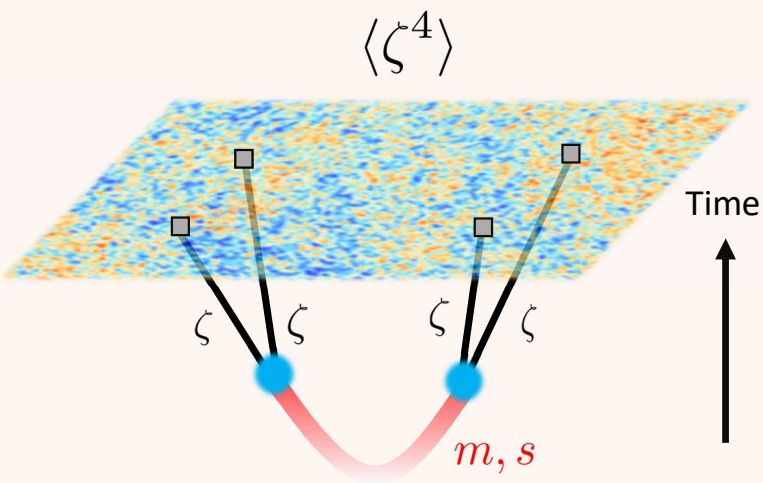
**Directl ytest able ?**

(in principle)

## Cosmological Correlators

From

## Wavefunction of Universe



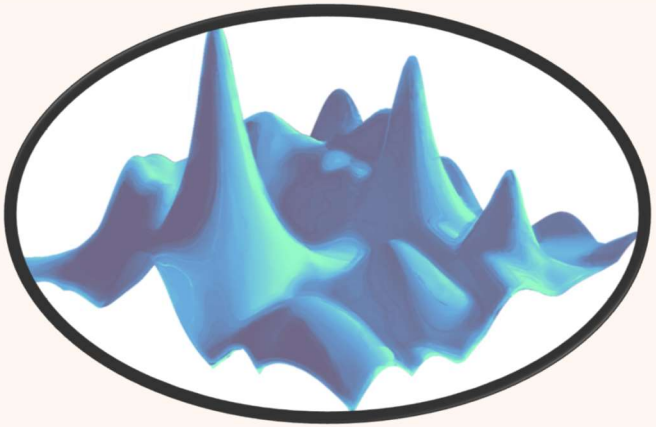



Figure from: Baumann, Joyce, Lectures on Cosmological correlations

$$f(B_n) = 0\Psi$$

Expand the wavefunction in powers of the field fluctuations:

Wavefunction coefficients

$$\Psi[\varphi] = \exp\left[+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{\{\mathbf{k}\}}\psi_n(\mathbf{k}_1,\cdots,\mathbf{k}_n)\,\varphi(\mathbf{k}_1)\cdots\varphi(\mathbf{k}_n)\,(2\pi)^3\delta^3(\mathbf{k}_1+\cdots+\mathbf{k}_n)\right]$$

**Parametrization**

- "Wavefunction of the universe":

$$\Psi[\varphi] = \langle\varphi|U(\eta_0,-\infty)|\mathrm{BD}\rangle = \int_{\mathrm{BD}}^{\Phi(\eta_0)=\varphi}\mathcal{D}\Phi\, e^{iS[\Phi]}$$

- Correlators from the Born rule:

$$B_n \equiv \langle\varphi(\mathbf{k}_1)\cdots\varphi(\mathbf{k}_n)\rangle = \frac{\int\mathcal{D}\varphi\left|\Psi[\varphi]\right|^2\varphi(\mathbf{k}_1)\cdots\varphi(\mathbf{k}_n)}{\int\mathcal{D}\varphi\left|\Psi[\varphi]\right|^2}$$

"Cross section"

$$\rho_n(\{\mathbf{k}\}) = \psi_n(\{\mathbf{k}\}) + \psi_n^*(\{-\mathbf{k}\})$$

"Scattering Amplitude"

- Correlators from the Born rule:

$$B_n \equiv \langle \varphi(\mathbf{k}_1) \cdots \varphi(\mathbf{k}_n) \rangle = \frac{\int \mathcal{D}\varphi \left|\Psi[\varphi]\right|^2 \varphi(\mathbf{k}_1) \cdots \varphi(\mathbf{k}_n)}{\int \mathcal{D}\varphi \left|\Psi[\varphi]\right|^2}$$

$$\Psi[\varphi] = \exp\left[ + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\{\mathbf{k}\}} \psi_n(\mathbf{k}_1, \cdots, \mathbf{k}_n)\, \varphi(\mathbf{k}_1) \cdots \varphi(\mathbf{k}_n) \right]$$

$$\rho_n(\{\mathbf{k}\}) = \psi_n(\{\mathbf{k}\}) + \psi_n^*(\{-\mathbf{k}\})$$

Dictionary between $\rho$ and $B$

$$B_2 = -\frac{1}{\rho_2} \,,$$

$$B_3 = -\frac{1}{{\rho_2}^3}\, \rho_3 \,,$$

$$B_4 = \frac{1}{{\rho_2}^4} \left[ \rho_4 - \left( \rho_3 \frac{1}{\rho_2} \rho_3 + 2 \text{ perms} \right) \right] \,,$$

$$B_5 = -\frac{1}{{\rho_2}^5} \left[ \rho_5 - \left( \rho_4 \frac{1}{\rho_2} \rho_3 + 9 \text{ perms} \right) + \left( \rho_3 \frac{1}{\rho_2} \rho_3 \frac{1}{\rho_2} \rho_3 + 14 \text{ perms} \right) \right]$$

General relation:

$$B_n = \frac{1}{(-\rho_2)^n} \sum_{k=0}^{n-3} (-1)^k \Big( k\text{-cuts} \Big)_\rho$$

$$\Big( k\text{-cuts} \Big)_\rho \equiv \sum_{n-k \geq n_1 \cdots n_{k+1} \geq 3} \left[ \rho_{n_1} \frac{1}{\rho_2} \rho_{n_2} \cdots \rho_{n_k} \frac{1}{\rho_2} \rho_{n_{k+1}} + (\pi_{n_1 \cdots n_{k+1}} - 1)\text{-perms} \right] .$$

## Dictionary between $\rho$ and $B$

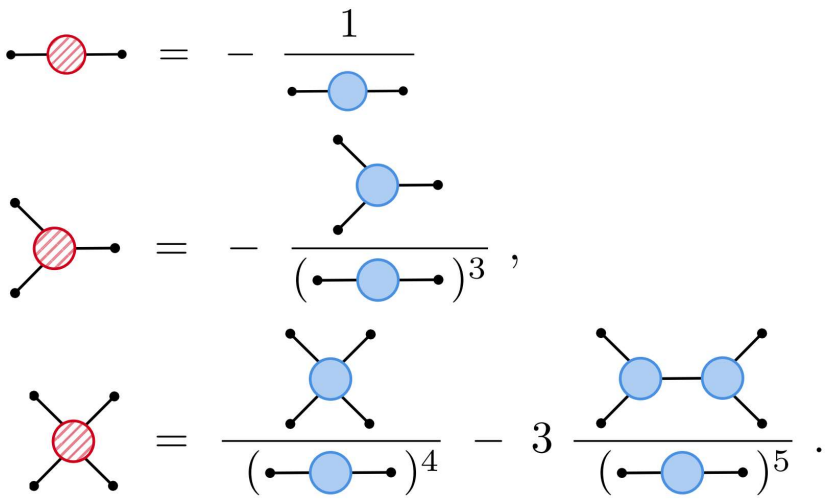
$$B_2 = -\frac{1}{\rho_2}\,,$$

$$B_3 = -\frac{1}{{\rho_2}^3}\,\rho_3\,,$$

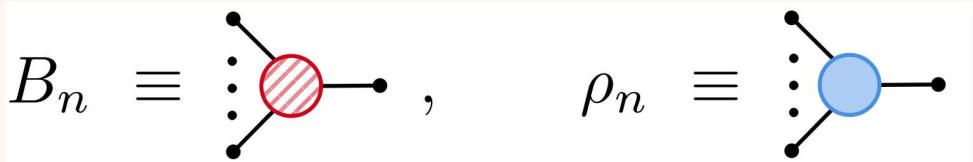
$$B_4 = \frac{1}{{\rho_2}^4}\left[\rho_4 - \left(\rho_3\frac{1}{\rho_2}\rho_3 + 2\text{ perms}\right)\right]\,,$$

$$B_5 = -\frac{1}{{\rho_2}^5}\left[\rho_5 - \left(\rho_4\frac{1}{\rho_2}\rho_3 + 9\text{ perms}\right) + \left(\rho_3\frac{1}{\rho_2}\rho_3\frac{1}{\rho_2}\rho_3 + 14\text{ perms}\right)\right]$$

## Dictionary between $\rho$ and $B$

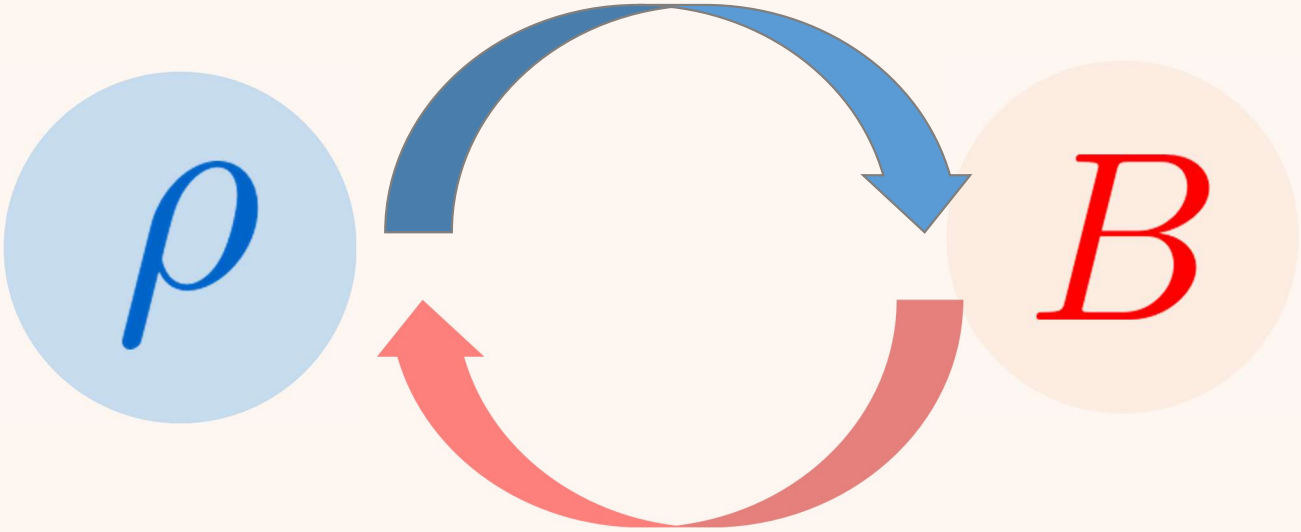


diagrammatic notations:



General relation:

$$B_n = \frac{1}{(-\rho_2)^n} \sum_{k=0}^{n-3} (-1)^k \Big(k\text{-cuts}\Big)_\rho$$




How about the inverse relation ?

$$B_n \equiv \langle \varphi(\mathbf{k}_1) \cdots \varphi(\mathbf{k}_n) \rangle = \frac{\int \mathcal{D}\varphi \left| \Psi[\varphi] \right|^2 \varphi(\mathbf{k}_1) \cdots \varphi(\mathbf{k}_n)}{\int \mathcal{D}\varphi \left| \Psi[\varphi] \right|^2}$$

- Use the generating functional

$$B_n = \left( \frac{\partial}{i\,\partial J} \right)^n \ln Z[J] \Big|_{J=0}$$

with $Z[J] = \int \mathcal{D}\varphi \, |\Psi[\varphi]|^2 e^{iJ\cdot\varphi}$

- Or the integration form:

$$\exp\left[ + \sum_{n=1}^{\infty} \frac{i^n}{n!} B_n J^n \right] = \int \mathcal{D}\varphi \, \exp\left[ + \sum_{n=1}^{\infty} \frac{1}{n!} \rho_n \varphi^n \right] e^{iJ\cdot\varphi}$$

$$\exp\left[ + \sum_{n=1}^{\infty} \frac{i^n}{n!} B_n J^n \right] = \int \mathcal{D}\varphi \, \exp\left[ + \sum_{n=1}^{\infty} \frac{1}{n!} \rho_n \varphi^n \right] e^{iJ\cdot\varphi}$$

Fourier transform

$$f(J) = \int d\varphi \tilde{f}(\varphi) e^{iJ\varphi}$$

- Inverse Fourier transform

$$\exp\left[ + \sum_{n=1}^{\infty} \frac{1}{n!} \rho_n \varphi^n \right] = \int \mathcal{D}J \, \exp\left[ + \sum_{n=1}^{\infty} \frac{i^n}{n!} B_n J^n \right] e^{-iJ\cdot\varphi}$$

- Change variable $J \to -J$

$$\exp\left[ + \sum_{n=1}^{\infty} \frac{1}{n!} \rho_n \varphi^n \right] = \int \mathcal{D}J \, \exp\left[ + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} B_n J^n \right] e^{iJ\cdot\varphi}$$

# Duality Relation

$$\exp\left[+\sum_{n=1}^{\infty}\frac{i^n}{n!}B_n J^n\right]=\int \mathcal{D}\varphi\,\exp\left[+\sum_{n=1}^{\infty}\frac{1}{n!}\rho_n\varphi^n\right]e^{iJ\cdot\varphi}$$

$$\exp\left[+\sum_{n=1}^{\infty}\frac{1}{n!}\rho_n\varphi^n\right]=\int \mathcal{D}J\,\exp\left[+\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}B_n J^n\right]e^{iJ\cdot\varphi}$$

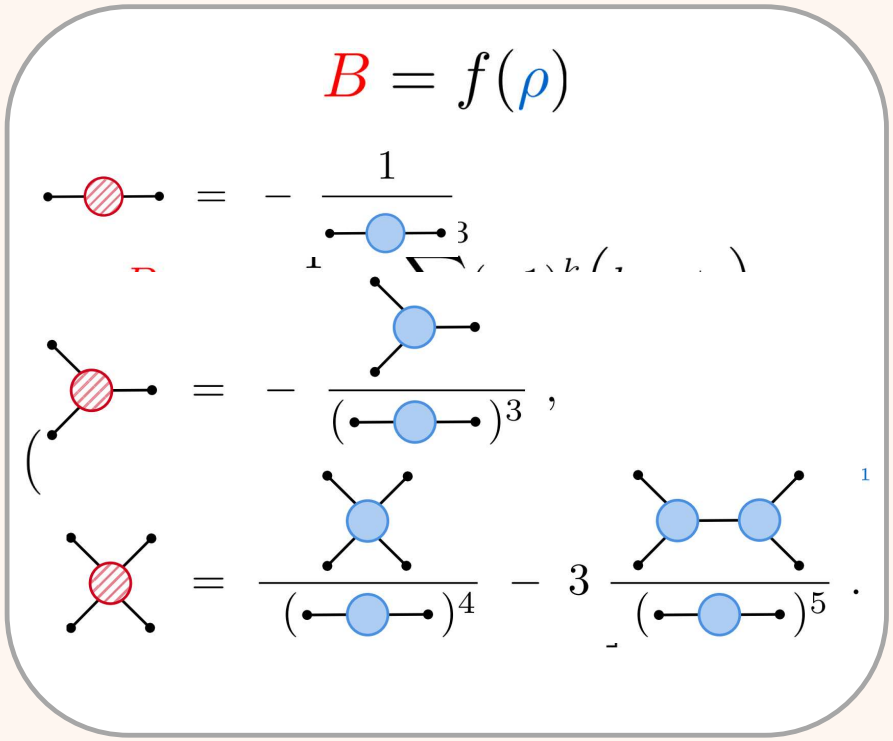
**Hidden symmetry**

$$g:\begin{pmatrix}\rho_n\\ B_n\end{pmatrix}\mapsto(-i)^n\begin{pmatrix}B_n\\ \rho_n\end{pmatrix}$$

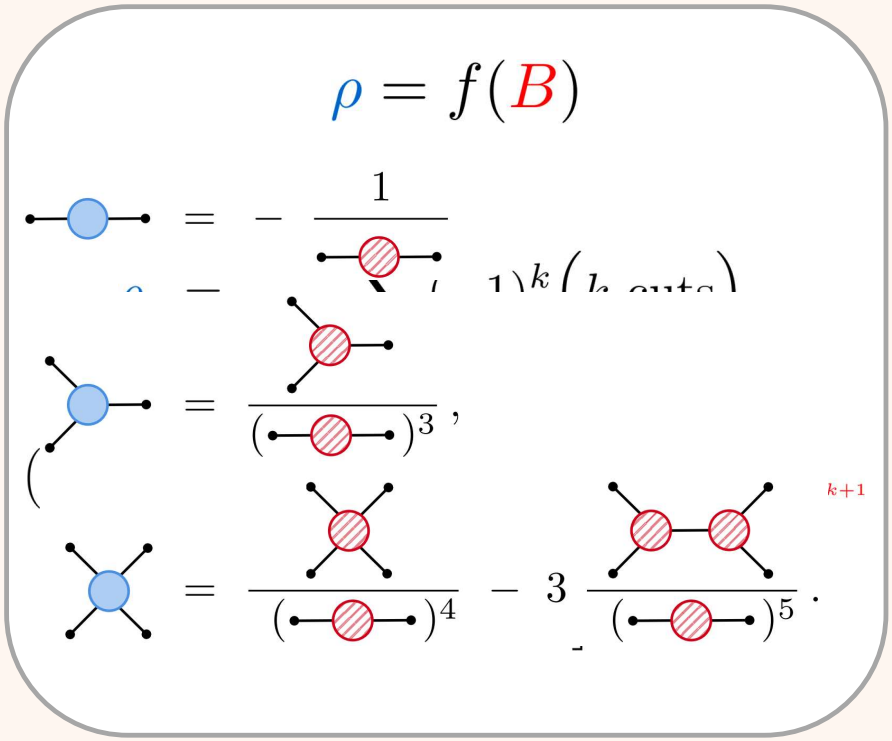
$$g^4=1\ ,\ g\in\mathbb{Z}_4$$

- Valid to all orders in perturbation theory

# Duality Relation




$g$

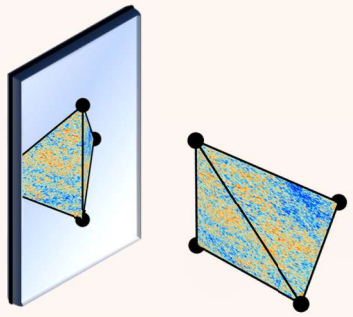


# Reality and Parity

**Fundamental Laws**

**Symmetry**

$$\rho_n = \frac{1}{{B_2}^n} \sum_{k=0}^{n-3} (-1)^k \Big(k\text{-cuts}\Big)_B$$



**Theorem 4.1.** ($\psi_n$**-reality**) *The tree-level wavefunction coefficient of massless scalar fields is purely real, i.e.* $\mathrm{Im}\,\psi_n = 0$*, in theories containing an arbitrary number of fields of any light mass, spin, coupling, sound speed and chemical potential, under the assumption of locality, unitarity, scale invariance, IR convergence and a Bunch-Davies vacuum.*

Stefanyszyn ,Tong, YHZ, [2309.07769]

- Unitarity & locality
- BD vacuum
- Tree level
- Scale invariance

$$\mathrm{Im}\,\psi = 0$$

$$\rho_n(\{\mathbf{k}\}) = \psi_n(\{\mathbf{k}\}) + \psi_n^*(\{-\mathbf{k}\})$$

For the parity-odd case:

$$\rho_n^{\mathrm{PO}}(\{\mathbf{k}\}) = \psi_n(\{\mathbf{k}\}) - \psi_n^*(\{\mathbf{k}\})$$

$$\rho_n^{\mathrm{PO}}(\{\mathbf{k}\}) = 0$$

# Reality + Duality

$$\rho_n^{\mathrm{PO}}(\{\mathbf{k}\}) = 0 \quad + \quad \rho_n = \frac{1}{{B_2}^n}\sum_{k=0}^{n-3}(-1)^k\Big(k\text{-cuts}\Big)_B$$

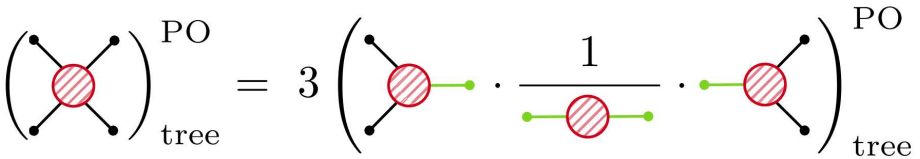
$$0 = \frac{1}{{B_2}^n}\sum_{k=0}^{n-3}(-1)^k\Big(k\text{-cuts}\Big)_B$$

**Correlator-to-correlator factorisation (CCF)**

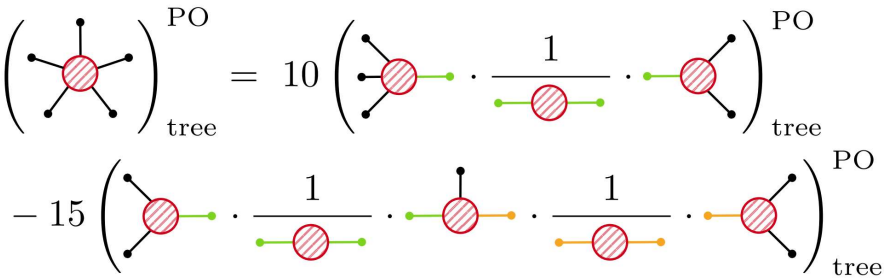
$$B_n^{\mathrm{PO}} = \sum_{k=1}^{n-3}(-1)^{k-1}\Big[\Big(k\text{-cuts}\Big)_B\Big]^{\mathrm{PO}}\,,\ \forall\, n \geq 4$$

*Examples*

**For four points:**




**For Five points:**



- Unitarity Cutting Rule
(Cosmological Optical Theorem)
- Manifestly Local Test
- Connection to scattering amplitude
- Connection to geometry

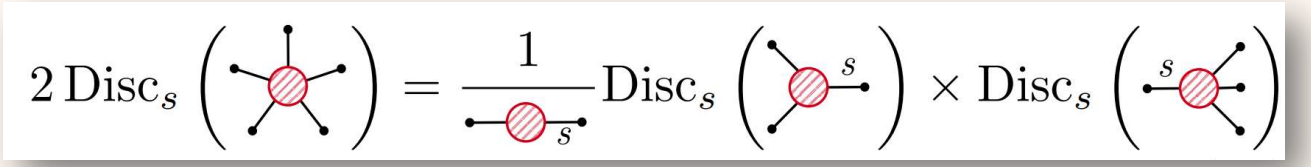
.....

$$g:\begin{pmatrix}\rho_n\\ B_n\end{pmatrix}\mapsto(-i)^n\begin{pmatrix}B_n\\ \rho_n\end{pmatrix}$$

$$\rho_n=\frac{1}{{B_2}^n}\sum_{k=0}^{n-3}(-1)^k\Big(k\text{-cuts}\Big)_B$$

?

Unitarity Cutting Rule for Correlators (5pts):




$$\text{Disc}_y f(\{x\},y)=f(\{x\},y)-f(\{x\},-y)$$

$$\text{Disc}_s B_3 = B_3(k_1,k_2,k_3,s)-B_3(k_1,k_2,k_3,-s)$$

# Summary

## Mapping

$$g : \begin{pmatrix} \rho_n \\ B_n \end{pmatrix} \mapsto (-i)^n \begin{pmatrix} B_n \\ \rho_n \end{pmatrix} \qquad g \in \mathbb{Z}_4$$

## Factorisation

